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2-Independent Sets in Complementary Prisms

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Introduction

We consider a generalization of the concepts of domination and independence in graphs. For a positive integer k, a subset S of vertices in a graph $G = (V, E)$ is *k-dominating* if every vertex of $V - S$ is adjacent to at least k vertices in S. The subset S is *k-independent* if the maximum degree of the subgraph induced by the vertices of S is at most $k-1$. Thus for $k = 1$, the 1-independent and 1-dominating sets are the classical independent and dominating sets. The minimum and maximum sizes of a maximal k-independent set in G are denoted $i_k(G)$ and $\alpha_k(G)$, respectively. The minimum and maximum sizes of a minimal k-dominating set in G are denoted γ_k and Γ_k, respectively.

The *complementary prism* of a graph G, denoted by $G\overline{G}$, is a graph obtained by the disjoint union of G and its complement $\overline{G}$ by adding edges of a perfect matching between the corresponding vertices. The Petersen graph is a complementary prism of the cycle on 5 vertices, as shown in Figure 1.

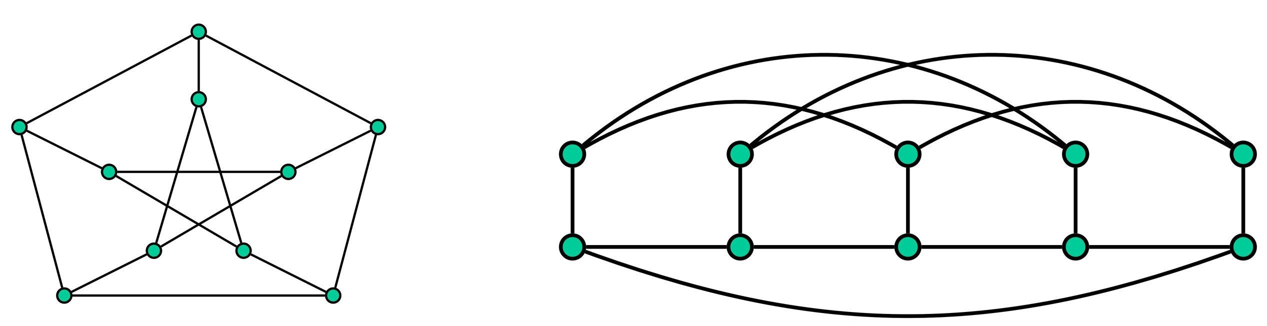

Figure 1: Two representations of the Petersen graph, the $C_5\bar{C}_5$ graph.

Haynes et al. [4] show upper and lower bounds for the maximum cardinality of 1-independent sets and for the minimum cardinality of 1-dominating sets. For a graph G, Chellali et al. [1] present a survey with relations and bounds between $\alpha(G)$, $i(G)$, $\gamma(G)$ and $\Gamma(G)$. Duarte et al. [2] prove that finding α_1 of a complementary prism $G\overline{G}$ is an NP-complete problem.

We present sharp lower and upper bounds for maximum 2-independent sets in complementary prism of any graph, characterize the graphs for which the upper and lower bound holds, and present closed formulas for the complementary prism of paths, cycles and complete graphs.

Relationships between the parameters

Since every set which is both 1-independent and 1-dominating is a minimal 1-dominating set of G, it is easy to see that

$$\gamma_1(G) \leq i_1(G) \leq \alpha_1(G) \leq \Gamma_1(G)$$

for any graph G.

Favaron [3] shows that, for any graph G and positive integer k, $\gamma_k(G) \leq \alpha_k(G)$ and $i_k(G) \leq \Gamma_k(G)$.

Some general properties:

- Every k-dominating set of a graph G contains at least k vertices and all vertices of degree less than k; so $\gamma_k(G) \geq k$ when $n \geq k$.
- Every set with k vertices is k-independent; so $i_k(G) \geq k$ when $n \geq k$.
- Every set S that is both k-independent and k-dominating is a maximal k-independent set and a minimal k-dominating set.
- Every $(k+1)$-dominating set is also a k-dominating set.
- Every k-independent set is also a $(k+1)$-independent set.

Results on 2-independent sets in complementary prisms

Haynes et al. [4] show that, for any graph G, $\alpha_1(G) + \alpha_1(\overline{G}) - 1 \leq \alpha_1(G\overline{G}) \leq \alpha_1(G) + \alpha_1(\overline{G})$, and both these bounds are sharp. In Theorem 1, we generalize this result for $\alpha_2(G\overline{G})$.

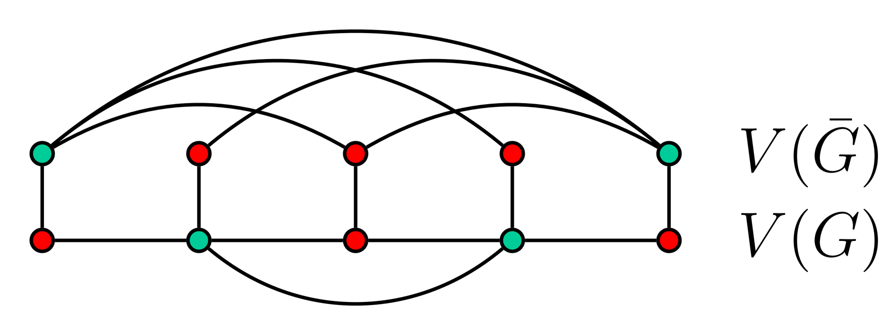

Figure 2: Graph with a maximum 2-independent set highlighted (red vertices) with $\alpha_2(G\overline{G}) = \alpha_1(G) + \alpha_1(\overline{G})$.

Theorem 1

For any graph G,

$$\alpha_1(G) + \alpha_1(\overline{G}) \leq \alpha_2(G\overline{G}) \leq \alpha_2(G) + \alpha_2(\overline{G}),$$

and both these bounds are sharp.

The graph G whose complementary prism $G\overline{G}$ is shown in Figure 2 attains the lower bound of Theorem 1, and the graph C_5 attains the upper bound. In the following result, we characterize the graphs for which the upper bound holds.

Theorem 2

A graph G has $\alpha_2(G\overline{G}) = \alpha_2(G) + \alpha_2(\overline{G})$ if and only if there exist disjoint vertex sets S and T in $V(G)$ such that S is $\alpha_2(\overline{G})$-set and T induces a maximum multipartite graph in G such that every partition has size at most two.

Now we show exact values for α_2 for some particular graph classes.

Theorem 3

Let $n \geq 5$. Then, $\alpha_2(K_n\overline{K}_n) = n + 1$,

$$\alpha_2(P_n\overline{P}_n) = \begin{cases} 2\lfloor n/3 \rfloor + 4, & n \equiv 2 \pmod 3, \\ 2\lfloor n/3 \rfloor + 3, & \text{otherwise}, \end{cases}$$

$$\alpha_2(C_n\overline{C}_n) = \begin{cases} 2\lfloor n/3 \rfloor + 3, & n \equiv 2 \pmod 3, \\ 2\lfloor n/3 \rfloor + 2, & \text{otherwise}. \end{cases}$$

Future work

As future work, we plan to characterize graphs attaining the lower bound on Theorem 1; to extend the presented results for α_k, for $k \geq 3$; and to study k-dominating sets in complementary prisms.

References

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