New results on intersecting families of subsets

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Abstract

Let $[n] = \{1, 2, \ldots, n\}$ be the underlying set. A family $\mathcal{F} \subset 2^{[n]}$ of its subsets is called intersecting if $F, G \in \mathcal{F}$ implies $F \cap G \neq \emptyset$. It is trivial that the largest intersecting family has 2^{n-1} members. The situation is different when only k-element subsets are considered, that is $\mathcal{F} \subset \binom{[n]}{k}$. The celebrated Erdős-Ko-Rado theorem says that if $k \leq \frac{n}{2}$ then an intersecting family of k-element sets cannot have more than $\binom{n-1}{k-1}$ members. Equality can be obtained for the family of sets containing one fixed element.

We will survey some of the results of the history of the area and show some new developments. One such direction is the problem of "two-part intersecting" families. The underlying set $[n]$ is partitioned into X_1 and X_2. Is it still true that the largest intersecting family is the one consisting of members containing one fixed element? It is perhaps surprising that the answer is yes. Even in the following very general form. Some positive integers $k_i, \ell_i (1 \leq i \leq m)$ are given. We prove that if $\mathcal{F}$ is an intersecting family containing members F such that $|F \cap X_1| = k_i, |F \cap X_2| = \ell_i$ holds for one of the values $i (1 \leq i \leq m)$ then $|\mathcal{F}|$ cannot exceed the size of the largest subfamily containing one element. The statement was known for the case $m = 2$ as a result of Frankl.

The shadow $\sigma(\mathcal{F})$ is defined for the case when $\mathcal{F} \subset \binom{[n]}{k}$. It is a family of all $k-1$-element subsets of members of $\mathcal{F}$. The shadow theorem determines the minimum size of the shadow family for fixed n, k and $|\mathcal{F}|$. The optimal family consists of sets being at the "beginning" of $[n]$. This family often contains disjoint pairs. Therefore to find the smallest shadow of an intersecting family is very different from the traditional problem. We will introduce some old and new results of this kind.