Reduced indifference graphs are type 1

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Introduction

In the year that Celina Figueiredo, João Meidanis and Célia de Mello celebrate another decade of life, we point out the following result which is an immediate consequence of their papers.

Corollary 1

All reduced indifference graphs are type 1.

Let G be a simple graph. A *total coloring* is an assignment of colors to the vertices and edges of G such that no two adjacent or incident elements receive the same color. See Fig. 1.

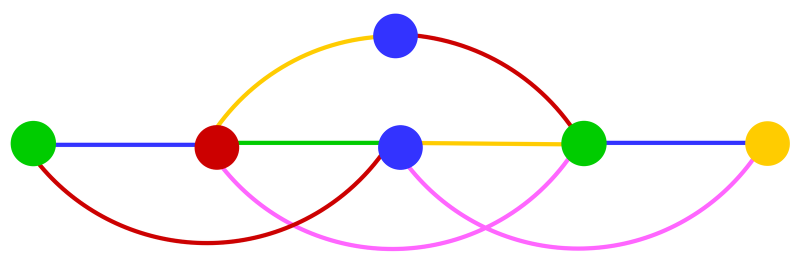

Figure 1: A total coloring of the Hajós graph.

The minimum number of colors for a total coloring of G is the *total chromatic number*, $\chi''(G)$. By definition, $\chi''(G) \geq \Delta(G)+1$. Vizing and Behzad posed the famous Total Coloring Conjecture.

Total Coloring Conjecture (TCC) [1, 2]

$$\chi''(G) \leq \Delta(G) + 2$$

If G has $\chi''(G) = \Delta(G) + 1$, it is *type 1*, otherwise it is *type 2*. By Theorem 1, it is NP-complete to decide if a graph is type 1 for the general case.

Theorem 1 [3]

To decide if a cubic bipartite graph G has $\chi''(G) = \Delta(G) + 1$ is NP-complete.

Total coloring of dually chordal graphs

A graph is *dually chordal* if it is the clique of a chordal graph. Dually chordal graphs generalizes known subclasses of chordal graphs such as interval graphs and indifference graphs. Celina Figueiredo, João Meidanis and Célia de Mello [4] presented the following result.

Theorem 2 [4]

If G is dually chordal, the TCC holds. Moreover, if $\Delta(G)$ is even, G is type 1.

The proof of Theorem 2 gives a polynomial-time algorithm that yields an optimum total coloring of dually chordal graphs with even maximum degree.

Reduced indifference graphs

G is an *indifference graph* if and only if its vertices can be ordered such that those that belong to the same maximal clique are consecutive. This order is known as *indifference order*. Two vertices are *true twins* if they are adjacent and belong to the same maximal cliques. A graph is *reduced* if it does not contain true twins. See Fig. 2.

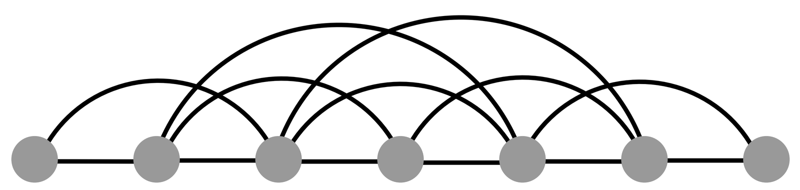

Figure 2: A reduced indifference graph.

Celina Figueiredo, Célia de Mello and Carmen Ortiz [5] presented the following interesting property on indifference graphs.

Theorem 3 [5]

If G is an indifference graph that does not contain maximum degree true twins, then G has a matching M that covers every maximum degree vertex. Moreover, the graph $G-M$, obtained from G by removing the edges of M, is an indifference graph.

Fig. 3 exhibits an indifference graph and a matching that covers its maximum degree vertices.

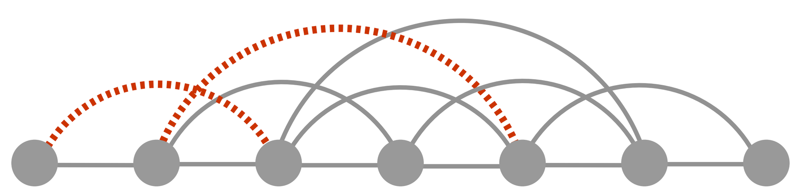

Figure 3: A matching according to Theorem 3.

We use the same technique presented in the proof of Theorem 2 and the property presented in Theorem 3 to prove Theorem 4 and, consequently, Corollary 1. Our proof also gives a polynomial-time algorithm for an optimum total coloring of reduced indifference graphs.

New result

Theorem 4

If G is an indifference graph that does not contain maximum degree true twins, then G is type 1.

Sketch of proof. If $\Delta(G)$ is even, $\chi''(G) = \Delta(G) + 1$, by Theorem 2. Suppose that $\Delta(G)$ is odd. Since G does not contain maximum degree true twins, it has a matching M that covers all maximum degree vertices, by Theorem 3. By Theorem 2, $\chi''(G-M) = \Delta(G)$. Consider an optimum total coloring of $G-M$ as in the proof of Theorem 2. Assign a new color for the edges of M. If the endvertices of an edge in M receive the same color, G has maximum degree true twins, a contradiction. So, $\chi''(G) = \Delta(G) + 1$. □

Fig. 4 presents a total coloring for the graph of Fig 2.

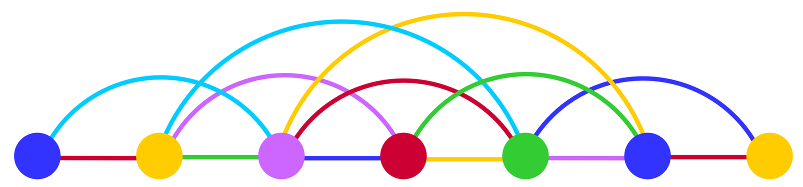

Figure 4: An optimum total coloring according to Theorem 4.

Corollary 1 is an immediate consequence of Theorem 4.

References

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