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The sandpile group of outerplanar graphs

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Introduction

We compute the sandpile groups of outerplanar planar graphs. The method can be used to determine the algebraic structure of the sandpile groups of other planar graph families.

Sandpile groups

The sandpile group was originated in statistical physics. It was the first model of a dynamical system exhibiting self-organized criticality.
The dynamics of the sandpiles are developed over a graph G in the following way. Consider a graph G with a special vertex q, called *sink*. A *configuration* c is a vector whose entries are associated with the number of grains of sand at each vertex of G. The sink vertex collects the sand quitting the system. A vertex is *stable* if the number of sand grains on it is lower than its *degree*, that is, the number of edges incident to the vertex. Otherwise, the vertex is *unstable*. A configuration is *stable* if all the non-sink vertices of G are stable. A *toppling* of an unstable configuration consists of selecting an unstable vertex v and moving $\deg(v)$ grains from v to its neighbors, such that each neighbor u receives $m(u,v)$ grains, where $m(u,v)$ denotes the number of edges between u and v. In Figure 1, we show a sequence of topplings.

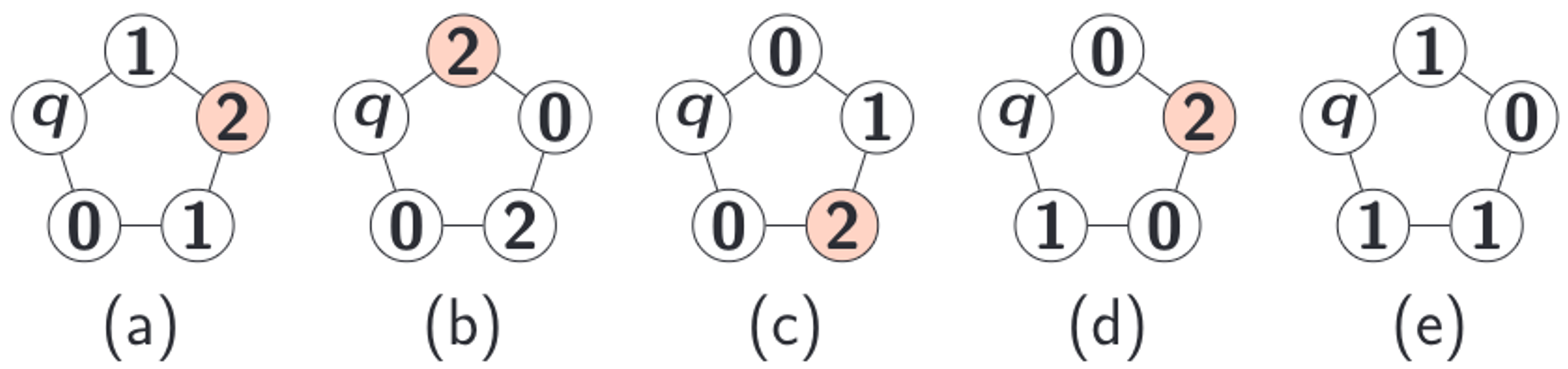

Figure 1: The sequence of topplings starts in (a) with the configuration $(1,2,1,0,q)$ over the cycle with 5 vertices. At each step, the toppled vertex is highlighted in red.

Over connected graphs with a sink, we will always obtain a stable and unique configuration after a finite sequence of topplings. The stable configuration obtained from the configuration c will be denoted by $s(c)$. The sum of two configurations c and d is performed entry-by-entry. Let $c \oplus d := s(c+d)$. A configuration c is recurrent if there exists a non-zero configuration d such that $c = c \oplus d$. Recurrent configurations play a central role in the dynamics of the Abelian sandpile model since recurrent configurations together with the $\oplus$ operation form an Abelian group known as *sandpile group* and denoted $K(G)$. An introduction to the topic can be found in [1].
For example, the recurrent configurations for the cycle with 5 vertices and sink vertex q are $(0,1,1,1,q)$, $(1,0,1,1,q)$, $(1,1,0,1,q)$, $(1,1,1,0,q)$ and $(1,1,1,1,q)$. Could the reader verify that these configurations form an Abelian group with the $\oplus$ operation? Which configuration is the identity?

Smith normal form and graphs

Let $GL_n(\mathbb{Z})$ denote the group of $n \times n$ invertible matrices with entries in the integers whose inverses also have entries in the integers. Two matrices M and N are *equivalent* if there exist two matrices $P, Q \in GL_n(\mathbb{Z})$ such that $M = QNP$. The *Smith normal form* of the matrix M is the unique diagonal matrix $diag(d_1, \dots, d_r, 0, \dots, 0)$ equivalent to M such that r is the rank of M and $d_i | d_j$ for $i < j$. The integers $d_1, \dots, d_r$ are called *invariant factors*.
Let G be a planar graph with s interior faces $F_1, \dots, F_s$, let $c(F_i)$ denote the number of edges in the cycle bounding F_i. We define the *cycle-intersection matrix*, $C(G) = (c_{ij})$ to be a symmetric $s \times s$ matrix, where $c_{ii} = c(F_i)$, and c_{ij} is the negative of the number of common edges in the cycles bounding F_i and F_j, when $i \neq j$.

Lemma [2]. Let $d_1, \dots, d_r$ be the invariant factors of $C(G)$, where G is a planar graph. Then $K(G) \approx \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r}$.

Sandpile groups of outerplanar graphs

We call a graph *outerplanar* if it has a planar embedding with the outer face containing all the vertices. The *weak dual* graph G_* is constructed the same way as the dual graph but without placing the vertex associated with the outer face. A graph G is biconnected outerplanar if and only if its weak dual is a tree. Note that $C(G) + A(G_*) = diag(c(F_1), \dots, c(F_s))$, where $A(G)$ is the adjacency matrix of G.
A *2-matching* M is a set of edges of a graph G such that each vertex of G is incident with at most 2 edges of M. Let denote by G°, the graph G where each vertex has a loop added. Given a *2-matching* M of G°, let $\Omega(M)$ denote the set of loops in M. A 2-matching M of G° is *minimal* if there is no 2-matching M' of G° such that $\Omega(M')$ is not contained in $\Omega(M)$ and $|M'| = |M|$. The set of minimal 2-matchings of a tree with loops T° with k edges will be denoted by $2M_k(T^\circ)$. Let $d(M)$ denote the determinant of the submatrix of $C(G) = diag(c(F_1), \dots, c(F_s)) - A(T)$ created by taking the rows and columns associated with the loops of M of T°.

Theorem [2]. Let G be a planar biconnected graph whose weak dual is the tree T with n vertices. Let $\Delta_k = gcd(\{d(M) : M \in 2M_k(T^\circ)\})$. Then the spanning-tree number $\tau(G)$ coincides with Δ_n and $K(G) \approx \mathbb{Z}_{\Delta_1} \oplus \mathbb{Z}_{\frac{\Delta_2}{\Delta_1}} \oplus \cdots \oplus \mathbb{Z}_{\frac{\Delta_n}{\Delta_{n-1}}}$.

References

[1] C.J. Klivans. The Mathematics of Chip-Firing. CRC Press, Taylor & Francis Group, 2018.
[2] C.A. Alfaro y R.R. Villagrán. The structure of sandpile groups of outerplanar graphs. [arXiv:2005.01314]