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November 25th, 2020

Arc-disjoint Branching Flows: a study of necessary and sufficient conditions

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Introduction

In this work, we investigate a conjecture by [1] that aims to characterize networks admitting k arc-disjoint s-branching flows, generalizing a result from [2] that provides such characterization when all arcs have capacity $n-1$, based on Edmonds' branching theorem [3].

- **Network**: $\mathcal{N} = (D, c)$, where $D = (V, A)$ is a digraph and $c : A(D) \to \mathbb{Z}_+$ is the *capacity function*. For an integer $\lambda \geq 0$, we write $c \equiv \lambda$ to state that $c(a) = \lambda,\ \forall a \in A(D)$. For an arc $a \in A(D)$ with tail u and head v, we may refer to a as uv.
- A **flow** f on a network $\mathcal{N}$ is a function $f : A(D) \to \mathbb{Z}_+$ such that $f(a) \leq c(a),\ \forall a \in A(D)$.
  Two flows f_1 and f_2 on a network $\mathcal{N}$ are **arc-disjoint flows** if $f_1(a) \times f_2(a) = 0,\ \forall a \in A(D)$.
- The **balance** of a vertex v with respect to a flow f is $bal_f(v) = \sum_{vu \in A(D)} f(vu) - \sum_{uv \in A(D)} f(uv)$. That is, $bal_f(v)$ is the sum of the flow leaving v minus the sum of the flow entering v.
- **s-branching flow**: flow f such that $bal_f(s) = n-1$ and $bal_f(v) = -1$ for all $v \in V(D) \setminus \{s\}$.

The hardness of the problem of finding k arc-disjoint s-branching flows in a network $\mathcal{N} = (D, c)$ where $c \equiv \lambda$, in general, depends on the choice of λ. Table 1 summarizes those results.

λ	Hardness
$\lambda \geq n - \ell$	Poly-time solvable for fixed ℓ [5]
$(\log n)^{1+\varepsilon} \leq \lambda \leq n - (\log n)^{1+\varepsilon}$	No poly-time algorithm (unless ETH[1]fails) [1,5]
$\lambda \leq \ell$	$\mathcal{NP}$-complete [5]

Table 1: *Summary of known hardness and algorithmic results for the problem of finding k arc-disjoint s-branching flows in a network $\mathcal{N} = (D, c)$ with $c \equiv \lambda$. Here, ℓ is a non-negative integer, $\varepsilon > 0$, and $n = |V(D)|$.*

In [1], the authors showed that the following property is a necessary condition satisfied by any network admitting k arc-disjoint s-branching flows.

$$d^-_D(X) \geq k \cdot \left\lceil \frac{|X|}{\lambda} \right\rceil, \forall X \subseteq V(D) \setminus \{s\}. \qquad \text{(Property 1)}$$

They also conjectured that Property 1 is a sufficient condition for the existence of k arc-disjoint s-branching flows in a network $\mathcal{N} = (D, c)$ with $c \equiv \lambda$, for any choices of k, λ, and s. In this work, we prove that their conjecture is true for some graphs, but false in general. An *out-branching with root* r is a digraph where $d^-_D(r) = 0$ and $d^-_D(v) = 1$ for every $v \in V(D) \setminus \{r\}$. Let a *multi out-branching with root* r be a digraph D formed by adding parallel arcs to an out-branching with root r. Observe that the underlying simple graph of D, constructed by discarding the orientation of the edges of D and removing parallel edges, is a tree. See Figure 1 for an example of a multi out-branching with root r and its underlying simple graph.

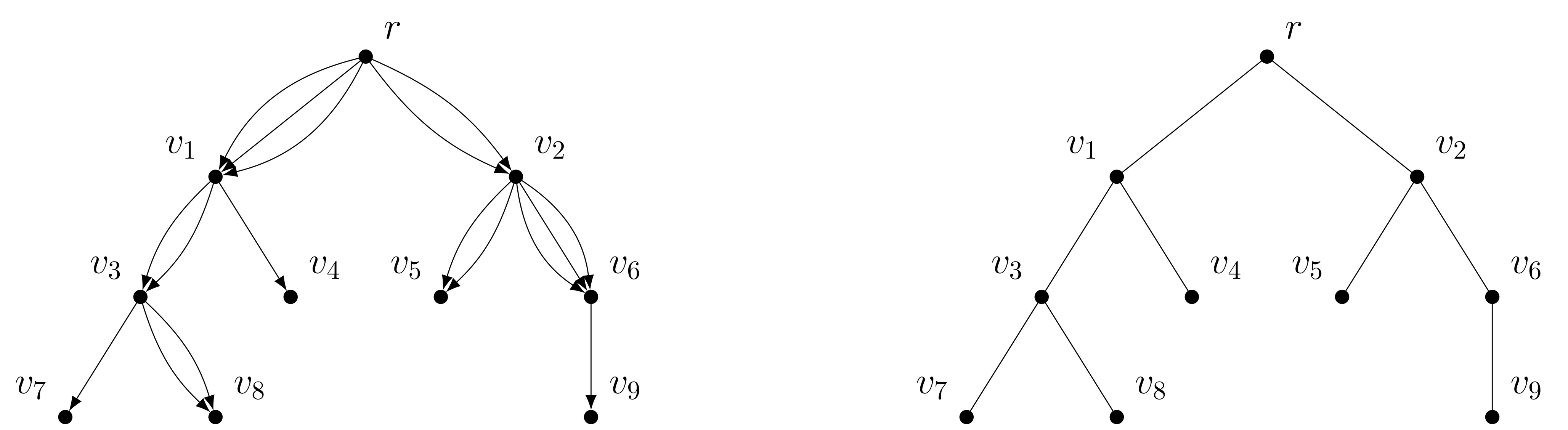

Figure 1: *Example of a multi out-branching with root r and its underlying simple graph.*

[1] Exponential Time Hypothesis [4].

Arc-disjoint branching flows on networks satisfying Property 1

We now state our results.

Theorem 1. *Let $\mathcal{N} = (D, c)$ be a network, where D is a multi out-branching with root s and $c \equiv \lambda$. If Property 1 holds for D with respect to k, λ and s then $\mathcal{N}$ admits k arc-disjoint s-branching flows.*

Figure 2 shows a network satisfying Property 1 for $k = \lambda = 2$ that does not contain 2 arc-disjoint s-branching flows. This statement is formalized by Theorem 2.

Theorem 2. *Let D be the digraph shown in Figure 2 and $\mathcal{N} = (D, c)$ be a network with $c \equiv 2$. Then Property 1 holds for $\mathcal{N}$ with respect to $\lambda = 2$, s,and $k = 2$, and there are no 2 arc-disjoint s-branching flows in $\mathcal{N}$.*

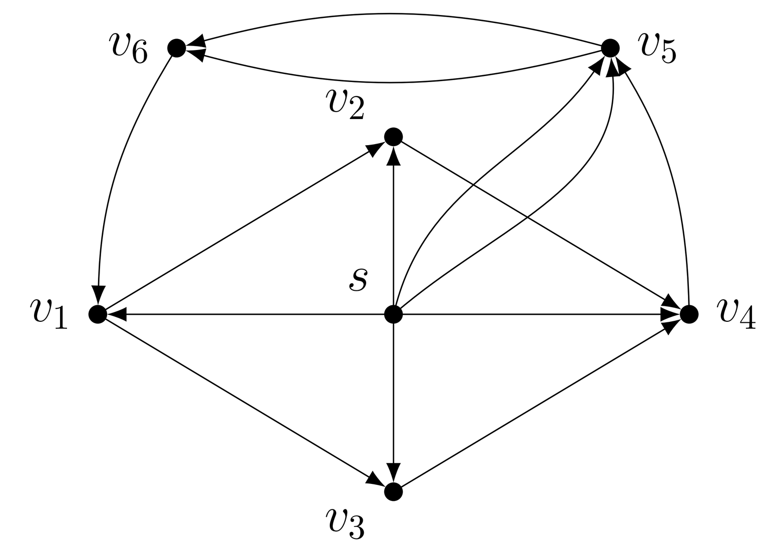

Figure 2: *A network for which Property 1 holds with respect to $k = \lambda = 2$ and the vertex s, but not containing 2 arc-disjoint s-branching flows.*

Future works

In future works, it will be interesting to consider whether there is a version of Theorem 1 for larger classes of digraphs, or whether there is a stronger necessary and sufficient condition that applies to all cases. We remark that, by the results shown in Table 1, we do not expect this condition to be easily verifiable in a given digraph.
In [5] the authors left open the question of whether the problem of finding k arc-disjoint s branching flows in a network $\mathcal{N} = (D, c)$ with $c \equiv n - \ell$ is *fixed-parameter tractable* with respect to ℓ. To our knowledge, this question remains open.

References

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Acknowledgement

CNPq Conselho Nacional de Desenvolvimento Científico e Tecnológico
CAPES
FUNCAP
Universidade Federal do Ceará
ParGO
Université Côte d'Azur
Inria