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On Embedding Trees in Grids

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Introduction

The problem of *grid embedding* is that of drawing a graph G onto a rectangular two-dimensional grid (called simply *grid*) such that each vertex $v \in V(G)$ corresponds to a grid point (an intersection of a horizontal and a vertical grid line) and the edges of G correspond to paths of the grid. Grid embedding of graphs has been considered with different perspectives [2, 5, 6]. In [5], linear-time algorithms are described for embedding planar graphs having their edges drawn as non-intersecting paths in the grid, such that the maximum number of bends of any edge is minimized, as well as the total number of bends.

Objective

We are interested in embedding trees T with $\Delta(T) \leq 4$ in a rectangular grid, such that the vertices of T correspond to grid points, while edges of T correspond to non-intersecting straight segments of the grid lines. The aim is to minimize the maximum number of bends of a path of T. We provide a quadratic-time algorithm for this problem. With this algorithm, we obtain an upper bound on the number of bends of EPG models of VPT∩EPT graphs [3, 4].

Embedding trees in a grid

Let T be a tree such that $\Delta(T) \leq 4$. Consider the problem of embedding such a tree in a grid $\mathcal{G}$, so that the vertices must be placed at grid points and the edges drawn as non-intersecting paths of $\mathcal{G}$ with no bends, which we will call a *model* of T. See Figures 1-5 for key notations.

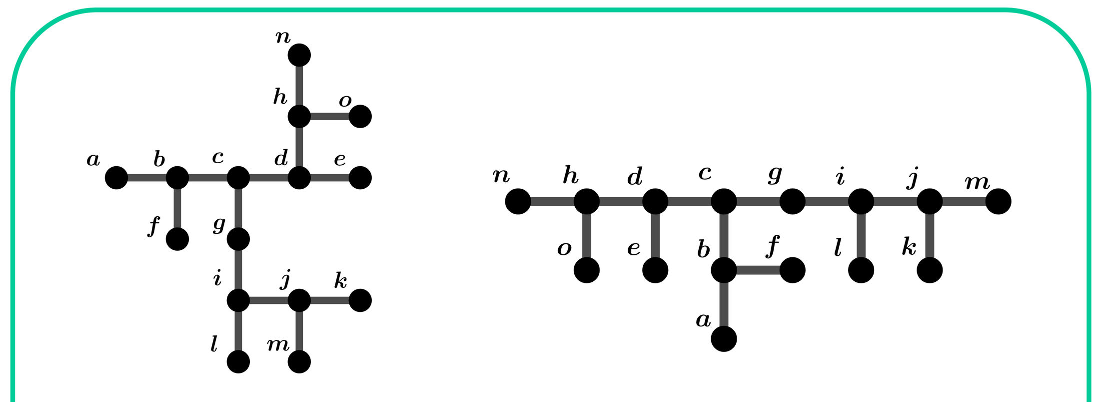

Figure 1: Two possible models M_1 (left) and M_2 (right) of the same tree T.

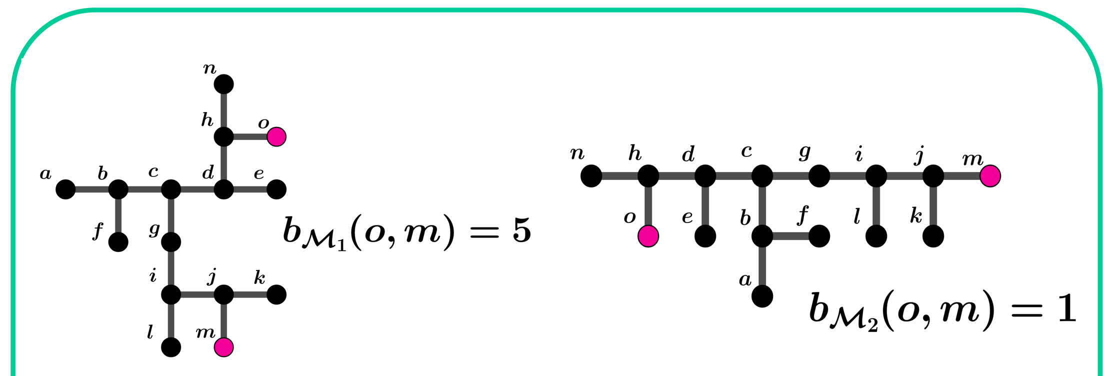

Figure 2: The number of bends of the path connecting u and v in M is denoted by $b_M(u, v)$.

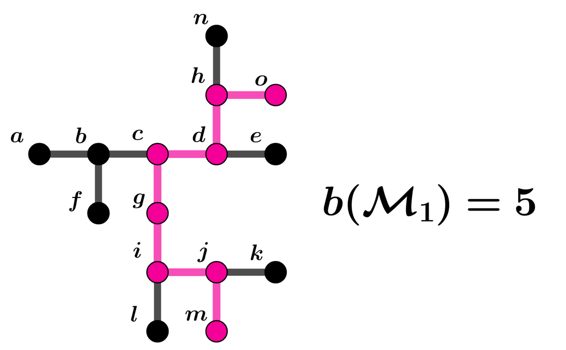

Figure 3: The number of bends of model M is $b(M) = \max\{b_M(u, v) \mid u$ and v are leaves of $T\}$.

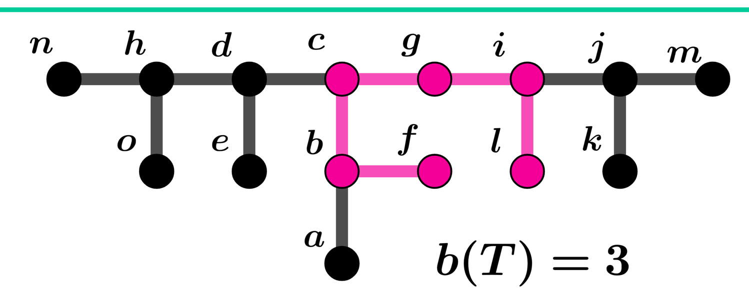

Figure 4: The number of bends of tree T is $b(T) = \min\{b(M) \mid M$ is a model of $T\}$.

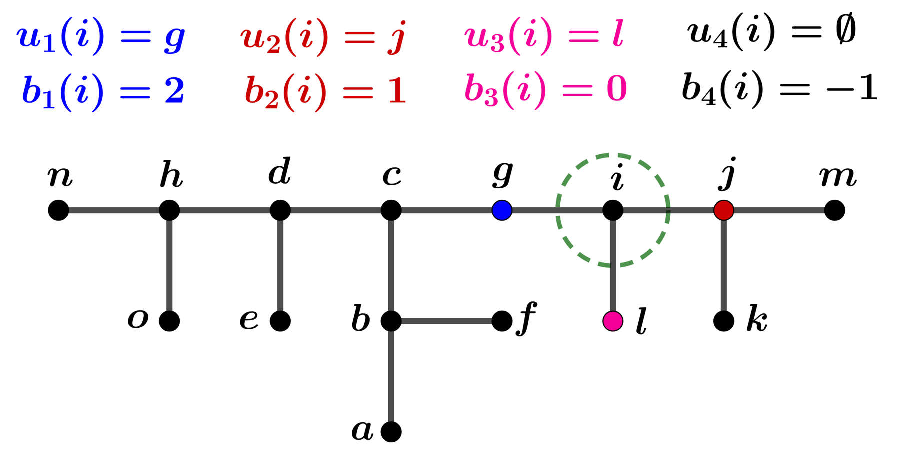

Figure 5: The neighborhood of vertex i ordered according to $b_\ell(i, j)$ for all $j \in N(i)$.

Given a model M, let $b_\ell(p,v)$ be the maximum number of bends of a path in M having as extreme vertices p and a leaf $\ell \in V(T)$, over all paths that contain $v \in V(T)$.

Let M be a model of T and $v \in V(T)$. Let $N(v) = \{u_i(v) \mid 1 \leq i \leq d(v)\}$ be the neighborhood of v and $b_i(v) = b_\ell(v, u_i(v))$.

For $d(v) < i \leq 4$, define "virtual" neighbors $u_i(v) = \emptyset$ for which $b_i(v) = -1$. Assume that the neighbors (both real and virtual) are ordered so that $b_i(v) \geq b_{i+1}(v)$ for all $1 \leq i < 4$. See example in Figure 5.

Let $v \in V(T)$ and M a model of T. We say that v is *balanced* if $u_1(v)$ and $u_2(v)$ are mutually in the same horizontal or vertical grid line in M (and, therefore, so are $u_3(v)$ and $u_4(v)$).

Question

Over all possible models, consider the problem of finding one in which the maximum number of bends of a path of T, over all of them, is minimum.

The algorithm

Algorithm 1: Determining $b(T)$

Input : a tree T such that $\Delta(T) \leq 4$
Output: a model $\mathcal{M}$ of T such that $b(\mathcal{M}) = b(T)$
Let $S = (v_0, \emptyset), (v_1, p_1), \ldots, (v_{n-1}, p_{n-1})$ be such that T is incrementally built by S
Let $\mathcal{M}$ be a model having a single vertex v_0 at some grid point
for $i \leftarrow 1$ **to** $n-1$ **do**
  Add to $\mathcal{M}$ the vertex v_i attached to the grid point of p_i, in any free horizontal or vertical grid line of p_i
  BALANCE ($\mathcal{M}$, v_i, p_i)

Procedure BALANCE ($\mathcal{M}$, p, v) **:**
  for $u \in N(v) \setminus \{p\}$ **do**
    BALANCE ($\mathcal{M}$, v, u)
  If v is not balanced, then make it balanced by rearranging in $\mathcal{M}$ the drawing of the four subtrees of v rooted at $u^i(v)$ (for $1 \leq i \leq 4$), potentially rotating and rescaling them to fit [***balance step***]

A tree T can be built from a single vertex v_0 by a sequence $v_1, v_2, \ldots, v_{n-1}$ of vertex additions, each new vertex v_i adjacent to exactly one vertex p_i of T for all $1 \leq i < n$. We will call that T is *incrementally built* by $(v_0, \emptyset), (v_1, p_1), \ldots, (v_{n-1}, p_{n-1})$.

Algorithm 1 consists of iteratively adding vertices to T and, for each new vertex v, traversing T in post-order having v as the root. The operation to be carried out in each visited vertex is to balance v if it is not balanced.

Theorem

Given a tree T, let M be the model produced by the execution of the algorithm on input T. Then, $b(M) = b(T)$.

EPG models of VPT∩EPT graphs

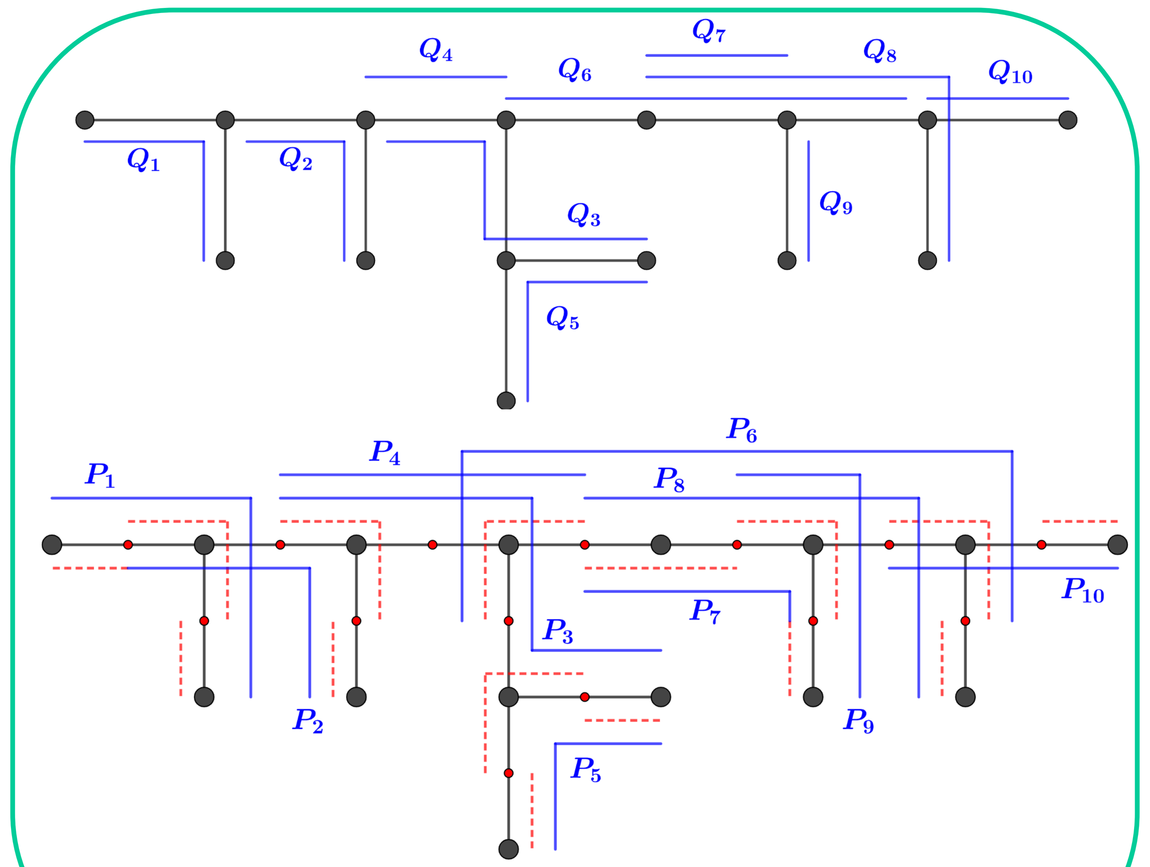

Figure 6: Construction of a B_k-EPG representation with $k \leq b(T)$.

We provide an upper bound on the number of bends of an EPG representation of VPT∩EPT graphs. The VPT∩EPT graphs are those that can be represented in host trees with maximum degree at most 3 [3]. In [1], this class is characterized by a family of minimal forbidden induced subgraphs. An EPG model $R = \{P_i \mid 1 \leq i \leq 10\}$ is shown in Figure 6, obtained from the family $P = \{Q_i \mid 1 \leq i \leq 10\}$.

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