Introduction

For a graph $G$ we denote by $\alpha(G)$ the maximum size of an independent set in $G$ and by $i(G)$ the minimum size of a maximal independent set in $G$. The independence gap of a graph $G$, denoted by $\mu(G)$, is the difference $\alpha(G) - i(G)$. Well-covered graphs have independence gap zero. We present characterizations of some graphs with independence gap at least 1 that are of girth at least 6, including graphs with independent gap $r-1$, for $r \geq 2$, with $r$ distinct and consecutive sizes of maximal independent sets.

Finbow et al. [3] define the set $\mathcal{M}_r$, for every positive integer $r$, to be the set of graphs that have maximal independent sets of exactly $r$ different sizes. If the $r$ different sizes of its maximal independent sets are consecutive, then it is also a member of $\mathcal{I}_r$ defined by Barbosa and Hartnell [1].

We present results related to the number of trees with specific maximum and minimum sizes of maximal independent sets (MIS). For a graph $G$, $\text{miss}(G) = \{|I| : I \text{ is a MIS of } G\}$. See Figure 1. A vertex is said to be of type $r$ if it is adjacent to exactly $r$ leaves.

![Graphs G1, G2, and G3](image)

**Figure 1**: Graph $G_1$ is well-covered, with $\text{miss}(G_1) = \{4\}$, and $\mu(G_1) = 0$; $G_2 \in \mathcal{M}_2$, but $G_2 \notin \mathcal{I}_2$ with $\text{miss}(G_2) = \{2, 4, 5\}$, and $\mu(G_2) = 4$; $G_3 \in \mathcal{I}_3$, with $\text{miss}(G_3) = \{3, 4, 5\}$, and $\mu(G_3) = 2$.

Results

Before we show some results regarding trees, we present in Table 1 the distribution in the set $\mathcal{I}_r$ of trees with $n$ vertices, where $6 \leq n \leq 20$. Not all trees in $\mathcal{M}_r$ belong to $\mathcal{I}_r$. The data were obtained via a computational program.

In Theorem 1, we show the number of non-isomorphic trees having specific sizes of MIS and prove that there are exactly $\left\lceil \frac{n}{2} \right\rceil - 1$ non-isomorphic trees $T$ with $n$ vertices having $\mu_r(T) = n - 4$.

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**Table 1**: Quantity of Trees of a given order in $\mathcal{I}_r$.

**Theorem 1**

Let $n \geq 3$ and $T$ be a tree with $n$ vertices.

1. There are exactly $n - 3$ trees with $\alpha(T) = n - 2$.
2. There are exactly $n - 3$ trees with $i(T) = 2$.
3. There are exactly $\left\lceil \frac{n}{2} \right\rceil - 1$ trees $\mu_r(T) = n - 4$.

Next result is a generalization of a result in [2] for graphs $G$ of girth at least 6 with $\mu_r(G) = 1$. We adapt our proof considering $\mu_r(G) \geq 1$. Additionally, we present the different sizes of MIS of $G$. Its proof gives a polynomial-time algorithm and it has some consequences to the class $\mathcal{I}_r$. In the following cases the sizes of MIS of $G$ are not consecutive: if $r \geq 3$ and the girth of $G$ is at least 7, and if $r \geq 4$ and the girth of $G$ is at least 6. We summarize these conditions in Corollary 3. We denote $G_i$ the subgraph of $G$ induced by internal vertices of $G$ that are type $i$.

**Theorem 2**

Let $G$ be a connected graph of girth at least 6, with exactly two vertices $u_1$ and $u_2$ of type $r$, and with no type $k$ vertices for $k \geq r + 1$. Then, $\mu_r(G) = r - 1$ if and only if $u_1$ and $u_2$ are adjacent, any other support vertex of $G$ is type 1, and one of the following two conditions holds:

1. $V(G_i) = \emptyset$.
2. $G_i \cong K_r$, neither of $u_1$ and $u_2$ has a neighbor in $G_0$, and the two vertices of $G_0$ are of degree 2 in $G$ and are contained in an induced 6-cycle containing $u_1$ and $u_2$.

Moreover, if $V(G_i) = \emptyset$, then $\mu_r(G) = \{\{V(G_i)\} + r + 1, |V(G_i)| + 2r\}$ otherwise $\mu_r(G) = \{\{V(G_i)\} + r + 2, |V(G_i)| + 2r, |V(G_i)| + 2r + 1\}$.

**Proof of (Sketch)**

Let $F_1$ and $F_2$ be the sets of leaves, respectively, of vertices $u_1$ and $u_2$. Suppose $\mu_r(G) = r - 1$. We claim that the other neighbors of vertices $u_1$ and $u_2$ are vertices of type 1, and $u_1$ and $u_2$ are adjacent. Suppose $V(G_i) = \emptyset$. Let $L_i = N_G(u_1) - (F_1 \cup \{u_2\})$ and $L_2 = N_G(u_2) - (F_2 \cup \{u_1\})$. Let $L_i^2$ the set of leaves adjacent to vertices of $L_i$, $i = 1, 2$. Now, let $I = F_1 \cup F_2 \cup L_i^1 \cup L_i^2$ and let $G^* = G - N_G[I]$. See Figure 2. We also claim that: $\mu_r(G^*)$ is well-covered and has a perfect matching formed by its pendant edges. $G_0$ has only one component that is isomorphic to $K_2$ and its vertices are under a 6-cycle containing $u_1$ and $u_2$. For the converse, we show all possible sizes of MIS considering the two cases: $V(G_0) = \emptyset$ and $V(G_0) \neq \emptyset$. If $V(G_0) = \emptyset$, then $\mu_r(G) = \{\{V(G_i)\} + r + 1, |V(G_i)| + 2r\}$ otherwise $\mu_r(G) = \{\{V(G_i)\} + r + 2, |V(G_i)| + 2r, |V(G_i)| + 2r + 1\}$.

Therefore, $\mu_r(G) = r - 1$.

![Graph G of girth 6 and two vertices of type r](image)

**Figure 2**: Graph $G$ of girth 6 and two vertices of type $r$.

**Corollary 3**

Let $r \geq 3$ and let $G$ be a graph of girth at least 6 with $\mu_r(G) = r - 1$ such that $G$ contains exactly two vertices of type $r$. Then, $G \in \mathcal{I}_r$, only if $r = 3$ and the girth of $G$ is exactly 6.

Acknowledgements

To CNPq and CAPES for the partial support.

References