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A_α-Spectrum of some Matrogenic Graphs

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A_α-Spectral Theory

Definition 1.([3]) Let $G = G(V, E)$ be a simple graph. The matrix $A_\alpha(G)$ is defined by

$$A_\alpha(G) = \alpha \cdot D(G) + (1-\alpha) \cdot A(G),\ \alpha \in [0,1],$$

where $A(G)$ denotes the adjacency matrix of G and $D(G) = (d_{ij})$, is a matrix of order n, where $d_{ij} = 0$, if $i \neq j$ and $d_{ij} = d(v_i)$, if $i = j$.

Matrogenic Graphs

Definition 2. Let $G = G(V, E)$ be a graph. Given $u, v \in V$, we say that u *dominates* v if $N_G(v) - \{u\} \subseteq N_G(u) - \{v\}$. When neither u dominates v nor v dominates u, then u and v are called *incomparable*.

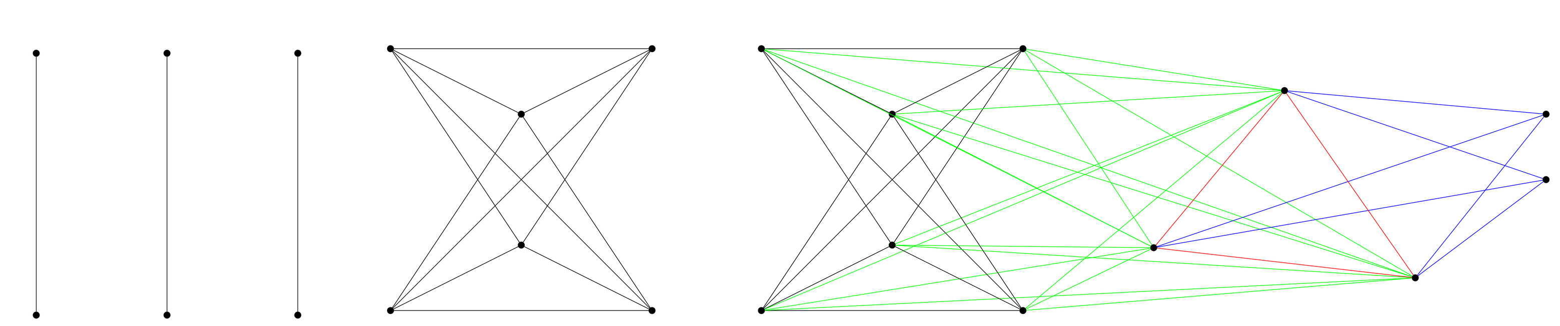
Definition 3. A graph G is *matrogenic* if for any two vertices u and v, incomparable in G, we have $|(N_G(u) - \{v\}) \oplus (N_G(v) - \{u\})| = 2$, where the symbol $\oplus$ denotes the symmetric difference.

Definition 4. A split graph $S(r, s)$ is a graph whose vertices can be partitioned into a clique of size r, and a independent set of size s. A split graph is called complete if every vertex in the independent set is adjacent to every vertex in the clique; it is denoted by $CS(s, r)$.

Definition 5. A graph G is threshold if for $u, v \in V(G)$, either u dominates v or v dominates u.

Properties of Matrogenic Graphs

Definition 6. A perfect matching, tK_2, is the union of t copies of K_2 and a cocktail party graph, $CP(2t)$, is the complement of a perfect matching.



$3K_2$ $\quad$ $CP(6)$ $\quad$ $G_{11}(CS(3,2), CP(6))$

Some properties of the matrogenic graphs: all induced subgraphs of a matrogenic graph are matrogenic; the complement of a matrogenic graph is matrogenic and the class of matrogenic graphs contains the class of threshold graphs. In particular, as the split complete graph is threshold, it is matrogenic.

Theorem 1.([2]) A graph $G = G(V, E)$ of order n is matrogenic if and only if V can be partitioned into three distinct sets K, S, and C such that

(i) $K \cup S$ induces a matrogenic split subgraph in which K is a clique and S is a independent set;

(ii) C induces a perfect matching, or a cocktail party, or a C_5;

(iii) every vertex of C is adjacent to every vertex of K and to no vertex in S.

Theorem 1 gives us a way to characterize matrogenic graphs from a partition of its vertex set V. Thus, we can denote every matrogenic graph as $G_n([K \cup S], [C])$. In the previous figure we show the matrogenic graph $G_{11}(CS(3,2), CP(6))$.

A_α-Spectrum

In this work, we analyze the A_α-spectrum of a subclass of matrogenic graphs.

Theorem 2. If $H = G_n(CS(k, s), CP(2t))$ then A_α-characteristic polynomial of H is given by

$$P_{A_\alpha(H)}(x) = f(x)[x - \alpha(2t+k) + 2]^{t-1}(x - \alpha n + 1)^{k-1}(x - \alpha k)^{s-1}[x - \alpha(2t+k-2)]^t,$$

where $f(x) = det(xI - \overline{A_\alpha}(H))$,

$$\overline{A_\alpha}(H) = \begin{pmatrix} \alpha(k+2t-2) + (1-\alpha)(2t-2) & (1-\alpha)k & 0 \\ (1-\alpha)2t & \alpha(k-1+s+2t) + (1-\alpha)(k-1) & (1-\alpha)s \\ 0 & (1-\alpha)k & \alpha k \end{pmatrix}.$$

Sketch of proof. There is a labeling of the vertices of the graph H, so that the matrix A_α can be written

$$A_\alpha(H) = \begin{pmatrix} B_\alpha & (1-\alpha)J_{2t\times k} & 0_{2t\times s} \\ (1-\alpha)J_{k\times 2t} & C_\alpha & (1-\alpha)J_{k\times s} \\ 0_{s\times 2t} & (1-\alpha)J_{s\times k} & \alpha k I_s \end{pmatrix},$$

where we denote the all-ones matrix by J, the all-zeros matrix by 0, the identity matrix by I, $B_\alpha = \alpha(k+2t-2)I_{2t} + (1-\alpha)(J_{2t} - I_{2t} - A(tK_2))$ and $C_\alpha = \alpha(k-1+s+2t)I_k + (1-\alpha)(J_k - I_k)$. Denote by e_k the vector with $2t$ coordinates whose k-th entry is equal to 1 and the others entries are zero. For each j, ℓ and i, with $1 \leq j \leq t$, $2 \leq \ell \leq k$ and $2 \leq i \leq s$, consider the vectors $z_j = (e_{2j-1} - e_{2j}|0|0)^T$, $w_\ell = (0|e_{2t+k+1} - e_{2t+k+\ell}|0)^T$ and $v_i = (0|0|e_{2t+k+1} - e_{2t+k+i}|0)^T$. We have,

$$A_\alpha(H)z_j = \alpha(2t+k-2)z_j,\quad A_\alpha(H)w_\ell = (\alpha n - 1)w_\ell \quad \text{and} \quad A_\alpha(H)v_i = \alpha k v_i.$$

Now, consider the vector $v^{(i)} = e_{2i-1} + e_{2i}$. Some calculations show that the $t-1$ vectors of the form $\left(v^{(1)} - v^{(i)}|0|0\right)^T$, $2 \leq i \leq t$, are the eigenvectors of $A_\alpha(H)$ associated with the eigenvalue $\alpha(k+2t) - 2$. The other eigenvalues are the roots of the polynomial $f(x)$, which follows from the matrix reduction technique (see Theorem 1.3.14 of [1]).

Conclusion

As it was claimed in [3], the matrix A_α can underpin a unified theory of the spectral study of the adjacency and singless Laplacian matrices of a graph. In this work, we obtain a partial factorization of the A_α-characteristic polynomial of a subfamily of matrogenic graphs which explicitly gives some eigenvalues of the graph.

References

[1] D. Cvetković, P. Rowlinson, and S. Simić. *An Introduction to the Theory of Graph Spectra.* Cambridge University Press. Cambridge. 2010.

[2] S. Foldes and P. Hammer. *On a class of matroid-producing graphs.* Colloq. Math. Soc. J. Bolyai (Combinatorics). 18: 331-352. 1978.

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