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Number of spanning trees of a subclass of matrogenic graphs

Robson Carlos de Moura Junior - IMECC/Unicamp - robson.ejr@gmail.com
Francisca Andrea Macedo França - ICEx/UFF - francisca_franca@id.uff.br
André Ebling Brondani - ICEx/UFF - andrebrondani@id.uff.br

Introduction

Laplacian Matrix of a Graph

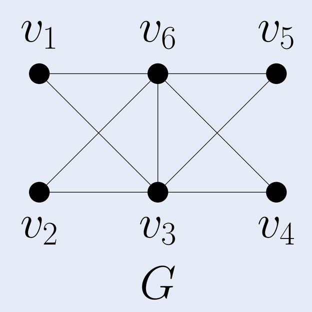
Definition ([1]) Let $G = G(V, E)$ be a simple graph with n vertices. The *adjacency matrix of* G is the matrix $A(G) = (a_{ij})$ with order n, whose entries are given by

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E \text{ for } v_i, v_j \in V; \\ 0, & \text{otherwise.} \end{cases}$$

Let $D(G)$ be the diagonal matrix given by the degree of vertices of G. The *Laplacian matrix* of G is the matrix $L(G)$ defined by

$$L(G) = D(G) - A(G).$$

For example,



$$L(G) = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{bmatrix}.$$

Preliminaries

Matrix-Tree Theorem

Theorem 1.1 ([3]) *The number of spanning trees of a graph G with order n is equal to any co-factor of $L(G)$. In symbols*

$$adj(L(G)) = \tau(G) J_{n\times n},$$

where $adj(L(G))$ is the classical adjoint of $L(G)$, $\tau(G)$ is the number of spanning trees of G and $J_{n\times n}$ is the matrix with order $n \times n$ whose entries are all equal to one.

We emphasize that this counting does not disregard isomorphic trees, that is, the number of non-isomorphic spanning trees is less than or equal to the number of spanning trees.

Corollary 1.2 ([1]) *Let G be a connected graph which n vertices. If $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are all the non-zero eigenvalues of $L(G)$, then*

$$\tau(G) = \frac{\mu_1 \mu_2 \cdots \mu_{n-1}}{n}.$$

This is the spectral version of the Matrix-Tree Theorem, which is very useful, since we've reduced the problem of finding the number of spanning trees in a graph to a problem of characterization of Laplacian eigenvalues.

For more reference see [4] and [5].

Matrogenic Graphs

Literature Results

The *symmetric difference* between two sets A, B is given by $A \oplus B = (A \cup B) \backslash (A \cap B)$.

Definition. If u, v are vertices of a graph G, we say that u *dominates* v if $N_G(v)\backslash\{u\} \subseteq N_G(u)\backslash\{v\}$. When neither u dominates v, nor v dominates u, we say that u and v are *incomparable*.

Definition. A *split graph* is a graph in which the set of vertices can be partitioned into a clique and an independent set. A graph is a *complete split graph* if it is a split graph such every vertex in the independent set is adjacent to every vertex in the clique.

Definition. A graph G is *matrogenic* if and only if for any incomparable vertices, u and v in G, we have that the cardinality of symmetric difference between the sets $N_G(v)\backslash\{u\}$ and $N_G(u)\backslash\{v\}$ is 2.

Proposition 2.1 The split complete graph is matrogenic.

Theorem 2.2 ([2]) *A graph $G = G(V, E)$ is matrogenic if and only if its vertex set V can be partitioned into three disjoint sets K, S and C such that*

(i) $K \cup S$ induces a matrogenic split graph in which K is a clique and S is an independent set.

(ii) C induces a crown, where a crown is either a perfect matching or a hyperoctahedron or a C_5.

(iii) Every vertex in C is adjacent to every vertex in K and to no vertex in S.

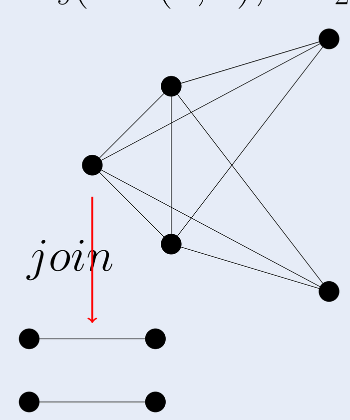
A Subclass of Matrogenic Graphs

From Theorem 2.2 every matrogenic graph of order n can be denoted by $G_n(K \cup S, C)$, where K, S and C are defined in the same theorem.

Given the non-negative integers r, s and t, we consider the class of graphs, $\mathcal{G}$, constituted by the matrogenic graphs of the form $G_n(K \cup S, C)$, where $K \cup S$ induces the complete split graph, $CS(r, s)$, and the subset of vertices C induces t copies of the complete graph K_2, that is,

$$\mathcal{G} = \{G_n(CS(r, s), tK_2) \,|\, r, s, t \in \mathbb{N} \wedge n = r + s + 2t\}.$$

The figure below shows the graph $G_9(CS(3, 2), 2K_2)$.



Application

Main Result

Theorem 3.1.
Let $H = G_n(CS(r, s), tK_2)$, then $\tau(H) = (r + s + 2t)^{r-1}(r + 2)^t r^{s+t-1}$.

Sketch of proof. We have

$$L(H) = \begin{bmatrix} D(tK_2) - A(tK_2) & -J_{2t\times r} & 0_{2t\times s} \\ -J_{r\times 2t} & D(K) - J_{r\times r} + I_{r\times r} & -J_{r\times s} \\ 0_{s\times 2t} & -J_{s\times r} & D(S) \end{bmatrix},$$

where $D(tK_2)$ is the diagonal matrix of the induced subgraph by tK_2, $D(K)$ is the diagonal matrix of induced subgraph by K, $D(S)$ is the diagonal matrix of induced subgraph by S, $I_{r\times r}$ is the identity matrix with order $r \times r$ and $0_{a\times b}$ is the matrix with order $a \times b$ with all entries is equal to 0.

Through eigenvalue calculation techniques we obtain $r \in Spec(L(H))$ with $m(r) \geq s - 1$, when $m(r)$ is the algebraic multiplicity of r as eigenvalue. On other hand, $r + s + 2t \in Spec(L(H))$ with $m(r + s + 2t) \geq r - 1$. In addition, we obtain $r + 2 \in Spec(L(H_1))$ with $m(r + 2) \geq t$.

By a result about reduced matrices, we obtain that $\{r+s+2t, r, 0\} \subset Spec(L(H))$.
So, $Spec(L(H)) = \{(r + s + 2t)^{[r]}, (r + 2)^{[t]}, r^{[s+t-1]}, 0\}$.
By the Corollary 1.2, the number of spanning trees of H is

$$\tau(H) = (r + s + 2t)^{r-1}(r + 2)^t r^{s+t-1}.$$

Corollary 3.2. The number of spanning trees of H depends of the cardinality of each cell of the partition of vertices in H.

For example, if $H = G_9(CS(3, 2), 2K_2)$, then $\tau(H) = (3 + 2 + 4)^2(3 + 2)^2(3)^3 = 54675$.

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