The 3-flow conjecture for almost even graphs with up to six odd vertices

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1. Integer flows

Let $G = (V(G), E(G))$ be an undirected graph. Let D be an orientation for $E(G)$, and f an assignment of non-negative integer weights to each edge of $E(G)$. We say that (D, f) is a *k-flow* for G if:

1. $0 < f(e) < k$, for each $e \in E(G)$;
2. the *flow balance* $\sum_{e \in \partial^+(v)} f(e) - \sum_{e \in \partial^-(v)} f(e) = 0$, for each $v \in V(G)$,

where $\partial^+(v)$ ($\partial^-(v)$) is the set of edges leaving (entering) vertex v.

In a *mod-k flow*, the flow balance at each vertex v is $\sum_{e \in \partial^+(v)} f(e) - \sum_{e \in \partial^-(v)} f(e) \equiv 0 \pmod{k}$. Figure 1 shows two graphs that admit a mod-3 flow.

A graph G admits a k-flow if and only if it admits a mod-k flow. Also, if G admits a mod-k flow, then it admits a mod-k flow for any given orientation. See [1], [2] and [3] for more on k-flows.

Figure 1: Examples of mod-3 flows for graphs $K_{3,3}$ and K_4 plus an edge. In both cases, all weights are equal to 1.

2. Tutte's 3-flow Conjecture and equivalent formulations

A *3-cut* is an edge cut of size three. A *bridge* is an edge cut of size one. Tutte's 3-flow conjecture is

Conjecture (Tutte's 3-flow conjecture)

Every bridgeless graph with no 3-cuts admits a 3-flow.

Two equivalent forms of this conjecture are:

- Every bridgeless 5-regular graph with no 3-cuts admits a 3-flow.
- Every bridgeless graph with at most three 3-cuts admits a 3-flow.

3. Objective

In this work, our objective is to characterize classes of graphs with up to four 3-cuts that admit a 3-flow. K_4, the complete graph on four vertices, is the smallest bridgeless graph that does not admit a 3-flow. We focus on *essentially 4-edge connected graphs*, i.e., whose edge cuts of size less than four are associated with vertices of degree three (3-vertices). Also, our graphs are *almost even*, i.e., having at most six odd vertices.

We obtain a characterization for such graphs with up to four odd vertices. We also obtain a partial characterization for graphs with up to four 3-vertices and two odd vertices of higher degree.

4. Motivation

Our motivation is to provide tools for a possible inductive approach to prove Tutte's 3-flow conjecture.

5. Graphs with exactly four vertices of odd degree

Let G be an essentially 4-edge connected, almost even, graph having at most four odd vertices, with S its set of odd vertices. We say that G has a *forbidden configuration* if: (i) the vertices of S all have degree three; (ii) $G[S]$ contains $K_{1,3}$; and (iii) every even-degree vertex v of G is separated from S by an edge cut of size at most four. We abuse this definition by saying that K_4 has a forbidden configuration.

Theorem 1

An essentially 4-edge connected, almost even, graph G with at most four odd-degree vertices admits a 3-flow, if and only if G does not have a forbidden configuration.

6. Graphs with exactly six vertices of odd degree

We give a partial characterization of almost even graphs with six odd-degree vertices that admit a 3-flow. By using the same definition of forbidden configuration to graphs with four 3-vertices and two odd-degree vertices of degree greater than 3, we obtain

Theorem 2

Let G be an essentially 4-edge-connected, almost even, graph with four 3-vertices and two other odd vertices of degree greater than 3, and assume G has a forbidden configuration. Then, G admits a 3-flow if and only if there are no 4-cuts separating the 3-vertices from the remaining odd vertices.

Sketch of proof: (if) We contract a set X that contains the two odd vertices with degree higher that three, and having an associated edge-cut of size six (e.g. V(G) minus the vertices of degree three). By Theorem 1, the resulting graph admits a 3-flow, that can be extended to $G/\overline{X}$. This is a 3-flow for G.

(only if) We contract a set X that contains the two odd vertices of degree higher than three, with an associated edge-cut of size four. By the previous theorem, G/X does not admit a 3-flow, and so neither does G.

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