Partitioning Graphs into Monochromatic Trees

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Introduction

This work presents complexity results about the NP-Completeness of Partition edge-coloured Graphs into vertex disjoint Monochromatic Trees (PGMT) when we restrict the frequency with each color occurs at the edges of the graph. Jin and Li [3] defined the PGMT problem as follows:

THE PGMT PROBLEM

Instance: An edge-coloured graph G and a positive integer k.
Question: Are there k or less vertex disjoint monochromatic trees which cover the vertices of the graph G?

Figure 1 shows an example of a graph partitioned into monochromatic trees; even in the colored graph with 3 colors, only two trees are sufficient to cover the vertices.

Figure 1

Figure 2

Related Works

Jin and Li [4] considered a version of PGMT where the number of different colors of the graph is fixed. In their work, Jin and Li [3] showed that PGMT is NP-Complete and there is no constant factor approximation algorithm.

In this version, the number of distinct colors of the graph is fixed. In this work we consider another kind of restriction to the input graph. In this version, instead of fixing the number of different colors, we only guaranteed that each color appears at most f times. We define this version as follows:

THE /MAX-PM GMT PROBLEM

Instance: An edge-coloured graph G, where each color occurs at most f times, and a positive integer k.
Question: Are there k or less vertex disjoint monochromatic trees which cover the vertices of the graph G?

The /MAX-PM GMT Problem

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NP-Completeness Results

We now show that f /MAX-PM GMT is NP-Complete, when f = 3, by reducing from Exact Cover by 3-Sets - X3C [1], which is defined as follows:

THE X3C Problem

Instance: An set \( X = \{v_1, \ldots, v_n\} \), |X| = 3k; an family of subsets \( F = \{S_1, S_2, \ldots, S_m\} \), \( S_i \subseteq X \) and \( |S_i| = 3, i \in \{1, 2, \ldots, |F|\} \).

Question: Is there \( F' \subseteq F \), such that \( \bigcup_{S \in F'} S = X \)?

We build an instance \((G, k + m - 2)\) of f /MAX-PM GMT that is equivalent to an instance \((X, F)\) of X3C as follows: The set of vertices is \( V(G) = \{v_1, \ldots, v_3k, v_3k + 1, \ldots, v_3k + m\} \). The set of edges is

\[
E(G) = \begin{cases}
  uS_i, & \text{if } u \in S_i \\
  \emptyset, & \text{otherwise}
\end{cases}
\]

for all \( i \in \{1, \ldots, n\} \), \( j \in \{1, \ldots, m\} \), \( p \in \{i, i + 1, i + 2\} \). And coloring the edges as follow:

\[
c(e) = \begin{cases}
  c_j, & \text{if } e = v_pS_i \\
  c_{mp}, & \text{if } e = v_pS_j
\end{cases}
\]

for all \( e \in E(G) \), \( i \in \{1, \ldots, n\} \), \( j \in \{1, \ldots, m\} \), \( p \in \{1, \ldots, m - 2\} \), \( q \in \{p, p + 1, p + 2\} \). Figure 2 shows an example of the transformation described: (a) X3C instance and (b) colored graph from that instance.

References

