Decomposing cubic graphs into locally irregular subgraphs

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Introduction

A decomposition of a graph $G$ is a set $\mathcal{D} = \{H_1, \ldots, H_k\}$ of edge-disjoint subgraphs of $G$ such that $\bigcup_{i=1}^k E(H_i) = E(G)$. A locally irregular graph is a graph in which adjacent vertices have distinct degrees.

A locally irregular decomposition (or locally irregular coloring) of a graph $G$ is a decomposition in which every element is locally irregular. We say that $G$ is decomposable if it admits a locally irregular decomposition. Equivalently, a locally irregular decomposition is a coloring of $E(G)$ in which every color class induces a locally irregular subgraph in $G$. If $k$ colors are used, then we say locally irregular $k$-edge-coloring or $k$-LIC for short.

![Figure 1: A locally irregular graph](image1)

Given a decomposable graph $G$, the irregular chromatic index of $G$ is the smallest number $k$ for which $G$ admits a $k$-LIC. We denote the irregular chromatic index of $G$ by $\chi_{irr}(G)$. The problem of computing the irregular chromatic index was proven to be an NP-complete problem [2].

In this work we explore the following conjecture posed by Baudon et al. [1].

**Conjecture 1** (O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak, 2015). For every decomposable graph $G$, we have $\chi_{irr}(G) \leq 3$.

Results toward confirming Conjecture 1 include that graphs whose set of vertices can be partitioned into a clique and an independent set admit a 3-LID [3] and graphs with maximum degree at most 3 admit a 4-LID [4]. We explore Conjecture 1 for graphs in which all vertices have degree 3, which are called cubic graphs.

**Contribution**

In this poster we verify Conjecture 1 for a class of cubic graphs; and we present a condition for a graph not to be 2-LIC.

Locally irregular coloring of some cubic graphs

A proper edge-coloring of a graph $G$ is an assignment of colors to the edges of $G$ in which edges that share a vertex are colored with different colors. A $P_2$-decomposition of a graph $G$ is a decomposition of $G$ into paths of length 2. Let $G$ be a cubic graph, and let $\mathcal{P}$ be a $P_2$-decomposition of $G$. Given a vertex $v \in V(G)$, let $P(v)$ denote the number of paths $P \in \mathcal{P}$ for which $d_{\mathcal{P}}(v) = 1$, and let $V_i$ be the set of vertices $v$ of $G$ for which $P(v) = i$.

**Theorem 1.** If $G$ is a cubic graph that admits a $P_2$-decomposition $\mathcal{P}$ for which $G[V_i^2]$ is a set of vertex-disjoint cycles, then $\chi_{irr}(G) \leq 3$.

**Proof:** First note that $P(v) \in \{1, 3\}$ for every $v \in V(G)$. In particular every vertex of $V_i^2$ is the interior of precisely one path of $\mathcal{P}$. Since $G[V_i^2]$ is a set of vertex-disjoint cycles, every vertex in $V_i^2$ is adjacent to precisely one vertex of $V_{i-1}$ and two vertices of $V_{i+1}$. Given a cycle $C \in G[V_i^2]$, we partition the vertex set of $C$ into pairs and at most one triple of consecutive vertices.

Let $H$ be the graph obtained from $G \setminus E(G[V_i^2])$ by identifying vertices in the same pair or triple, and keeping parallel edges. Note that every path of $\mathcal{P}$ has exactly one edge in $H$. The graph $H$ is a bipartite graph with maximum degree exactly 3. It is not hard to prove that $G$ admits a proper edge-coloring with three colors.

Now, we use the proper edge-coloring above to obtain a locally irregular coloring of $E(G)$. By construction every path in $\mathcal{P}$ has precisely one edge already colored in $H$, and we color its remaining edge (which is in a cycle of $G[V_i^2]$) with the same color. Since each vertex of $G[V_i^2]$ is in the same pair or triple of at least one of its neighbors in $G[V_i^2]$, each path of $\mathcal{P}$ is colored with the same color of at most one path with which it shares a vertex. Therefore each color consists of vertex-disjoint paths of length 2 and trees with four edges and one vertex of degree 3, and hence, is a locally irregular graph.

![Figure 2: (a) A 2-LIC of $G$. (b) An induced subgraph of $G$ using the edges with color red. (c) An induced subgraph of $G$ using the edges with color blue.](image2)

In order to prove that some cubic graphs have locally irregular chromatic index at least 3, we define the gadget below which we call a strip. So we have the following theorem.

**Theorem 2.** If $G$ has a strip $S$ whose vertices with degree 3 are not adjacent to vertices in $V(G) \setminus S$, then $\chi_{irr}(G) > 2$.

**Proof:** The proof follows from the fact that any 2-LIC of a “half strip” must be as in the figure below, and then the two “half strips” of the same strip cannot be colored in a compatible manner.

![Figure 3: A 2-LIC of $G$.](image3)

By replacing one edge by strip, we can prove that there are infinitely many graphs that do not admit a 2-LIC. In particular, there are an infinite number of cubic graphs with chromatic index 4 and planar graphs that do not admit a 2-LIC, and hence the upper bound of Conjecture 1 is tight for these classes of graphs.

References