The strong pseudoachromatic number of split graphs

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Introduction

Given a graph G and a set of colors C, a vertex coloring $\alpha : V(G) \to C$ is an assignment of colors from C to the vertices of G. If there are no adjacent vertices with the same color, α is proper. Let β be a not necessarily proper vertex coloring of G such that for every two distinct colors, there are adjacent vertices in G assigned these colors. If β is proper, then it is an achromatic (or complete) coloring of G. If β is nonproper, then it is a pseudoachromatic (or nonproper complete) coloring of G. If β is a pseudoachromatic coloring of G and for every color i, there is an edge of G whose both vertices are colored i, then β is a strong pseudoachromatic (or strong nonproper complete) coloring of G. (See Figure 1.) The <u>maximum</u> number of colors of a strong pseudoachromatic coloring is its strong pseudoachromatic number (or strong achromatic number), $\psi^*(G)$.

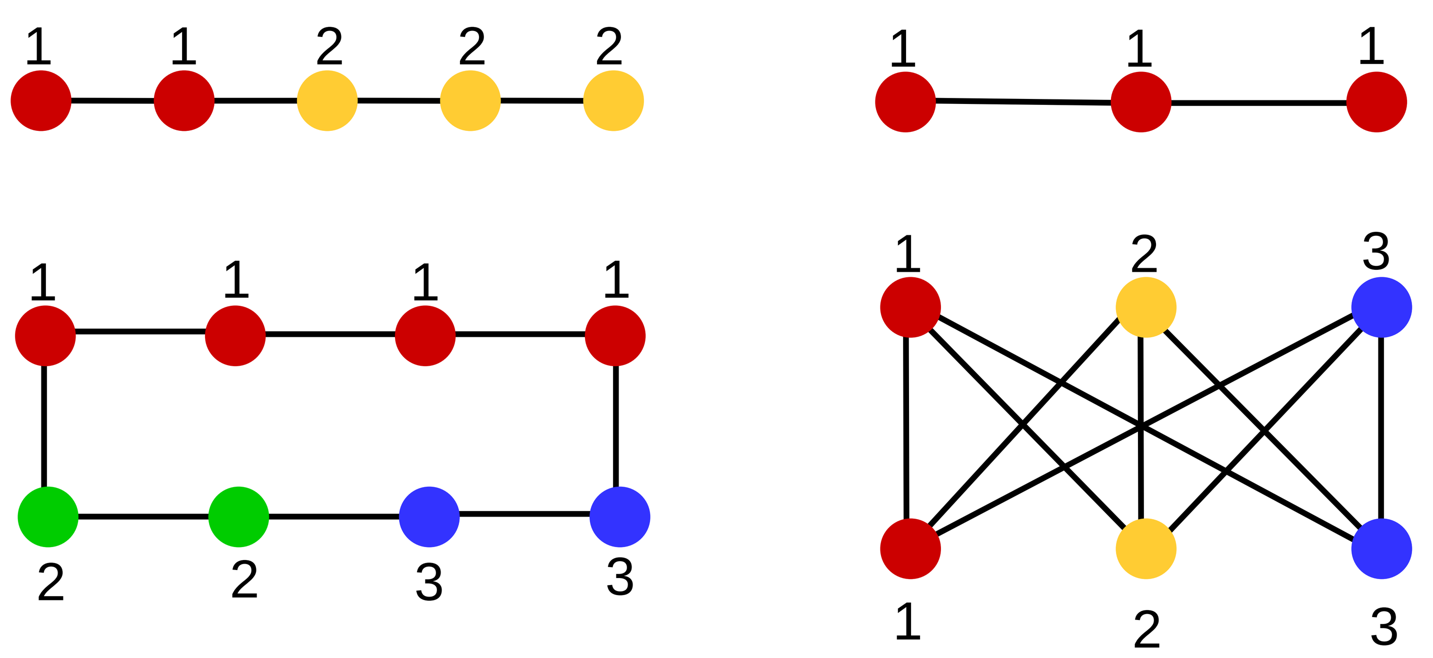

Figure 1: A strong pseudoachromatic coloring for P_5, P_3, C_8 and $K_{3,3}$.

Historical context

Chartrand and Zhang [1, p. 329] presented the strong pseudoachromatic coloring (they use the term "nonproper complete coloring") in the Study Project 6 [1, p. 442]. They ask for bounds to the pseudoachromatic number in terms of the number of edges and suggest investigating the strong pseudoachromatic number of paths and graphs in general.

Previous results

Although there are many studies of the achromatic coloring (see Chartrand and Zhang [1, p. 329]), the only published paper on strong pseudoachromatic coloring is by Liu, Li, and Liu [2]. They present bounds for the strong pseudoachromatic number in the general case and determine the strong pseudoachromatic number of complete graphs, paths, cycles, complete multipartite graphs, complete biequipartite graphs from which a perfect matching is deleted, wheels, fans, and some line graphs.

Motivation

Let G be a graph and $\beta : V(G) \to C$ be a pseudoachromatic coloring of G. By the definition of pseudoachromatic coloring, for each color $i \in C$, there must be an edge whose both vertices are colored i. So, $|C|$ is at most the size of a maximum matching of G, denoted by $\alpha'(G)$. Consequently, $\psi^*(G) \leq \alpha'(G)$. By the previous results [2], this upper bound is tight, since $\psi^*(G) = \alpha'(G)$ when G is a complete graph or a complete multipartite graph. (See Figure 2.)

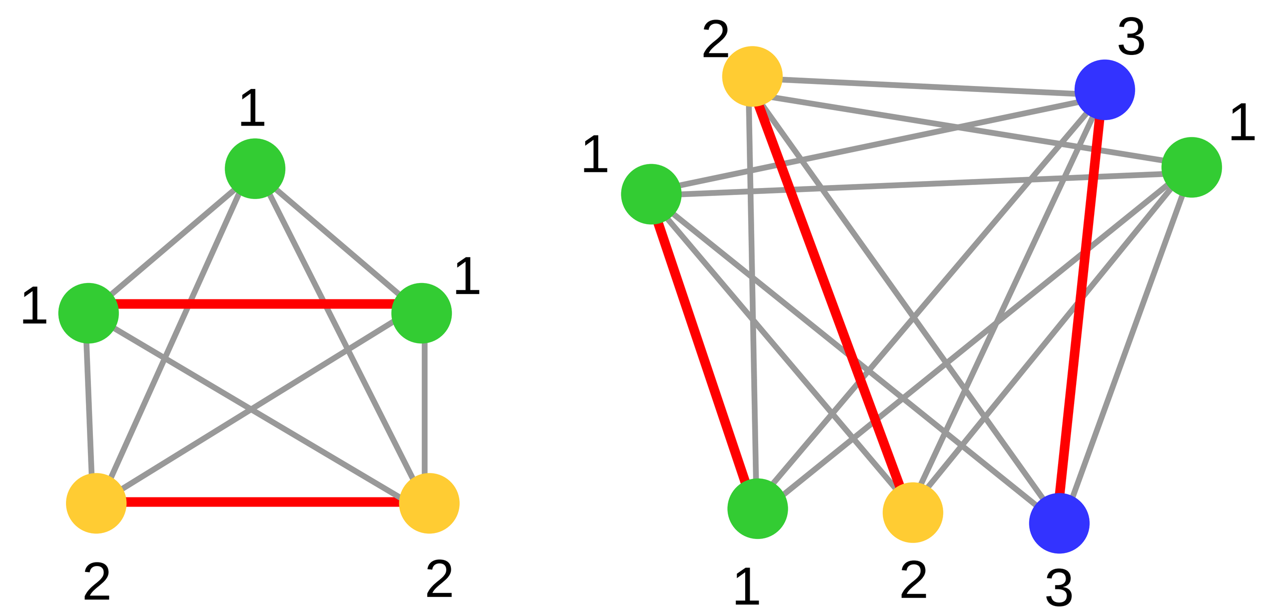

Figure 2: A maximum strong pseudoachromatic coloring and a maximum matching (in red) of K_5 and $K_{2,2,3}$.

A graph G is a split graph iff $V(G)$ can be partitioned into a maximum clique Q and a stable set S. Figure 3 exhibits a split graph. The size of Q is denoted $\omega(G)$. The bipartite subgraph of G obtained by removing the edges between vertices of Q is denoted B_G.

Our contribution

Theorem 2 If G is a split graph, then

$$\alpha'(G) = \alpha'(B_G) + \left\lfloor \frac{\omega(G) - \alpha'(B_G)}{2} \right\rfloor .$$

Theorem 3 If G is a split graph, then $\psi^*(G) = \alpha'(G)$.

Sketch of proof. Since $\psi^*(G) \leq \alpha'(G)$ for any graph G, it is sufficient to exhibit a strong pseudoachromatic coloring with $\alpha'(G)$ colors. Let Q be a maximum clique in G. Consider a maximum matching M_B in B_G and a maximum matching M_Q in $G[Q \setminus V(B_G)]$. For each edge in $M_B \cup M_Q$, assign a new color to its vertices (the same color for both vertices). Assign a color previously used to the remaining vertices. Since, for each color i, there is a vertex in Q colored i and an edge of $M_B \cup M_Q$ whose vertices are colored i, we have a strong pseudoachromatic coloring. □

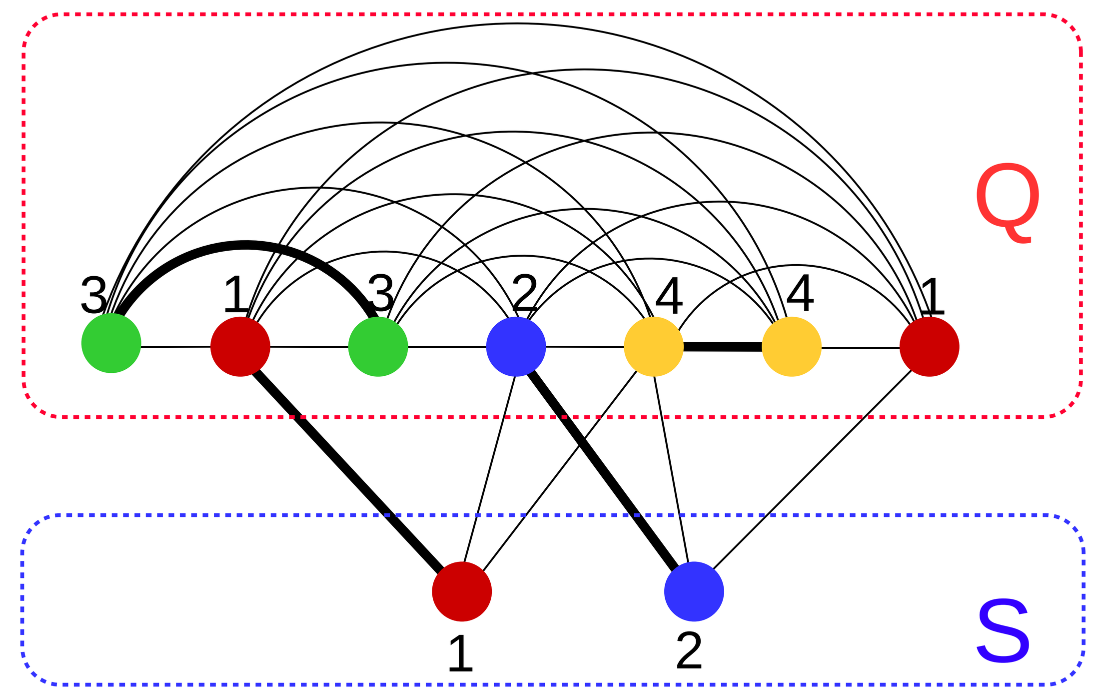

Figure 3: A maximum strong pseudoachromatic coloring of a split graph.

References

[1] Gary Chartrand and Ping Zhang.
Chromatic graph theory.
CRC press, 2009.

[2] Muhuo Liu, Rui Li, and Bolian Liu.
The strong pseudoachromatic number of a graph.
Utilitas Mathematica, 98:3--21, 2015.