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Maximal Independent Sets in Graphs of Girth at Least 6

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Introduction

For a graph G we denote by $\alpha(G)$ the maximum size of an independent set in G and by $i(G)$ the minimum size of a maximal independent set in G. The independence gap of a graph G, denoted by $\mu_\alpha(G)$ is the difference $\alpha(G) - i(G)$. Well-covered graphs have independence gap zero. We present characterizations of some graphs with independence gap at least 1 that are of girth at least 6, including graphs with independent gap $r-1$, for $r \geq 2$, with r distinct and consecutive sizes of maximal independent sets.

Finbow et al. [3] define the set $\mathcal{M}_r$, for every positive integer r, to be the set of graphs that have maximal independent sets of exactly r different sizes. If the r different sizes of its maximal independent sets are consecutive, then it is also a member of $\mathcal{I}_r$, defined by Barbosa and Hartnell [1].

We present results related to the number of trees with specific maximum and minimum sizes of maximal independent sets (MIS). For a graph G, $\text{miss}(G) = \{|I| : I \text{ is a MIS of } G\}$. See Figure 1. A vertex is said to be of *type* r if it is adjacent to exactly r leaves.

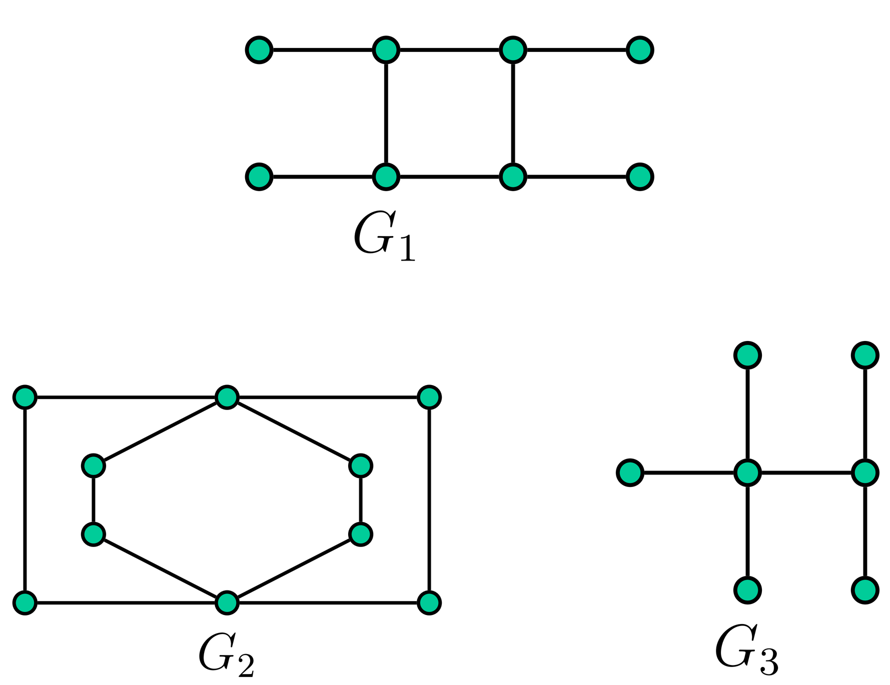

Figure 1: Graph G_1 is well-covered, with $\text{miss}(G_1) = \{4\}$, and $\mu_\alpha(G_1) = 0$; $G_2 \in \mathcal{M}_3$, but $G_2 \notin \mathcal{I}_3$, with $\text{miss}(G_2) = \{2,4,5\}$, and $\mu_\alpha(G_2) = 3$; $G_3 \in \mathcal{I}_3$, therefore $G_3 \in \mathcal{M}_3$, with $\text{miss}(G_3) = \{3,4,5\}$, and $\mu_\alpha(G_3) = 2$.

Results

Before we show some results regarding trees, we present in Table 1 the distribution in the set $\mathcal{I}_r$ of trees with n vertices, where $6 \leq n \leq 20$. Not all trees in $\mathcal{M}_r$ belong to $\mathcal{I}_r$. The data were obtained via a computational program.

In Theorem 1, we show the number of non-isomorphic trees having specific sizes of MIS and prove that there are exactly $\lceil \frac{n}{2} \rceil - 1$ non-isomorphic trees T with n vertices having $\mu_\alpha(T) = n-4$.

r \ Vertices	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1		2		3		6		11		23		47		106
2	2	5	4	12	14	31	40	78	122	202	351	522	1018	1370	2890
3	1	2	7	12	32	59	129	262	500	1063	1877	4069	6837	14817	24298
4				7	15	52	130	319	806	1737	4354	8812	21397	42069	98236
5					4	14	63	191	579	1654	4200	11561	27109	71181	160724
6						1	9	57	244	813	2856	7822	24781	63028	183301
7								4	55	266	1066	4206	12977	44759	125465
8									1	41	241	1206	5536	18954	72259
9											24	219	1282	6878	25945
10												10	184	1212	8079
11													3	134	1177
12															77

Table 1: Quantity of Trees of a given order in $\mathcal{I}_r$.

Theorem 1

Let $n \geq 3$ and T be a tree with n vertices.
1. There are exactly $n-3$ trees with $\alpha(T) = n-2$.
2. There are exactly $n-3$ trees with $i(T) = 2$.
3. There are exactly $\lceil \frac{n}{2} \rceil - 1$ trees $\mu_\alpha(T) = n-4$.

Next result is a generalization of a result in [2] for graphs G of girth at least 6 with $\mu_\alpha(G) = 1$. We adapt their proof considering $\mu_\alpha(G) \geq 1$. Additionally, we present the different sizes of MIS of G. Its proof gives a polynomial-time algorithm and it has some consequences to the class $\mathcal{I}_r$. In the following cases the sizes of MIS of G are not consecutive: if $r \geq 3$ and the girth of G is at least 7, and if $r \geq 4$ and the girth of G is at least 6. We summarize these conditions in Corollary 3. We denote G_i the subgraph of G induced by internal vertices of G that are type i.

Theorem 2

Let $r \geq 2$ and G be a connected graph of girth at least 6, with exactly two vertices u_1 and u_2 of type r, and with no type k vertices for $k \geq r+1$. Then $\mu_\alpha(G) = r-1$ if and only if u_1 and u_2 are adjacent, any other support vertex of G is type 1, and one of the following two conditions holds:
1. $V(G_0) = \emptyset$;
2. $G_0 \cong K_2$, neither of u_1 and u_2 has a neighbor in G_0, and the two vertices of G_0 are of degree 2 in G and are contained in an induced 6-cycle containing u_1 and u_2.

Moreover, if $V(G_0) = \emptyset$, then $\text{miss}(G) = \{|V(G_1)| + r + 1, |V(G_1)| + 2r\}$ otherwise $\text{miss}(G) = \{|V(G_1)| + r + 2, |V(G_1)| + 2r, |V(G_1)| + 2r + 1\}$.

Proof 1: (Sketch)

Let F_1 and F_2 be the sets of leaves, respectively, of vertices u_1 and u_2. Suppose $\mu_\alpha(G) = r-1$. We claim that the other neighbors of vertices u_1 and u_2 are vertices of type 1, and u_1 and u_2 are adjacent. Suppose $V(G_0) \neq \emptyset$; Let $L_1 = N_G(u_1) - (F_1 \cup \{u_2\})$ and $L_2 = N_G(u_2) - (F_2 \cup \{u_1\})$. Let L'_i the set of leaves adjacent to vertices of L_i, $i = 1, 2$. Now, let $I = F_1 \cup F_2 \cup L'_1 \cup L'_2$ and let $G' = G - N_G[I]$. See Figure 2. We also claim that: 1) graph G' is well-covered and has a perfect matching formed by its pendant edges. 2) G_0 has only one component that is isomorphic to K_2 and their vertices are under a 6-cycle containing u_1 and u_2. For the converse, we show all possible sizes of MIS considering the two cases: $V(G_0) = \emptyset$ and $V(G_0) \neq \emptyset$. If $V(G_0) = \emptyset$, then $\text{miss}(G) = \{|V(G_1)| + r + 1, |V(G_1)| + 2r\}$ otherwise $\text{miss}(G) = \{|V(G_1)| + r + 2, |V(G_1)| + 2r, |V(G_1)| + 2r + 1\}$. Therefore $\mu_\alpha(G) = r-1$.

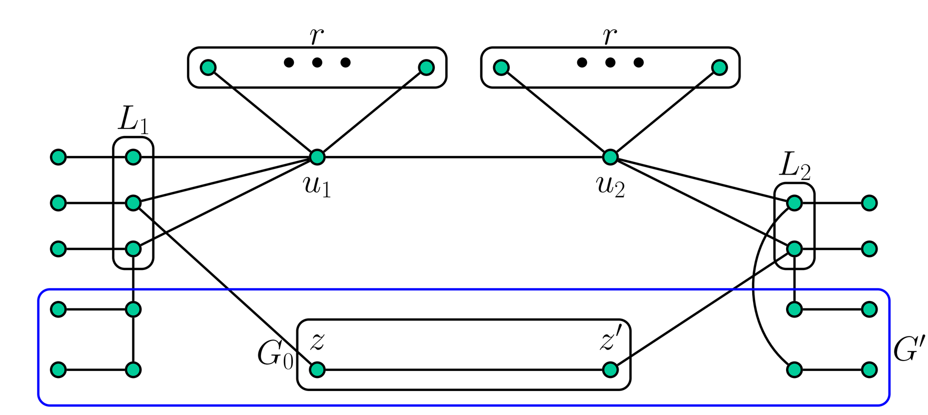

Figure 2: Graph G of girth 6 and two vertices of type r.

Corollary 3

Let $r \geq 3$ and let G be a graph of girth at least 6 with $\mu_\alpha(G) = r-1$ such that G contains exactly two vertices of type r. Then, $G \in \mathcal{I}_r$ only if $r = 3$ and the girth of G is exactly 6.

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References

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