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Sharp Bounds for the Annihilation Number of the Nordhaus-Gaddum type

Guilherme Porto
E.E.E.M. Dom Pedro II
guiporto.mat@gmail.com

Daniel Jaume
UNSL-ARG
jaume@unsl.edu.ar

Marco Puliti
UNSL-ARG
mlpuliti@unsl.edu.ar

INTRODUCTION

The **annihilation number** is a graph invariant used as a sharp upper bound for the independence number. In this poster, we present bounds and Nordhaus-Gaddum type inequalities for the annihilation number.

We also investigate the extremal behavior of the invariant and showed that both parameters satisfy the **interval property**. In addition, we characterize some extremal graphs, ensuring that the bounds obtained are the best possible.

ANNIHILATION NUMBER

The **independence number** of a graph is the cardinality of a largest set of mutually non-adjacent vertices. It is not always possible to determine the number of independence of a graph, since this is a well-known widely-studied NP-hard problem, and for this reason the approximation of the independence number through inequalities represents a relevant research topic.

The annihilation number is a polynomial time computable upper bound for the independence number introduced by R. Pepper and S. Fajtlowicz [1,2].

Definition

The annihilation number of G, denoted by $a(G)$, can be defined as the largest integer k such that the sum of the smallest k degrees of graph G was at most its number of edges $e(G)$, that is

$$a(G) = \max\left\{ k \in \mathbb{N} : \sum_{i=1}^{k} d_i \leq e(G) \right\},$$

where d_i is the i-th smallest degree of G.

The annihilation number and the independence number are used to investigate the relationship between the reactivity of an organic molecule, represented by a graph, and its independence number. More precisely, the research states that, for a fixed number of vertices, molecules with a lower independence number are, in general, less reactive than molecules with a greater independence number. This study is known in organic chemistry as the independence-stability hypothesis [2].

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NORDHAUS-GADDUM PROBLEM

The Nordhaus-Gaddum problem is related to find lower and upper bounds on the sum and the product of the invariant of a graph and its complement, denoted by G^c [3].

The Nordhaus-Gaddum problem was studied for several domination parameters associated with the annihilation number, such as the independence number, the domination number, the Roman domination number, the total domination number, among others. This establishes a valuable connection between the annihilation number and the Nordhaus-Gaddum problem.

INTERVAL PROPERTY

Let $\mathcal{G}$ be a collection of graphs and $\xi : \mathcal{G} \to \mathbb{R}$ be a graph parameter defined on $\mathcal{G}$. We say that ξ has the interval property on $\mathcal{G}$ if $\xi(\mathcal{G}) = I \cap \mathbb{Z}$, for some interval $I \subset \mathbb{R}$ [4].

In other words, a graph parameter satisfies the interval property if each integer value in an interval is realized by at least one graph. The interval property generalizes the behavior of a parameter in an interval making it a relevant research topic.

BOUNDS FOR ANNIHILATION NUMBER

We present bounds for the annihilation number of a graph and prove that those bounds are the best possible. To state the result, we denote by K_n the **complete graph** on n vertices.

Theorem

Let G be a graph of order n. Then

$$\left\lfloor \frac{n}{2} \right\rfloor \leq a(G) \leq n.$$

Equality holds in the upper bound if and only if G is isomorphic to nK_1.
If G is a non-empty k-regular graph then the equality holds in the lower bound.

As a consequence, we show that the annihilation number satisfies the interval property.

Interval Property for $a(G)$

Let n and k be integers such that $\lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n - 1$.
If G is isomorphic to

$$(n-k)K_2 \cup (2k-n)K_1,$$

then $a(G) = k$.

NORDHAUS-GADDUM FOR $a(G)$

We present Nordhaus-Gaddum inequalities associated with the annihilation number and ensure that they are the best possible. To state the result, we denote by S_n the **star graph** on n vertices.

Theorem

Let G be a graph of order n. Then

$$2\left\lfloor \frac{n}{2} \right\rfloor \leq a(G) + a(G^c) \leq n + \left\lfloor \frac{n}{2} \right\rfloor.$$

For n even, the equality holds in the upper bound if and only if G or G^c is isomorphic to nK_1.
For n odd, the equality holds in the upper bound if and only if G or G^c is isomorphic to nK_1 or $S_{d_n+1} \cup (n - d_n - 1)K_1$, for $\lfloor \frac{n}{2} \rfloor \leq d_n \leq n-1$.
If G and G^c are non-empty graphs and G is a k-regular graph then the equality holds in the lower bound.

We then show that $a(G) + a(G^c)$ satisfies the interval property.

Interval Property for $a(G) + a(G^c)$

Let n and k be integers such that $2\lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n + \lfloor \frac{n}{2} \rfloor - 1$.
If G is isomorphic to

$$\left(n + \left\lfloor \frac{n}{2} \right\rfloor - k\right) K_2 \cup \left(2k - 2\left\lfloor \frac{n}{2} \right\rfloor - n\right) K_1,$$

then $a(G) + a(G^c) = k$.

CONCLUSION

We obtained important structural information about the graphs that satisfy the equality in the upper bounds. In particular, we can observe that, in general, such graphs have few edges.

The lower bounds are satisfied by a large number of graphs and, consequently, their characterization is important for understanding the extremal behavior of the annihilation number.

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