Total coloring of some unitary Cayley graphs

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Unitary Cayley graphs

For a positive integer n, the *unitary Cayley* graph $X_n = \mathrm{Cay}(\mathbb{Z}_n, \mathbb{U}_n)$ is defined by the additive group of the ring $\mathbb{Z}_n$ of integers modulo n and the multiplicative group $\mathbb{U}_n$ of its units, where $\mathbb{U}_n = \{a \in \mathbb{Z}_n : \gcd(a,n) = 1\}$. The vertex set of X_n is the set $V(X_n) = \mathbb{Z}_n = \{0, 1, \cdots, n-1\}$ and its edge set is $E(X_n) = \{ab : a, b \in \mathbb{Z}_n$ and $\gcd(a-b, n) = 1\}$. The unitary Cayley graphs X_n are regular of degree $|\mathbb{U}_n| = \phi(n)$, where $\phi(n)$ is the Euler function.

Total coloring

A k-total coloring of G is an assignment of k colors to the edges and vertices of G, such that no adjacent elements (vertices and edges) receive the same color. The total chromatic number of G, denoted by $\chi_T(G)$, is the least k for which G has a k-total coloring. Let $\Delta(G)$ be the maximum degree of G, clearly, $\chi_T(G) \geqslant \Delta(G) + 1$ and the Total Coloring Conjecture (TCC) [1, 6] states that $\chi_T(G) \leqslant \Delta(G) + 2$. This conjecture has been verified for some classes but the general statement has remained open for more than fifty years and has not been settled even for regular graphs. If $\chi_T(G) = \Delta(G) + 1$, then G is said to be Type 1, and if $\chi_T(G) = \Delta(G) + 2$, then G is said to be Type 2. The problem of deciding if a graph is Type 1 has been shown NP-complete [5].

For more information, we refer to [3], which is the first PhD thesis on total coloring developed in Brazil.

Total coloring of unitary Cayley graphs

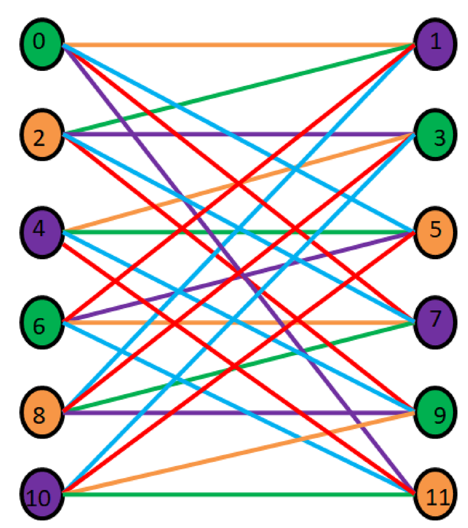
Prajnanaswaroopa et al. [4] established the TCC for all unitary Cayley graphs. Some unitary Cayley graphs are already known to be Type 1 or Type 2. If $n = p^r$ is a prime power, then X_{p^r} is a complete p-partite graph and the total chromatic number is well known: if p is odd, then X_{p^r} is Type 1, and if p is even, then X_{p^r} is Type 2 [3].

We determine the total chromatic number of all members of two families of unitary Cayley graphs X_n: when $n = 6s$, for a positive integer s, and when $n = 3p$, for prime $p \geq 5$.

Boggess et al. [2] proved that for $n \geq 3$, graph X_n can be decomposed into $\frac{\phi(n)}{2}$ edge-disjoint Hamiltonian cycles, denoted by H_n^j, with $j \in \mathbb{U}_n$; and this result is used to prove the following theorems. Consider directed edges $\{\langle i, i+j \mod n\rangle : 0 \leq i \leq n-1\}$ to indicate the direction used to construct the cycles H_n^j, as H_n^j and H_n^{n-j} are the same cycle.

Theorem 1. *For positive integer s, the graph X_{6s} is Type 1.*

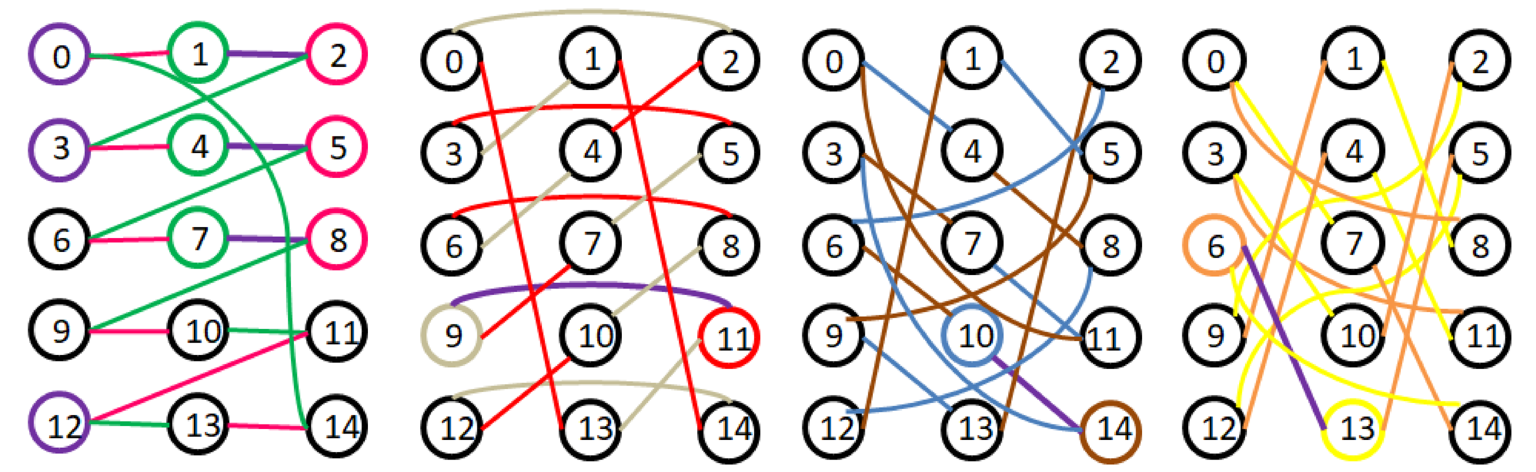
Proof. Graph X_{6s} is bipartite with parts $A = \{2i : 0 \leq i \leq \frac{6s-2}{2}\}$ and $B = \{2i+1 : 0 \leq i \leq \frac{6s-2}{2}\}$. Consider the Hamiltonian cycle H_{6s}^1, since it has $6s$ vertices, it is well known that admits a 3-total coloring T such that vertices i, with $i \equiv 0 \mod 3$ (resp. $i \equiv 1 \mod 3$ and $i \equiv 2 \mod 3$) receive the same color. As $3 \notin \mathbb{U}_{6s}$, the adjacent vertices in X_{6s} do not have the same color assigned by T. Now, remove from X_{6s} all the edges in H_{6s}^1. Clearly, the resulting bipartite graph is $(\Delta(X_{6s})-2)$-regular and, by Hall's theorem, it can be edge colored with $\Delta(X_{6s}) - 2$ colors. Therefore, X_{6s} is Type 1. The following figure presents a 5-total coloring of X_{12}. □



Theorem 2. *For prime $p \geq 5$, the graph X_{3p} is Type 1.*

Idea of the proof. Graph X_{3p} is a 3-partite graph with parts $A = \{3i : 0 \leq i \leq p-1\}$, $B = \{3i+1 : 0 \leq i \leq p-1\}$ and $C = \{3i+2 : 0 \leq i \leq p-1\}$. By Vizing's theorem, each Hamiltonian cycle H_{3p}^j admits a 3-edge coloring. For $j > 1$, assign 3 colors to the edges of every H_{3p}^j such that a special color c_0 is used in all cycles in a particular directed edge $\langle a, a+j \mod 3p\rangle$, and the endpoints $\{a, a+j \mod 3p\}$ receive 2 different colors already used in the respective cycle. For $j = 1 \in \mathbb{U}_{3p}$, assign 3 colors to the edges of H_{3p}^1 so that the special color c_0 is assigned to exactly 3 directed edges: $\langle 1, 2\rangle$, $\langle 4, 5\rangle$, $\langle 7, 8\rangle$; and the endpoints $\{1, 4, 7\} \in B$ and $\{2, 5, 8\} \in C$ receive the 2 colors already used in the respective cycle, one color to each part. The remaining vertices not colored in X_{3p} are in part A, and we assign color c_0 to these vertices.

Notice that the assignment of colors does not have conflict. We used 2 colors for the elements of each of the $p-1$ Hamiltonian cycles and used color c_0 in all cycles. Thus, we obtain a $2(p-1)+1 = \Delta(X_{3p})+1$-total coloring. The figure below presents the four edge-disjoint Hamiltonian cycles H_{3p}^1, H_{3p}^2, H_{3p}^4 and H_{3p}^7 of X_{15} with a 9-total coloring such that the color c_0 is represented by purple color. □



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