Total Coloring in Some Split-Comparability Graphs

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Luis Gustavo da Soledade Gonzaga[1] **Sheila Morais de Almeida**[1] **Cândida Nunes da Silva**[2] **Jadder Bismarck de Sousa Cruz**[3]

[1] Federal University of Technology - Paraná, Brazil [2] Federal University of São Carlos, Brazil [3] State University of Campinas, Brazil

lgonzaga@alunos.utfpr.edu.br, sheilaalmeida@utfpr.edu.br, candida@ufscar.br, bismarck@ic.unicamp.br

1 Introduction

Let G be a simple graph. For $S \subseteq V(G) \cup E(G)$ and $C\{1, 2, \ldots, k\}$, let $c : S \to C$ be a mapping such that $c(x) \neq c(y)$ for each adjacent or incident elements $x, y \in S$. We say c is a k-total coloring when $S = V(G) \cup E(G)$ and a k-edge coloring when $S = E(G)$. See Fig. 1 for an example. The least j and the least k for which G has a j-total coloring and a k-edge coloring are denoted by $\chi''(G)$ and $\chi'(G)$, respectively.

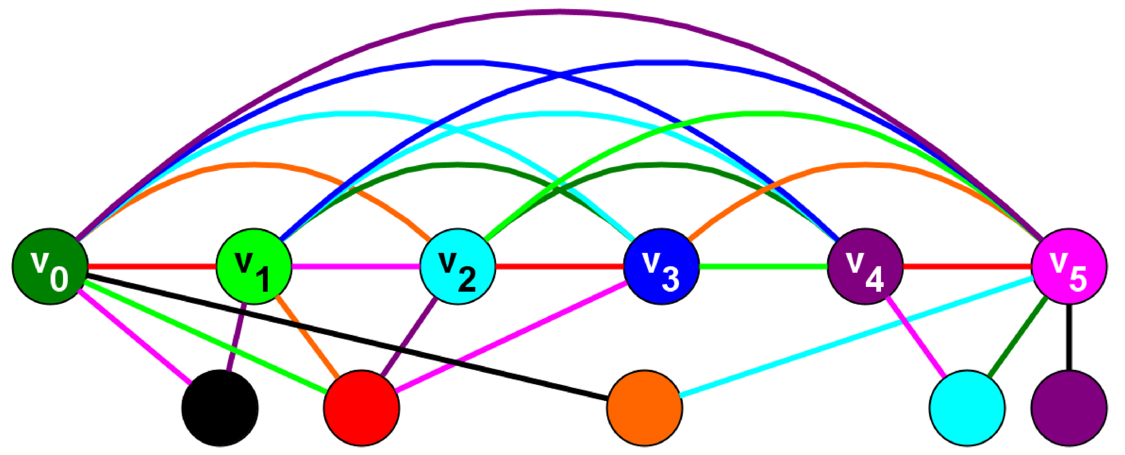

Figure 1: 9-total coloring for G

The Total Coloring Conjecture (TCC) [1, 7] asserts that $\chi''(G) \leq \Delta(G) + 2$ for any G. If $\chi''(G) = \Delta(G) + 1$, G is Type 1; otherwise it is Type 2. To decide if G is Type 1 is NP-Complete [6]. A graph $G[Q, S]$ is split if $V(G)$ can be partitioned into $[Q, S]$ so that Q is a clique and S an independent set.

Theorem 1 [2] Let G be a split graph. Then $\chi''(G) \leq \Delta(G) + 2$. In particular, when $\Delta(G)$ is even G is Type 1.

Ortiz and Villanueva [5] characterized the split-comparability graphs.

Theorem 2 [5] A split graph $G[Q, S]$ is a comparability graph iff Q has a partition $[Q_l, Q_t, Q_r]$ and its vertices can be ordered Q_l, Q_t, Q_r so that for any vertex $s \in S$: $N(s) \cap Q_t = \emptyset$; if $v_k \in (N(s) \cap Q_l)$ then $v_{k-1} \in (N(s) \cap Q_l)$; and if $v_k \in (N(s) \cap Q_r)$ then $v_{k+1} \in (N(s) \cap Q_r)$.

The subset of S whose vertices are not adjacent to Q_r are denoted as S_l, those not adjacent to Q_l denoted as S_r, and $S_t = S \setminus S_l \cup S_r$.

Here we show that certain split-comparability graphs with odd maximum degree are Type 1.

2 Previous Results

When $|E(G)| > \left\lfloor \frac{|V(G)|}{2} \right\rfloor \Delta(G)$ we say G is overfull and if G has a subgraph H with $\Delta(H) = \Delta(G)$ that is overfull, then it is subgraph-overfull. Whenever G is overfull or subgraph-overfull, then $\chi'(G) = \Delta(G) + 1$.

Theorem 3 [3] A split-comparability graph G has $\chi'(G) = \Delta(G)$ iff G is not subgraph-overfull.

Hilton proved the following result for graphs with a universal vertex, i.e. a vertex with degree $|V(G)| - 1$.

Theorem 4 [4] A graph G with a universal vertex is Type 1 iff

$$|E(\overline{G})| + \alpha'(\overline{G}) \geq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor.$$

3 Our Contribution

Theorem 5 A split-comparability graph G, with $|Q_l| \geq |Q_r|$, is Type 1 if

$$|Q| \geq \left(\frac{|S_l|}{|S_l| - 0.5} \right) |Q_l|.$$

Sketch of proof. We assume $|S_r| \neq 0$, $|S_l| \neq 0$ and $\Delta(G)$ is odd, otherwise $\chi''(G)$ is known by Theorems 1 and 4. By Theorem 2, $Q_l \cap Q_r = \emptyset$. Assume $|Q_l| \geq |Q_r|$; so $|Q_r| \leq \frac{|Q|}{2}$. We define a split-comparability supergraph G' of G by adding a vertex v_f twin to the largest degree vertex $v_0 \in Q_l$. Since $|Q_l| \geq |Q_r|$ and $|Q| - |Q_l| \geq \frac{|Q|}{2|S_l|}$, G' is not subgraph-overfull. So, it has a $\Delta(G')$-edge coloring c', by Theorem 3. Fig. 2 shows G' obtained from the graph of Fig. 1.

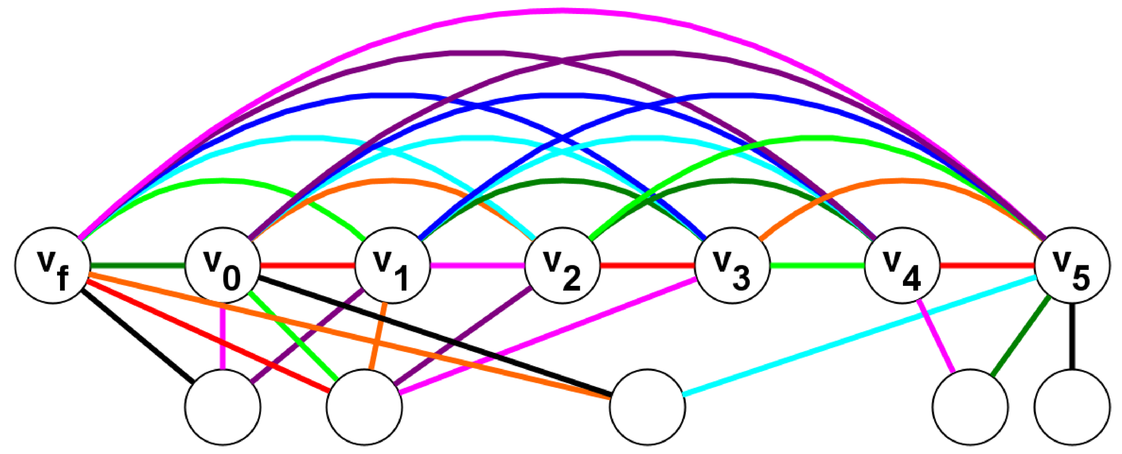

Figure 2: 9-edge coloring for G'

Assign the color $c'(v_f, x)$ to x, for all x in order to obtain a total coloring of $G - S_r$. (Fig. 3 exhibits a partial total-coloring for the graph of Fig. 1.) As $|Q_r| \leq \frac{|Q|}{2}$, at most $|Q|$ colors are used in vertices adjacent or edges incident to vertices of S_r. Since $|Q| < \Delta(G)$ some color is available to be assigned to each vertex $y \in S_r$, and $\chi''(G) = \Delta(G) + 1$.

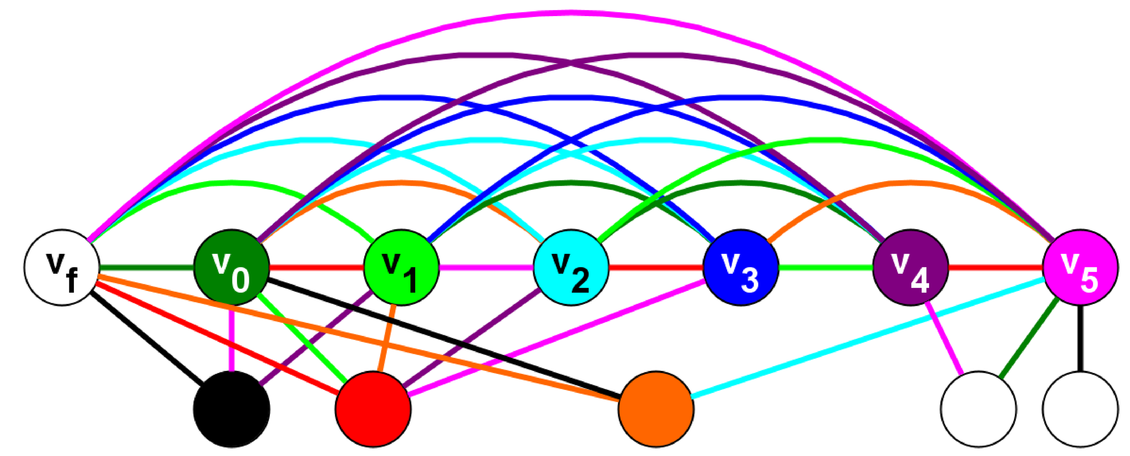

Figure 3: Extending to a total coloring

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