1 Introduction

Let $G$ be a simple graph. For $S \subseteq V(G) \cup E(G)$ and $C\{1, 2, \ldots, k\}$, let $c : S \to C$ be a mapping such that $c(x) \neq c(y)$ for each adjacent or incident elements $x, y \in S$. We say $c$ is a $k$-total coloring when $S = V(G) \cup E(G)$ and a $k$-edge coloring when $S = E(G)$. See Fig. 1 for an example. The least $j$ and the least $k$ for which $G$ has a $j$-total coloring and a $k$-edge coloring are denoted by $\chi^t(G)$ and $\chi^e(G)$, respectively.

![Figure 1: 9-total coloring for $G$](image)

The Total Coloring Conjecture (TCC) [1, 7] asserts that $\chi^t(G) \leq \Delta(G) + 2$ for any $G$. If $\chi^t(G) = \Delta(G) + 1$, $G$ is Type 1; otherwise it is Type 2. To decide if $G$ is Type 1 is NP-Complete [6]. A graph $G[Q, S]$ is split if $V(G)$ can be partitioned into $[Q, S]$ so that $Q$ is a clique and $S$ an independent set.

**Theorem 1** [2] Let $G$ be a split graph. Then $\chi^t(G) \leq \Delta(G) + 2$. In particular, when $\Delta(G)$ is even $G$ is Type 1.

**Theorem 2** [5] A split graph $G[Q, S]$ is a comparability graph if $Q$ has a partition $[Q_l, Q_r, Q_t]$ and its vertices can be ordered $Q_l, Q_r, Q_t$ so that for any vertex $s \in S$: $N(s) \cap Q_l = \emptyset$; if $v_k \in (N(s) \cap Q_t)$ then $v_{k-1} \in (N(s) \cap Q_l)$; and if $v_k \in (N(s) \cap Q_l)$ then $v_{k+1} \in (N(s) \cap Q_r)$.

The subset of $S$ whose vertices are not adjacent to $Q_l$ are denoted as $S_r$ those not adjacent to $Q_t$ denoted as $S_r$, and $S = S_l \cup S_l \cup S_r$.

Here we show that certain split-comparability graphs with odd maximum degree are Type 1.

2 Previous Results

When $|E(G)| \geq \lfloor \frac{|V(G)|}{2} \rfloor \Delta(G)$ we say $G$ is overfull and if $G$ has a subgraph $H$ with $\Delta(H) = \Delta(G)$ that is overfull, then it is subgraph-overfull. Whenever $G$ is overfull or subgraph-overfull, then $\chi^e(G) = \Delta(G) + 1$.

**Theorem 3** [3] A split-comparability graph $G$ has $\chi^e(G) = \Delta(G)$ iff $G$ is not subgraph-overfull.

Hilton proved the following result for graphs with a universal vertex, i.e. a vertex with degree $|V(G)| - 1$.

**Theorem 4** [4] A graph $G$ with a universal vertex is Type 1 iff $|E(G)| + \alpha'(G) \geq \lceil \frac{\Delta(G)}{2} \rceil$.

3 Our Contribution

**Theorem 5** A split-comparability graph $G$, with $|Q| \geq |Q_l|$, is Type 1 if

$$|Q| \geq \left( \frac{|S_l|}{|S_l| - 0.5} \right) |Q|.$$ 

**Sketch of proof.** We assume $|S_l| \neq 0$, $|S_l| \neq 0$ and $\Delta(G)$ is odd, otherwise $\chi^t(G)$ is known by Theorems 1 and 4. By Theorem 2, $Q_l \cap Q_t = \emptyset$. Assume $|Q| \geq |Q_l|$, so $|Q| \leq \frac{|Q|}{2}$. We define a split-comparability supergraph $G'$ of $G$ by adding a vertex $v_l$ twin to the largest degree vertex $v_l \in Q_l$. Since $|Q| \geq |Q_l|$ and $|Q| - |Q_l| \geq \frac{|Q|}{2}$, $G'$ is not subgraph-overfull. So, it has a $\chi^t(G')$-coloring $\chi'$, by Theorem 3. Fig. 2 shows $G'$ obtained from the graph of Fig. 1.

![Figure 2: 9-edge coloring for $G'$](image)

(Fig. 3 exhibits a partial total-coloring for the graph of Fig. 1.) As $|Q| \leq \frac{|Q|}{2}$, at most $|Q|$ colors are used in vertices adjacent or edges incident to vertices of $S_r$. Since $|Q| < \Delta(G)$ some color is available to be assigned to each vertex $y \in S_r$, and $\chi^t(G) = \Delta(G) + 1$.

![Figure 3: Extending to a total coloring](image)

References


