The 3-flow conjecture for almost even graphs with up to six odd vertices

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1. Integer flows
Let \( G = (V(G), E(G)) \) be an undirected graph. Let \( D \) be an orientation for \( E(G) \), and \( f \) an assignment of non-negative integer weights to each edge of \( E(G) \). We say that \( (D, f) \) is a \( k \)-flow for \( G \) if:
1. \( 0 < f(e) < k \), for each \( e \in E(G) \);
2. the flow balance \( \sum_{e \in \partial^+(v)} f(e) - \sum_{e \in \partial^-(v)} f(e) = 0 \), for each \( v \in V(G) \), where \( \partial^+(v) (\partial^-(v)) \) is the set of edges leaving (entering) vertex \( v \).
In a mod-\( k \) flow, the flow balance at each vertex \( v \) is \( \sum_{e \in \partial^+(v)} f(e) - \sum_{e \in \partial^-(v)} f(e) \equiv 0 \) (mod \( k \)). Figure 1 shows two graphs that admit a mod-3 flow.

2. Tutte’s 3-flow Conjecture and equivalent formulations
A 3-cut is an edge cut of size three. A bridge is an edge cut of size one. Tutte’s 3-flow conjecture is

Conjecture (Tutte’s 3-flow conjecture)
Every bridgeless graph with no 3-cuts admits a 3-flow.

Two equivalent forms of this conjecture are:
- Every bridgeless 5-regular graph with no 3-cuts admits a 3-flow.
- Every bridgeless graph with at most three 3-cuts admits a 3-flow.

3. Objective
In this work, our objective is to characterize classes of graphs with up to four 3-cuts that admit a 3-flow. \( K_5 \), the complete graph on four vertices, is the smallest bridgeless graph that does not admit a 3-flow. We focus on essentially 4-edge connected graphs, i.e., whose edge cuts of size less than four are associated with vertices of degree three (3-vertices). Also, our graphs are almost even, i.e., having at most six odd vertices.

4. Motivation
Our motivation is to provide tools for a possible inductive approach to prove Tutte’s 3-flow conjecture.

5. Graphs with exactly four vertices of odd degree
Let \( G \) be an essentially 4-edge connected, almost even, graph having at most four odd vertices, with \( S \) its set of odd vertices. We say that \( G \) has a forbidden configuration if: (i) the vertices of \( S \) all have degree three; (ii) \( G[S] \) contains \( K_{5,3} \); and (iii) every even-degree vertex \( v \) of \( G \) is separated from \( S \) by an edge cut of size at most four. We abuse this definition by saying that \( K_4 \) has a forbidden configuration.

Theorem 1
An essentially 4-edge connected, almost even, graph \( G \) with at most four odd-degree vertices admits a 3-flow, if and only if \( G \) does not have a forbidden configuration.

6. Graphs with exactly six vertices of odd degree
We give a partial characterization of almost even graphs with six odd-degree vertices that admit a 3-flow.
By using the same definition of forbidden configuration to graphs with four 3-vertices and two odd-degree vertices of degree greater than 3, we obtain

Theorem 2
Let \( G \) be an essentially 4-edge-connected, almost even, graph with four 3-vertices and two other odd vertices of degree greater than 3, and assume \( G \) has a forbidden configuration. Then, \( G \) admits a 3-flow if and only if there are no 4-cuts separating the 3-vertices from the remaining odd vertices.

Sketch of proof: (i) We contract a set \( X \) that contains the two odd vertices with degree higher than three, and having an associated edge-cut of size six (e.g. \( V(G) \) minus the vertices of degree three). By Theorem 1, the resulting graph admits a 3-flow, that can be extended to \( G/X \). This is a 3-flow for \( G \).

(only if) We contract a set \( X \) that contains the two odd vertices of degree higher than three, with an associated edge-cut of size four. By the previous theorem, \( G/X \) does not admit a 3-flow, and so neither does \( G \).

References

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