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# L(2,1)-labeling of Loupekine Snarks


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## Introduction

An $L(2,1)$-*labeling* of a simple graph $G = (V, E)$ is a function $f\colon V \to \{0, \ldots, t\}$ such that $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$ and $f(u) \neq f(v)$ if $d(u, v) = 2$, where $d(u, v)$ denotes the distance between two vertices $u$ and $v$ of $G$ and $t \in \mathbb{N}$. We say that *a conflict occurs* if any of the necessary conditions to have an $L(2,1)$-labeling are not met. The *span* of an $L(2,1)$-labeling $f$ is the largest integer (label) assigned by $f$ to a vertex of $G$. The $\lambda$-*number* of $G$, denoted by $\lambda(G)$, is the smallest number $t$ such that $G$ has an $L(2,1)$-labeling with span $t$. Figure 1 exhibits an $L(2,1)$-labeling of the Petersen graph with the smallest span.

The $L(2,1)$-labeling problem was introduced by Griggs and Yeh [3] in 1992, motivated by problems of frequency assignment to transmitters. The main unsolved problem regarding $L(2,1)$-labelings is the *Griggs and Yeh's Conjecture*, which states that every simple graph $G$ with maximum degree $\Delta(G) \geq 2$ has $\lambda(G) \leq \Delta(G)^2$.

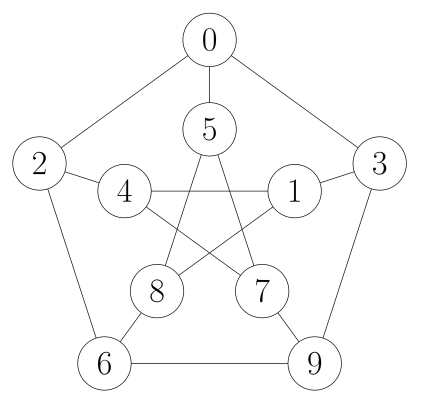


Figure 1: An $L(2,1)$-labeling of Petersen Graph with span 9.

Since Griggs and Yeh's seminal work, $\lambda(G)$ has been determined for various families of graphs [2, 3, 4]. In particular, Georges and Mauro [2] verified Griggs and Yeh's conjecture for some families of 3-regular graphs and, based on their results, posed Conjecture 1.

**Conjecture 1.** With the exception of the Petersen Graph, every connected 3-regular graph $G$ has $\lambda(G) \leq 7$.

In this work, we verify Conjecture 1 for a family of Loupekine snarks called $LP_1$-snarks and present a lower bound on $\lambda(G)$ for its members.

## Loupekine Snarks

A *snark* is a simple, connected, bridgeless 3-regular graph such that its edges cannot be colored with only three colors such that every two adjacent edges are assigned distinct colors. Snarks are related to fundamental problems in graph theory such as the *4-Color Problem* and the 5-*Flow Conjecture*.

Loupekine snarks were originally defined by Loupekine and first presented by Isaacs [1]. $LP_1$-snarks are an infinite family of Loupekine snarks and their construction is presented below.

Let $k$ be an odd positive integer. A $k$-$LP_1$-snark $G$ is constructed from $k \geq 3$ subgraphs called *blocks*, obtained from the Petersen graph $P$ as follows: given $k$ copies $R_0, \ldots, R_{k-1}$ of $P$, block $B_i$ is obtained from $R_i$ by deleting the vertices of an arbitrary path $P_3 \subset R_i$, for $0 \leq i \leq k-1$. Figure 2 illustrates an arbitrary block $B_i$ with its vertices named. Vertices $x_i, u_i, w_i, v_i, y_i$ are called *border vertices*.

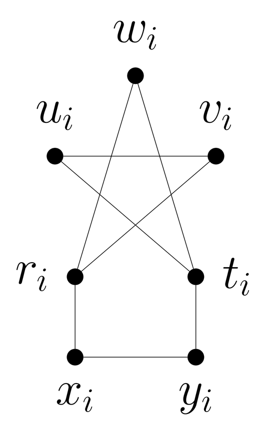


Figure 2: Block $B_i$.

For all $i \in \{0, \ldots, k-1\}$, the border vertices $v_i$ and $y_i$ of block $B_i$ are linked to the border vertices $u_{i+1}$ and $x_{i+1}$ of block $B_{i+1}$ (indices taken modulo $k$) by edges called *linking edges*. The linking edges can be $\{v_i x_{i+1}, y_i u_{i+1}\}$ or $\{v_i u_{i+1}, y_i x_{i+1}\}$, but not both.

Any three distinct border vertices $w_i, w_j, w_\ell$ are linked to a new vertex $u_{i,j,\ell}$, called *star vertex*, by adding $u_{i,j,\ell}$ and three new edges $w_i u_{ij\ell}$, $w_j u_{ij\ell}$ and $w_\ell u_{ij\ell}$ to $G$. The previous operation can be done an odd number $q$ of times, with $1 \leq q \leq k$. Since $k$ is odd, an even number $k - q$ of border vertices remain. If $k - q > 0$, the remaining border vertices are paired up and each pair $w_i$ and $w_j$ is linked by a new edge $w_i w_j$, thus concluding the construction of a $k$-$LP_1$-snark. Figure 3 shows a 3-$LP_1$-snark with an $L(2,1)$-labeling with span 7.

## Results

**Theorem 1.** Every $LP_1$-snark $G$ has $\lambda(G) \leq 7$.

*Sketch of the proof.* Given a $k$-$LP_1$-snark $G$, we construct an $L(2,1)$-labeling $f$ of $G$ with span 7. Initially, choose a block $B_i$ such that its border vertex $w_i$ is adjacent to another border vertex $w_j$ of $G$. Name this block by $B_{k-1}$ and name the remaining blocks consecutively from this one. If there is no such block, start the enumeration from any block. For every $i \in \{0, \ldots, k-1\}$, label the vertices of block $B_i$ as follows: $f(u_i) = f(x_i) = 2 \cdot (2i \bmod 3)$, $f(v_i) = f(y_i) = 2 \cdot (2i + 1 \bmod 3)$, $f(r_i) = 6$ and $f(t_i) = 7$. Conflicts occur in this partial labeling when $k \not\equiv 0 \pmod 3$ and, in order to resolve them, some vertex labels in $B_{k-1}$ are changed.

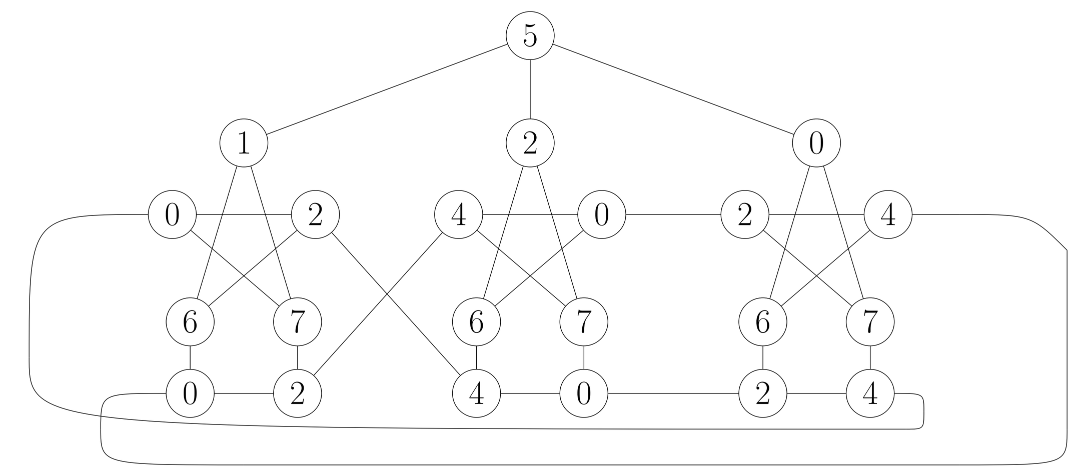


Figure 3: $L(2,1)$-labeling of a 3-$LP_1$-snark with span 7.

If $k \equiv 1 \pmod 3$, define $f(u_{k-1}) = f(x_{k-1}) = 1$ and $f(v_{k-1}) = f(y_{k-1}) = 3$. If $k \equiv 2 \pmod 3$, define $f(u_{k-1}) = f(x_{k-1}) = 5$, $f(v_{k-1}) = f(y_{k-1}) = 3$ and $f(r_{k-1}) = 1$. Other conflicts can occur depending on the adjacencies of the star vertices. All of them are resolved so that we finally verify that a valid label can always be assigned for every remaining unlabeled vertex without conflict.

**Theorem 2.** Every $LP_1$-snark $G$ has $\lambda(G) \geq 6$.

*Sketch of the proof.* It follows from $L(2,1)$-labeling's definition and $G$ being 3-regular that $\lambda(G) \geq 5$. If $\lambda(B_i) \geq 6$, then $\lambda(G) \geq 6$ since $B_i \subset G$. We suppose that $\lambda(B_i) = 5$. Then, we prove that this assumption restricts to 1 and 4 the labels that a border vertex $w_i$ can have. This restriction leads to a contradiction. Thus, $\lambda(B_i) \geq 6$.

## Acknowledgements


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