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Enumeration of cospectral and coinvariant graphs

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Introduction

Starting from the eigenvalues of a matrix associated to a graph, spectral graph theory seeks to deduce combinatorial properties of the graph. For this, we associate a graph G to a matrix M and analyze the eigenvalues of M. Motivated by the graph isomorphism problem, it is of interest to study, for a graph G, what fraction of all graphs is uniquely determined by the M-spectrum of G. We propose representing a graph using the Smith Normal Form (SNF) of certain distance matrices. We provide numerical evidence that this algebraic representation may do a better job in distinguishing graphs.

Spectrum and invariant factors

The eigenvalues of a matrix $M(G)$ associated with a graph G are called the *M-spectrum* of G, which is the multiset that allows multiple instances for each of its eigenvalues. *M-cospectral* graphs are graphs that share the same M-spectrum.

The *Smith Normal Form* of an integer matrix M, denoted by $SNF(M)$, is the unique diagonal matrix such that $SNF(M) = diag(\mathrm{d}_1, \dots, \mathrm{d}_\mathrm{r}, 0, \dots, 0) = PMQ$ for invertible matrices $P, Q \in GL(n, \mathbb{Z})$ such that r is the rank of M and $d_i | d_j$ for $i < j$. The *invariant factors* of M are the integers in the diagonal of $SNF(M)$. We say that graphs G and H are *M-coinvariant,* if the SNFs of integer matrices $M(G)$ and $M(H)$ are the same.

Enumeration

We focus on the following matrices for connected graphs: the adjacency matrix A, the Laplacian matrix L, the distance matrix D, the signless Laplacian matrix Q, the distance Laplacian matrix D^L and the distance signless Laplacian matrix D^Q.

Extensive research has been devoted to understand cospectral graphs, but much less has been dedicated to understand coinvariant graphs and its potential to characterize graphs. The reason for this could be that for matrices A, L, Q and D, there is a large proportion of connected graphs having a coinvariant graph, as Figure 1.1 shows.

Figure 1.2 displays the number of cospectral and coinvariant graphs for matrices D^L and D^Q. We also include the spectral graphs for matrix Q, since according to Figure 1.1, this would be the best invariant for distinguishing graphs using only the spectrum. According to our results, the SNF of D^Q performs better than the spectrum for distinguishing graphs for the considered matrices. Details can be found in [1].

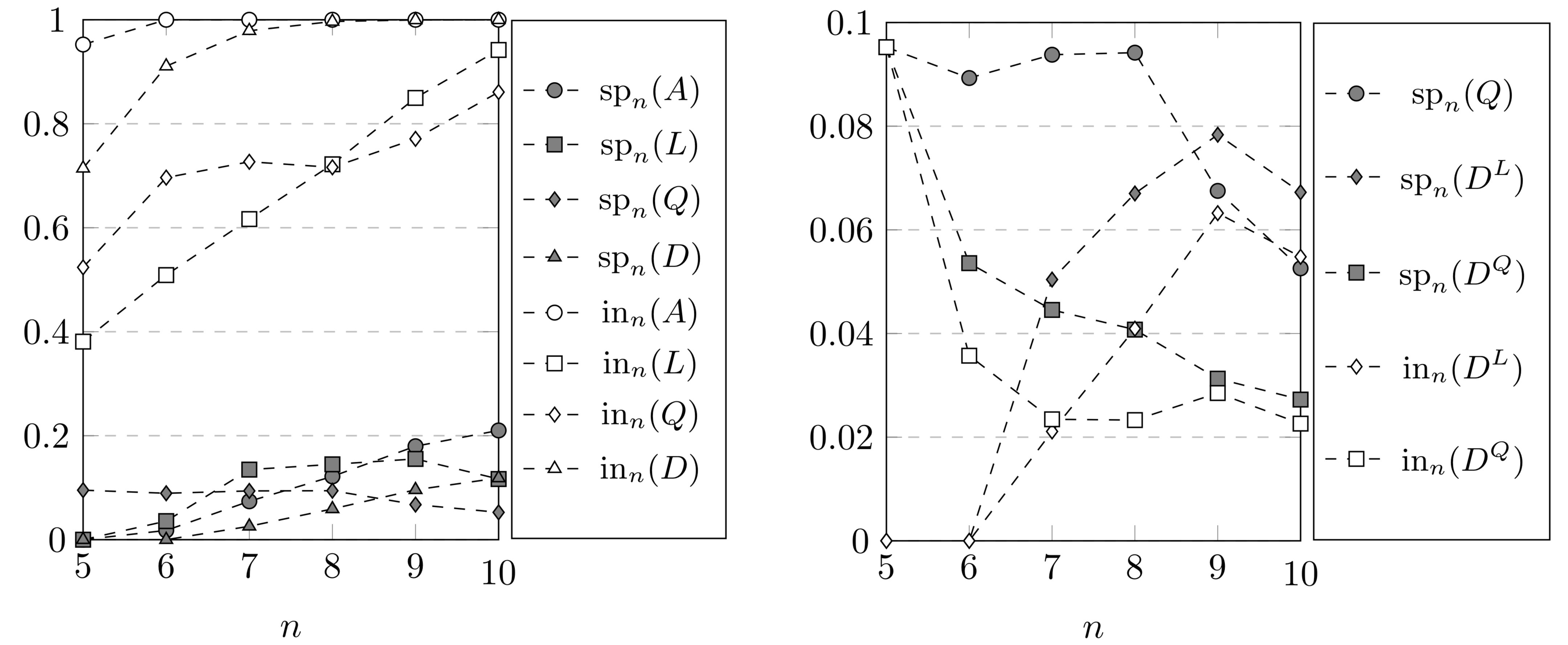

Figure 1: The fraction of connected graphs on n vertices that have at least one cospectral graph with respect to a certain associated matrix is denoted as sp_n. The fraction of connected graphs on n vertices with respect to a certain associated matrix that have at least one coinvariant graph is denoted as in_n.

Coinvariant trees

Aouchiche and Hansen reported in [2] enumeration results on cospectral trees with at most 20 vertices with respect to D, D^L and D^Q matrices. For D, they found that among the 123,867 trees on 18 vertices, there are two pairs of D-cospectral trees. Among the 317,955 trees on 19 vertices, there are six pairs of D-cospectral trees. There are 14 pairs of D-cospectral trees over all the 823,065 trees on 20 vertices. Surprisingly, after the enumeration of all 1,346,023 trees on at most 20 vertices, they found no D^L-cospectral trees and no D^Q-cospectral trees. This fact led Aouchiche and Hansen to conjecture that every tree is determined by its distance Laplacian spectrum, and by its distance signless Laplacian spectrum.

Analogously, for the SNF of D, D^L and D^Q of trees, one can obtain some similar insights. Hou and Woo obtained in [3] that the SNF of the distance matrix for any tree with $n+1$ vertices equals $I_2 \oplus I_{n-2} \oplus (2n)$. From which follows that all trees with n vertices are D-coinvariant graphs. On the other hand, after enumerating coinvariant trees with at most 20 vertices with respect to D^L and D^Q, we found no D^L-coinvariant trees and no D^Q-coinvariant trees among all trees with up to 20 vertices. This fact led us to conjecture that all trees are determined by the SNF of D^L, and, analogously, by the SNF of D^Q.

References

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[3] Y. Hou and C. Woo. Distance unimodular equivalence of graphs. Linear Multilinear Algebra 56 (2008) 611-626.