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Conflict Free Closed Neighborhood Coloring Game

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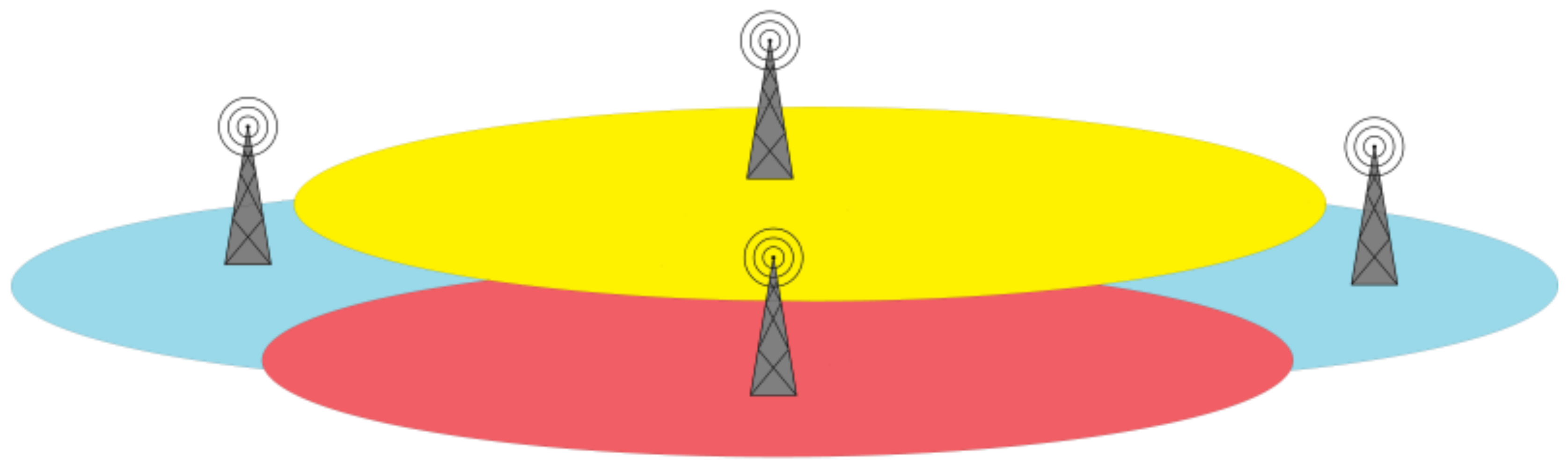
Introduction



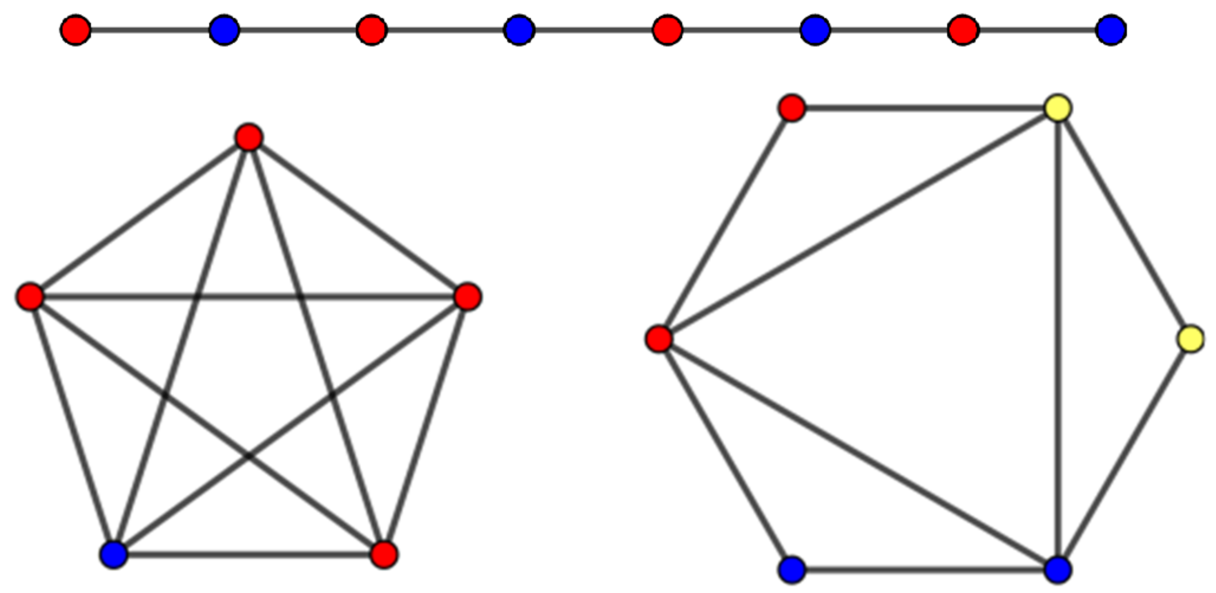
In Cellular Networks, communication between bases and mobile devices is established via radio frequencies. Interference occurs if one particular device communicates with two different bases that have the same frequency. So, every device must contact a base with an unique frequency and, since having a lot of different frequencies is expensive, it's important trying to minimize their quantity, in a way that there exists no interference.

With that motivation, in 2002, Even, Lotker, Ron and Smorodinsky [1] introduced the concept of *Conflict Free coloring* in a geometric scenario, which itself led to the study of CFCN coloring in graphs, and, in 2015, Gardano and Rescigno [2] proved that CFCN coloring is NP-complete.

Inspired by this problem, and by the well known coloring game, we introduce a game theoretical approach to CFCN coloring, and determine the minimum number of colors necessary for Alice to have a winning strategy in the case of Complete Graphs.

CFCN Coloring

A *CFCN coloring* of a graph G is an assignment of colors to the vertices of G such that each vertex v in G has an uniquely colored vertex in its *closed neighborhood* $N[v]$ (the set of all vertices adjacent to v including itself). A *CFCN k-coloring* of a graph G is a CFCN coloring with at most k colors. We say that $N[v]$ is *fully colored* if each vertex of $N[v]$ has a color assigned to it. A graph together with a CFCN k-coloring is said to be *CFCN k-colored.*



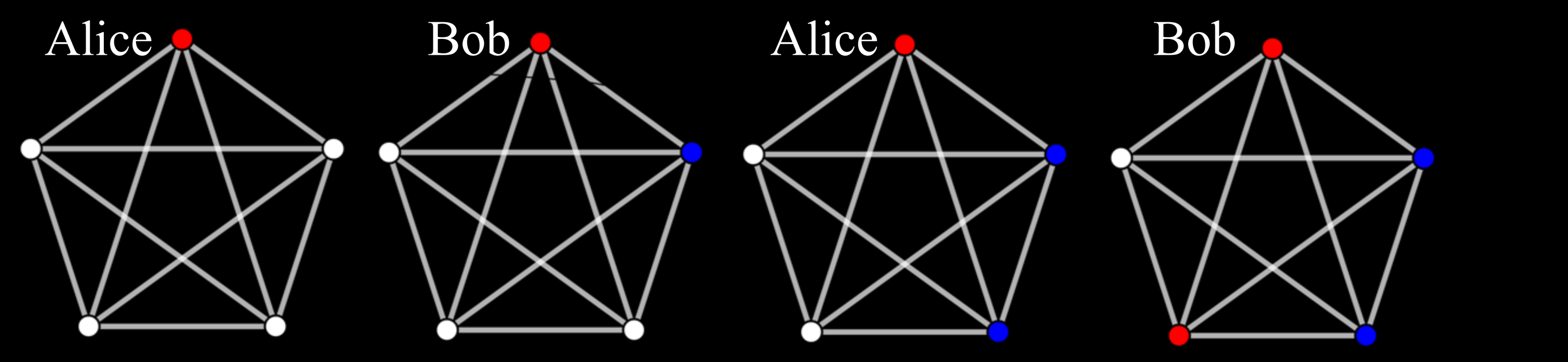
CFCN Coloring of Complete Graphs

Complete graphs have a CFCN 2-coloring, by coloring one vertex with the first color and the others with the second.

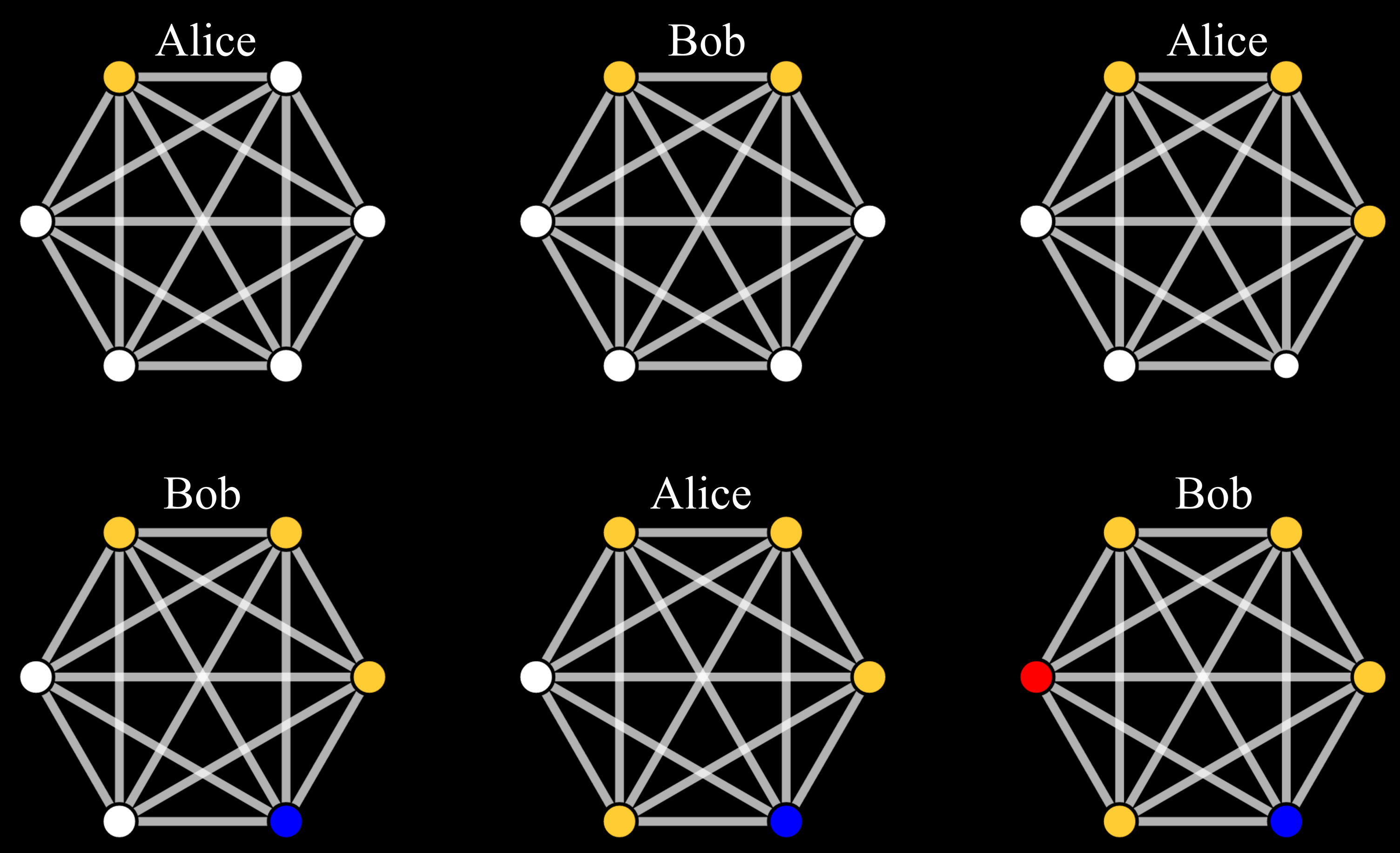
CFCN Coloring Game

Given a graph G and $k>1$ colors, two players, Alice and Bob, take turns coloring vertices of G such that at each turn for every v with a fully colored $N[v]$, the induced subgraph $G[N[v]]$ is CNCF k-colored. The goal of Alice is to obtain a CFCN k-coloring of G while Bob does his best to prevent it. Alice wins if at the end G has a CFCN k-coloring; otherwise Bob wins.

We refer to the next figure for a CFCN 2-coloring game on K_5, where white vertices are uncolored ones. The game ends on the 4th turn because, no matter which color Alice chooses for the 5th turn , it creates a fully colored neighborhood that is not CFCN 2-colored.



The figure below shows a CFCN 3-coloring game on K_6, where white vertices are uncolored ones. The game ends on the 6th turn because the Graph is CFCN 3-colored.



Results

Theorem: *Alice wins CFCN k-coloring game on a complete graph G on n vertices if and only if* $k > \lceil \frac{n}{4} \rceil$.

Sketch of the proof: Let $k>1$ be the number of available colors. Without loss of generality, Alice starts playing in any vertex with color 1.

We claim that Alice always wins if $n \leq 4$ and $k=2$ (winning for any k).

Indeed if $n=1$, Alice colors the vertex with 1. If $n=2$, Alice colors a vertex with 1 and then Bob is forced to color the other vertex with 2. If $n=3$, on the 1st turn she colors a vertex with 1 and on the 3rd with 2. If $n=4$, Alice guarantees that by the 3rd turn, without loss of generality, there are two vertices colored with 1 and one vertex colored with 2, thus Bob has to finish the coloring with 1 or another color different from 2.

If n>4, the proof is based on the following strategy.

Assume that $k \leq \lceil \frac{n}{4} \rceil$, Bob colors a vertex with 2. If Alice colors the next vertex with 1 (resp. 2), Bob colors a vertex with 2 (resp. 1). On the following turns, independently of the colors chosen by Alice, Bob chooses the other colors twice and the game ends. If Alice colors the next vertex with a color c not in $\{1,2\}$, then Bob colors the next ones with 1, 2, c, and then chooses the remaining colors twice. In any case Bob wins the game.

□

Now assume $k > \lceil \frac{n}{4} \rceil$. If the number of vertices is even then Alice always plays 1. If the number of vertices is odd then Alice does the same strategy until her last turn, in which she chooses 1 or one of the remaining colors. In both cases, Alice wins because the graph doesn't have enough vertices for Bob to guarantee that each colors is used twice.

References

[1] G. Even, Z. Lotker, D. Ron, S. Smorodinsky, Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks, SIAM J. Comput. 33 (1) (2004) 94–136.

[2] L. Gardano, A. Rescigno, Complexity of conflict-free colorings of graphs, Theor. Comput. Sci. 566 (2015) 39-49

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