Decomposing cubic graphs into locally irregular subgraphs

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Introduction

A **decomposition** of a graph G is a set $\mathcal{D} = \{H_1, \ldots, H_k\}$ of edge-disjoint subgraphs of G such that $\cup_{i=1}^{k} E(H_i) = E(G)$. A **locally irregular** graph is a graph in which adjacent vertices have distinct degrees.

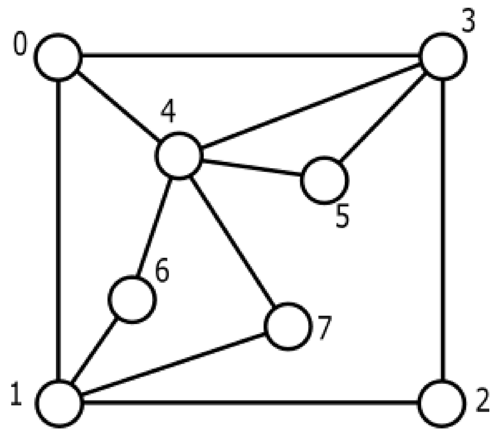


Figure 1: A locally irregular graph

A **locally irregular decomposition** (or **locally irregular coloring**) of a graph G is a decomposition in which every element is locally irregular. We say that G is **decomposable** if it admits a locally irregular decomposition. Equivalently, a locally irregular decomposition is a coloring of $E(G)$ in which every color class induces a locally irregular subgraph in G. If k colors are used, then we say **locally irregular k-edge-coloring** or **k-LIC** for short.

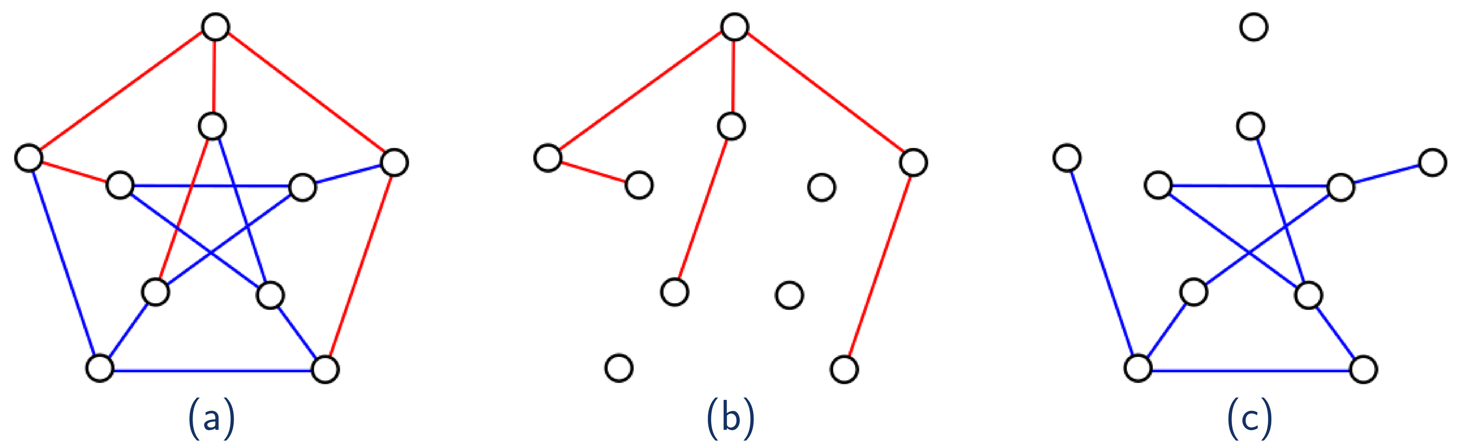


Figure 2: (a) A 2-LIC of G. (b) An induced subgraph of G using the edges with color red. (c) An induced subgraph of G using the edges with color blue.

Given a decomposable graph G, the **irregular chromatic index** of G is the smallest number k for which G admits a k-LIC. We denote the irregular chromatic index of G by $\chi'_{irr}(G)$. The problem of computing the irregular chromatic index was proven to be an NP-complete problem [2].

In this work we explore the following conjecture posed by Baudon et al. [1].

Conjecture 1 (O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak, 2015)**.** For every decomposable graph G, we have $\chi_{irr}(G) \leq 3$.

Results toward confirming Conjecture 1 include that graphs whose set of vertices can be partitioned into a clique and an independent set admit a 3-LID [3] and graphs with maximum degree at most 3 admit a 4-LID [4]. We explore Conjecture 1 for graphs in which all vertices have degree 3, which are called **cubic graphs**.

Contribution

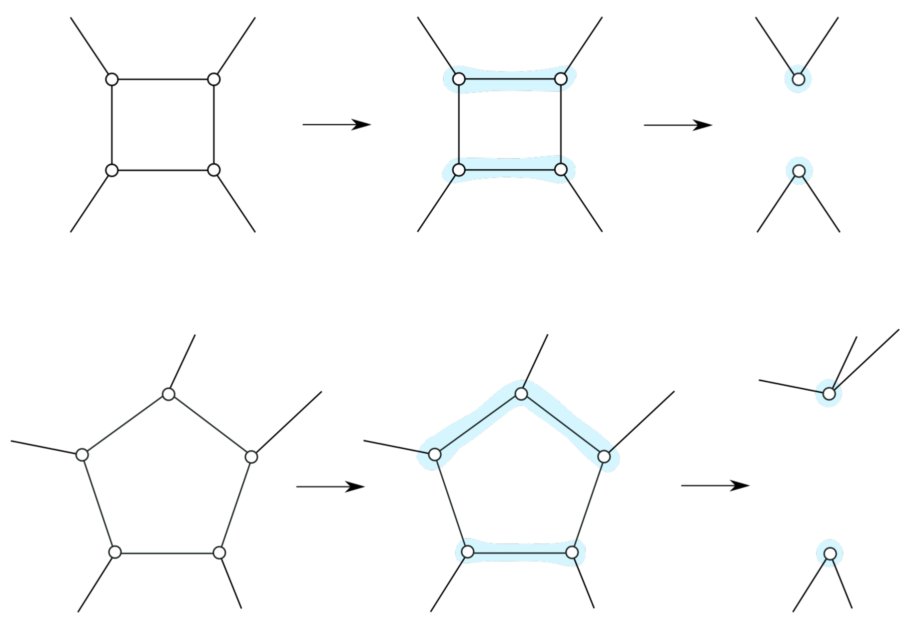
In this poster we verify Conjecture 1 for a class of cubic graphs; and we present a condition for a graph not to be 2-LIC.

Locally irregular coloring of some cubic graphs

A **proper edge-coloring** of a graph G is an assignment of colors to the edges of G in which edges that share a vertex are colored with different colors. A P_2-decomposition of a graph G is a decomposition of G into paths of length 2. Let G be a cubic graph, and let $\mathcal{P}$ be a P_2-decomposition of G. Given a vertex $v \in V(G)$, let $\mathcal{P}(v)$ denote the number of paths $P \in \mathcal{P}$ for which $d_P(v) = 1$, and let $V_i^{\mathcal{P}}$ be the set of vertices v of G for which $\mathcal{P}(v) = i$.

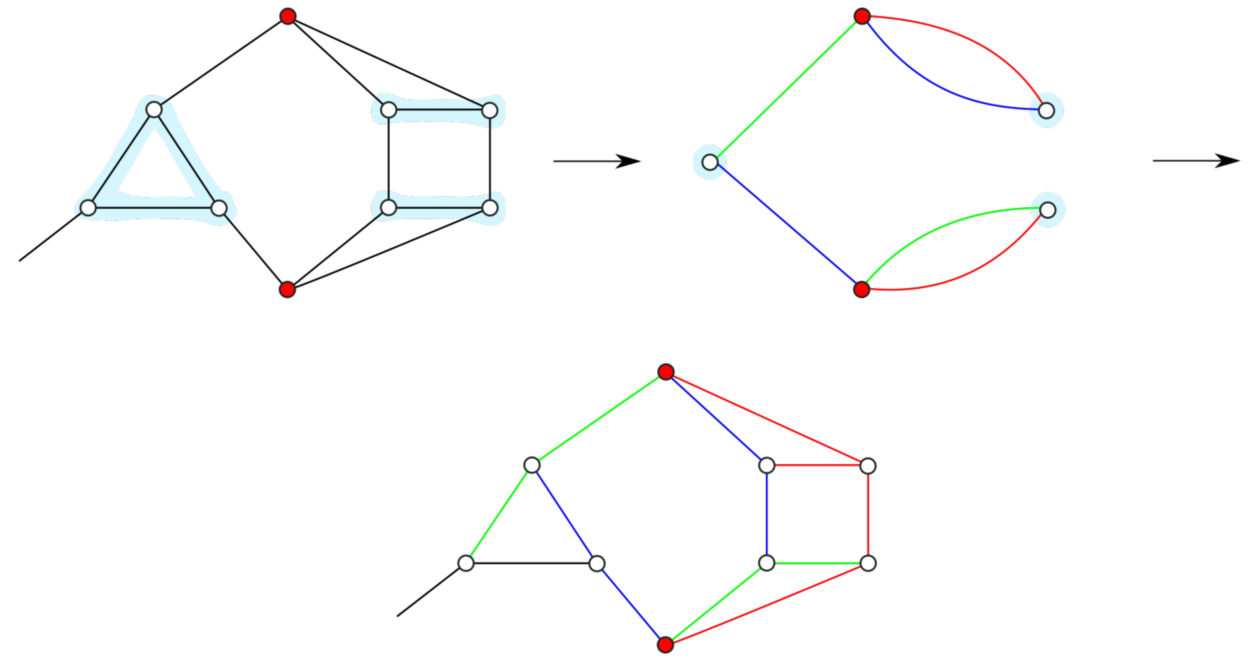
***Theorem* 1.** *If G is a cubic graph that admits a P_2-decomposition $\mathcal{P}$ for which $G[V_1^{\mathcal{P}}]$ is a set of vertex-disjoint cycles, then $\chi'_{irr}(G) \leq 3$.*

Proof: First note that $\mathcal{P}(v) \in \{1, 3\}$ for every $v \in V(G)$. In particular every vertex of $V_1^{\mathcal{P}}$ is the interior vertex of precisely one path of $\mathcal{P}$. Since $G[V_1^{\mathcal{P}}]$ is a set of vertex-disjoint cycles, every vertex in $V_1^{\mathcal{P}}$ is adjacent to precisely one vertex of $V_3^{\mathcal{P}}$ and two vertices of $V_1^{\mathcal{P}}$. Given a cycle $C \in G[V_1^{\mathcal{P}}]$, we partition the vertex set of C into pairs and at most one triple of consecutive vertices.

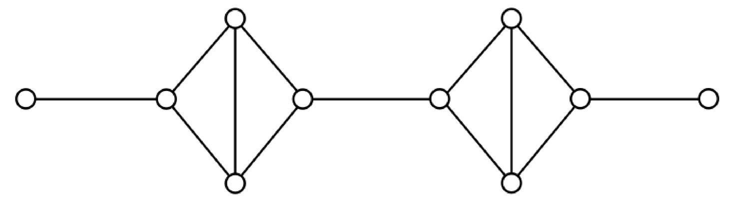


Let H be the graph obtained from $G \setminus E(G[V_1^{\mathcal{P}}])$ by identifying vertices in the same pair or triple, and keeping parallel edges. Note that every path of $\mathcal{P}$ has exactly one edge in H. The graph H is a bipartite graph with maximum degree exactly 3. It is not hard to prove that G admits a proper edge-coloring with three colors.

Now, we use the the proper edge-coloring above to obtain a locally irregular coloring of $E(G)$. By construction every path in $\mathcal{P}$ has precisely one edge already colored in H, and we color its remaining edge (which is in a cycle of $G[V_1^{\mathcal{P}}]$) with the same color. Since each vertex of $G[V_1^{\mathcal{P}}]$ is in the same pair or triple of at least one of its neighbors in $G[V_1^{\mathcal{P}}]$, each path of $\mathcal{P}$ is colored with the same color of at most one path with which it shares a vertex. Therefore each color consists of vertex-disjoint paths of length 2 and trees with four edges and one vertex of degree 3, and hence, is a locally irregular graph. $\square$

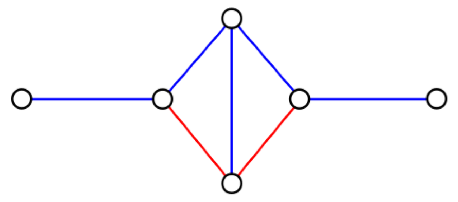


In order to prove that some cubic graphs have locally irregular chromatic index at least 3, we define the gadget below which we call a **strip**. So we have the following theorem.



***Theorem* 2.** *If G has a strip S whose vertices with degree 3 are not adjacent to vertices in $V(G) \setminus S$, then $\chi'_{irr}(G) > 2$.*

Proof: The proof follows from the fact that any 2-LIC of a "half strip" must be as in the figure below, and then the two "half strips" of the same strip cannot be colored in a compatible manner.



By replacing one edge by strip, we can prove that there are infinitely many graphs that do not admit a 2-LIC. In particular, there are an infinite number of cubic graphs with chromatic index 4 and planar graphs that do not admit an 2-LIC, and hence the upper bound of Conjecture 1 is tight for these classes of graphs.

References

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