Intersection models for 2-thin and proper 2-thin graphs

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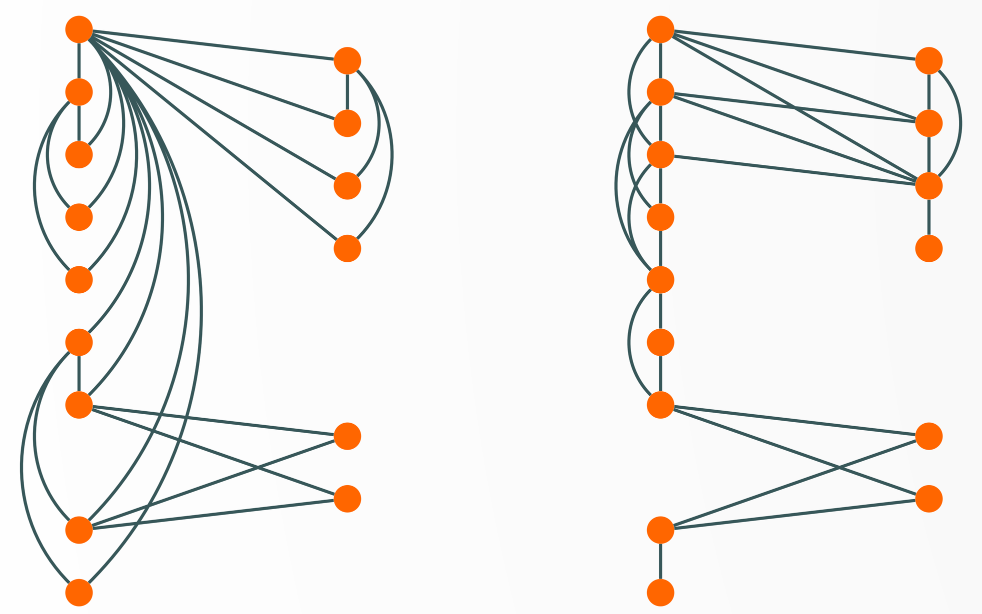
Thinness and proper thinness

A graph $G = (V, E)$ is k-thin if there exist an ordering and a k-partition of V s.t., for $u < v < w$, if u, v belong to the same class and $uw \in E$, then $vw \in E$. The minimum such k is called the *thinness* of G and denoted $\mathrm{thin}(G)$ [1].

Interval graphs[1] are exactly the 1-thin graphs, and 2-thin graphs include convex bipartite graphs. Complements of induced matchings have unbounded thinness.



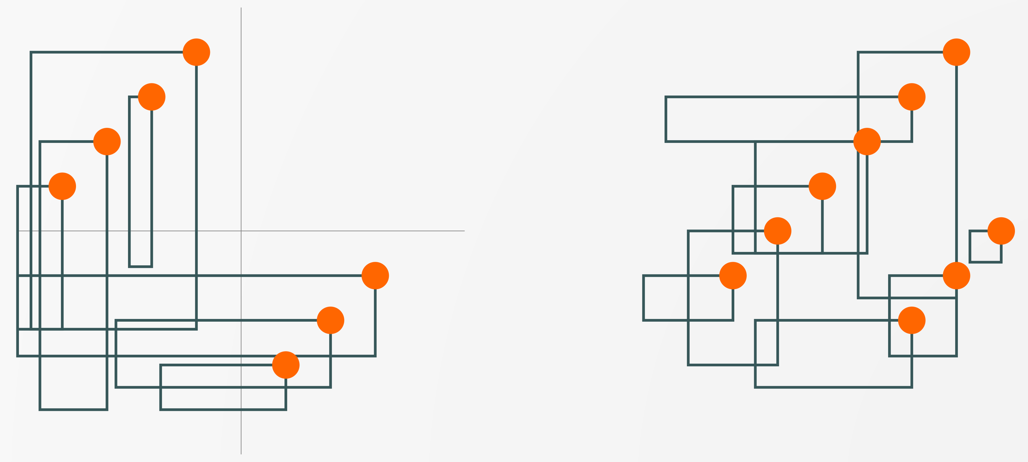
A 2-thin graph and a proper 2-thin graph.

A graph $G = (V, E)$ is proper k-thin if there exist an ordering and a k-partition of V s.t., for $u < v < w$, if u, v belong to the same class and $uw \in E$, then $vw \in E$, and if v, w belong to the same class and $uw \in E$, then $uv \in E$. The minimum such k is called the *proper thinness* of G and denoted $\mathrm{pthin}(G)$ [2].

Proper interval graphs are exactly the proper 1-thin graphs, and interval graphs have unbounded proper thinness.

2-diagonal box intersection models

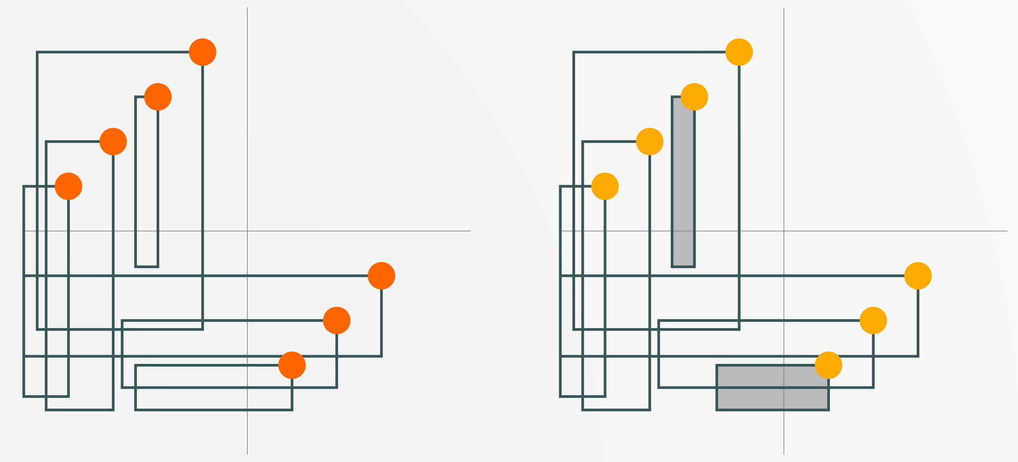
A set of boxes drawn with sides parallel to the Cartesian axes of the plane is 2-diagonal if their upper-right corners are pairwise distinct and lie in two diagonals $y = x + d_1$, $y = x + d_2$, either in the 2nd or in the 4th quadrant, and weakly 2-diagonal if there is no quadrant restriction.



A 2-diagonal and a weakly 2-diagonal model.

Blocking models

A model is blocking if for two non-intersecting boxes b_1, b_2 in the upper and lower diagonal, resp., either the vertical prolongation of b_1 intersects b_2 or the horizontal prolongation of b_2 intersects b_1.



Blocking 2-diagonal model and not.

Characterizations

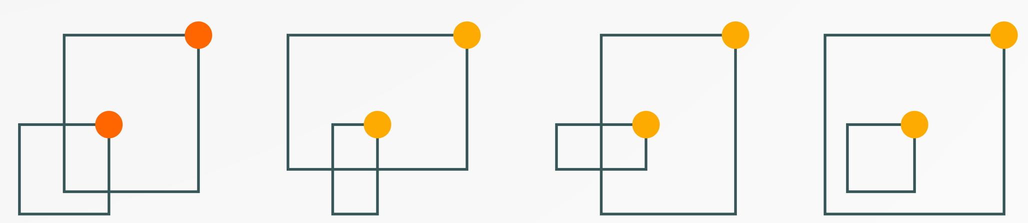
The main results of this work are the following characterizations of 2-thin and proper 2-thin graphs as intersection graphs of boxes drawn with sides parallel to the Cartesian axes of the plane.

Theorem. A graph is 2-thin if and only if it has a blocking 2-diagonal model.

The blocking property is necessary since there are graphs with thinness 3 and a 2-diagonal model.

A model is bi-semi-proper if for two boxes b, b' in the same diagonal, $x_2 < x'_2$ implies $x_1 \leq x'_1$ and $y_1 \leq y'_1$.

Theorem. The following statements are equivalent:
1. G is a proper 2-thin graph.
2. G has a bi-semi-proper blocking 2-diagonal model.
3. G has a bi-semi-proper weakly 2-diagonal model.



Example of bi-semi-proper (first situation) and not bi-semi-proper (last three situations).

The bi-semi-proper property is necessary as interval graphs may have arbitrarily large proper thinness. These models are based on a model by Mannino, Oriolo and Chandran, defined to show that k-thin graphs can be represented as intersection graphs of boxes in the k-dimensional Euclidean space.

2-thin graphs as VPG graphs

A graph is B_k-VPG if it is the vertex intersection graph of paths with at most k bends in a grid. An L-graph is a B_1-VPG graph admitting a representation with all the paths having the same of the four possible shapes L, ⅃, Γ, ꓶ.

- B_0-VPG graphs have unbounded thinness.
- 2-thin graphs are L-graphs (thus B_1-VPG).
- The wheel W_4 is 2-thin and not B_0-VPG.
- 3-thin graphs are B_3-VPG.

Bonus track: new upperbound

The pathwidth (resp. bandwidth) of a graph G can be defined as one less than the maximum clique size of an interval (resp. *proper* interval) supergraph of G, chosen to minimize its maximum clique size [3].

It was proved in [1] that

$$\mathrm{thin}(G) \leq \mathrm{pw}(G) + 1$$

We prove that, if $|E(G)| \geq 1$, then

$$\mathrm{pthin}(G) \leq \mathrm{bw}(G)$$

References

[1] C. Mannino, G. Oriolo, F. Ricci, and S. Chandran. The stable set problem and the thinness of a graph. *Oper. Res. Lett.*, 35:1–9, 2007.

[2] F. Bonomo and D. De Estrada. On the thinness and proper thinness of a graph. *Discrete Appl. Math.*, 261:78–92, 2019.

[3] H. Kaplan and R. Shamir. Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques. *SIAM J. Comput.*, 25(3):540–561, 1996.

[1] Due to lack of space, some standard graph classes, graph parameters, and small graphs are not defined here. The definitions of those concepts can be found in **http://graphclasses.org**