Total coloring of some unitary Cayley graphs

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Unitary Cayley graphs

For a positive integer \( n \), the unitary Cayley graph \( X_n = \text{Cay}(\mathbb{Z}_n, \mathbb{U}_n) \) is defined by the additive group of the ring \( \mathbb{Z}_n \) of integers modulo \( n \) and the multiplicative group \( \mathbb{U}_n \) of its units, where \( \mathbb{U}_n = \{ a \in \mathbb{Z}_n : \text{gcd}(a, n) = 1 \} \). The vertex set of \( X_n \) is the set \( V(X_n) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \) and its edge set is \( E(X_n) = \{ ab : a, b \in \mathbb{Z}_n \text{ and gcd}(a - b, n) = 1 \} \). The unitary Cayley graphs \( X_n \) are regular of degree \( \phi(n) \), where \( \phi(n) \) is the Euler function.

Total coloring

A \( k \)-total coloring of \( G \) is an assignment of \( k \) colors to the edges and vertices of \( G \), such that no adjacent elements (vertices and edges) receive the same color. The total chromatic number of \( G \), denoted by \( \chi_{t}(G) \), is the least \( k \) for which \( G \) has a \( k \)-total coloring. Let \( \Delta(G) \) be the maximum degree of \( G \), clearly, \( \chi_{t}(G) \geq \Delta(G) + 1 \) and the Total Coloring Conjecture (TCC) [1, 6] states that \( \chi_{t}(G) \leq \Delta(G) + 2 \). This conjecture has been verified for some classes but the general statement has remained open for more than fifty years and has not been settled even for regular graphs. If \( \chi_{t}(G) = \Delta(G) + 1 \), then \( G \) is said to be Type 1, and if \( \chi_{t}(G) = \Delta(G) + 2 \), then \( G \) is said to be Type 2. The problem of deciding if a graph is Type 1 has been shown NP-complete [5]. For more information, we refer to [3], which is the first PhD thesis on total coloring developed in Brazil.

Total coloring of unitary Cayley graphs

Prajnanaswaroopa et al. [4] established the TCC for all unitary Cayley graphs. Some unitary Cayley graphs are already known to be Type 1 or Type 2. If \( n = p^r \) is a prime power, then \( X_{p^r} \) is a complete \( p \)-partite graph and the total chromatic number is well known: if \( p \) is odd, then \( X_{p^r} \) is Type 1, and if \( p \) is even, then \( X_{p^r} \) is Type 2 [3].

We determine the total chromatic number of all members of two families of unitary Cayley graphs \( X_n \): when \( n = 6s \), for a positive integer \( s \), and when \( n = 3p \), for prime \( p \geq 5 \). Boggess et al. [2] proved that for \( n \geq 3 \), graph \( X_n \) can be decomposed into \( \frac{n}{2} \) edge-disjoint Hamiltonian cycles, denoted by \( H_i \), with \( i \in \mathbb{N} \); and this result is used to prove the following theorems. Consider directed edges \( \{ (i, i+j \mod n) : 0 \leq i \leq n-1 \} \) to indicate the direction used to construct the cycles \( H_i \), as \( H_i \) and \( H_{(i+1) \mod n} \) are the same cycle.

Theorem 1. For positive integer \( s \), the graph \( X_{6s} \) is Type 1.

Proof. Graph \( X_{6s} \) is bipartite with parts \( A = \{ 2i : 0 \leq i \leq \frac{6s-2}{2} \} \) and \( B = \{ 2i+1 : 0 \leq i \leq \frac{6s-2}{2} \} \). Consider the Hamiltonian cycle \( H_1 \), since it has 6s vertices, it is well known that admits a 3-total coloring \( T \) such that vertices \( i \), with \( i \equiv 0 \mod 3 \) (resp. \( i \equiv 1 \mod 3 \) and \( i \equiv 2 \mod 3 \)) receive the same color. As \( G \neq \mathbb{U}_s \), the adjacent vertices in \( X_{6s} \) do not have the same color assigned by \( T \). Now, remove from \( X_{6s} \) all the edges in \( H_{6s} \). Clearly, the resulting bipartite graph is \( (\Delta(X_{6s}) - 2) \)-regular and, by Hall’s theorem, it can be edge colored with \( \Delta(X_{6s}) - 2 \) colors. Therefore, \( X_{6s} \) is Type 1. The following figure presents a 5-total coloring of \( X_{12} \).

Theorem 2. For prime \( p \geq 5 \), the graph \( X_{3p} \) is Type 1.

Idea of the proof. Graph \( X_{3p} \) is a 3-partite graph with parts \( A = \{ 3i : 0 \leq i \leq p - 1 \} \), \( B = \{ 3i+1 : 0 \leq i \leq p - 1 \} \) and \( C = \{ 3i + 2 : 0 \leq i \leq p - 1 \} \). By Vizing’s theorem, each Hamiltonian cycle \( H_{3j} \) admits a 3-edge coloring. For \( j > 1 \), assign 3 colors to the edges of every \( H_{3j} \) such that a special color \( c_0 \) is used in all cycles in a particular directed edge \( (a, a+j \mod 3p) \), and the endpoints \( \{ a, a+j \mod 3p \} \) receive 2 different colors already used in the respective cycle. For \( j = 1 \in \mathbb{U}_{3p} \), assign 3 colors to the edges of \( H_{3p} \) so that the special color \( c_0 \) is assigned to exactly 3 directed edges: \( \{ 1, 2 \}, \{ 4, 5 \}, \{ 7, 8 \} \) and the endpoints \( \{ 1, 4, 7 \} \in B \) and \( \{ 2, 5, 8 \} \in C \) receive the 2 colors already used in the respective cycle, one color to each part. The remaining vertices not colored in \( X_{3p} \) are in \( A \), and we assign color \( c_0 \) to these vertices.

Notice that the assignment of colors does not have conflict. We used 2 colors for the elements of each of the \( p - 1 \) Hamiltonian cycles and used color \( c_0 \) in all cycles. Thus, we obtain a \((2p-1)+1=\Delta(X_{3p})+1\)-total coloring. The figure below presents the four edge-disjoint Hamiltonian cycles \( H_{3p} \), \( H_{3p} \), \( H_{3p} \) and \( H_{3p} \) of \( X_{15} \) with a 9-total coloring such that the color \( c_0 \) is represented by purple color.

References