Generalized Packing Functions of Graphs with few P_4's

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Abstract

We introduce a concept of packing of graphs which generalizes all those previously defined in the literature and we study the computational complexity of computing the associated parameter, the generalized packing number of the graph. We find that this new packing parameter comes to be much more complicated to handle than those previously defined, even on particular graph classes as spider and quasi-spider graphs. Nevertheless, we prove that the associated optmization problem can be solved in linear time for some graph classes with few P_4's.

General packing functions

Let $G = (V, E)$ be a graph and $\mathbf{k}, \boldsymbol{\ell}, \mathbf{u} \in \mathbb{Z}_+^V$ with $\boldsymbol{\ell} \leq \mathbf{u}$. A $(\mathbf{k}, \boldsymbol{\ell}, \mathbf{u})$-*packing function of* G is a function $f : V \to \mathbb{Z}_+$ satisfying the following conditions for all $v \in V$: $\ell(v) \leq f(v) \leq u(v)$ and $f(N[v]) \leq k(v)$. In addition, we define $\mathcal{L}_{\mathbf{k},\boldsymbol{\ell},\mathbf{u}}(G) = \{f : f \text{ is a } (\mathbf{k}, \boldsymbol{\ell}, \mathbf{u}) - \text{packing function of } G\}$. Then, the $(\mathbf{k}, \boldsymbol{\ell}, \mathbf{u})$- generalized packing number of G is

$$L_{\mathbf{k},\boldsymbol{\ell},\mathbf{u}}(G) = max\{f(V) : f \in \mathcal{L}_{\mathbf{k},\boldsymbol{\ell},\mathbf{u}}(G)\}.$$

Reduction to instances with $\boldsymbol{\ell} = 0$: $L_{\mathbf{k},\boldsymbol{\ell},\mathbf{u}}(G) = \boldsymbol{\ell}(V) + L_{\tilde{\mathbf{k}},0,\tilde{\mathbf{u}}}(G)$ for $\tilde{\mathbf{k}}(v) = \mathbf{k}(v) - \boldsymbol{\ell}(N[v])$, $\tilde{\mathbf{u}}(v) = \mathbf{u}(v) - \boldsymbol{\ell}(v)$

$$L_{\mathbf{k},\ell,\mathbf{u}}(G) \to L_{\tilde{\mathbf{k}},0,\tilde{\mathbf{u}}}(G) \to L_{\tilde{\mathbf{k}},\tilde{\mathbf{u}}}(G).$$

Given $f \in \mathcal{L}_{\mathbf{k},\mathbf{u}}(G)$ such that $f(V) = L_{\mathbf{k},\mathbf{u}}(G)$ we say that f is an *optimal* $(\mathbf{k}, \mathbf{u})$-packing function of G.
The *Packing Function Problem* (PFP) has a graph G and vectors $\mathbf{k}, \mathbf{u} \in \mathbb{Z}_+^{V(G)}$ as input and the objective is to obtain an optimal $(\mathbf{k}, \mathbf{u})$-packing function of G.

Modular decomposition

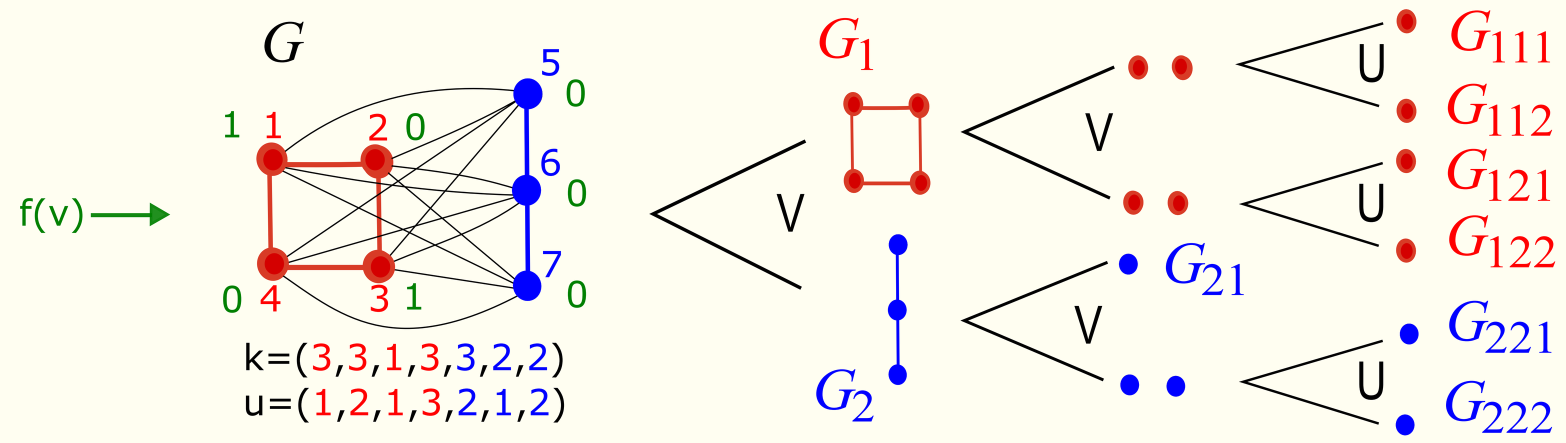
The modular decomposition of graphs involves two graph operations, *union* ($\cup$) and *join* ($\vee$). If a graph is not connected, it is the union of two graphs and if the complement of a graph is not connected, the graph is the join of two graphs (figure below). A graph is *modular* if it is connected and its complement is connected.

The parameter for these two operations can be computed as follows. Let $\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2)$, $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, $k_i^* = \min\{k_i(u) : u \in V_i\}$ for $i = 1, 2$, $\ell_1(r) = \min\{L_{\mathbf{k}_1 - r.\mathbf{1},\mathbf{u}_1}(G_1), k_2^*\}$, $\ell_2(r) = \min\{L_{\mathbf{k}_2 - r.\mathbf{1},\mathbf{u}_2}(G_2), k_1^*\}$, and $\Delta(s) = \min\{\Delta : s \leq \ell_1(\ell_2(s) - \Delta), \Delta \in [0, \ell_2(s)]\}$. Then,

$$L_{\mathbf{k},\mathbf{u}}(G_1 \cup G_2) = L_{\mathbf{k}_1,\mathbf{u}_1}(G_1) + L_{\mathbf{k}_2,\mathbf{u}_2}(G_2),$$

$$L_{\mathbf{k},\mathbf{u}}(G_1 \vee G_2) = \max\{s + \ell_2(s) - \Delta(s) : s \in [0, \ell_1(0)]\}.$$

Example:



$$L_{\mathbf{k},\mathbf{u}}(G) = L_{\mathbf{k},\mathbf{u}}(G_1 \vee G_2) = L_{\mathbf{k},\mathbf{u}}(((G_{111} \cup G_{112}) \vee (G_{121} \cup G_{122})) \vee (G_{21} \vee (G_{221} \cup G_{222})) = 2$$

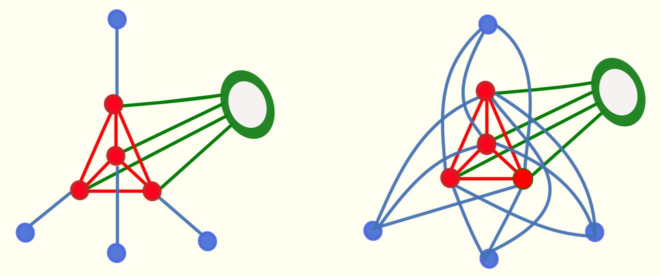
For a graph class $\mathcal{F}$, we denote by $M(\mathcal{F})$ the class of graphs in $\mathcal{F}$ which are modular. From the previous formulas we have the following result.

Lemma 1. *Let $\mathcal{F}$ be a hereditary family of graphs such that the PFP can be solved in polynomial (resp. linear) time for graphs in $M(\mathcal{F})$. Then, the PFP can be solved in polynomial (resp. linear) time for every graph in $\mathcal{F}$.*

Thus, let us study the graphs in $\mathcal{F}$ for graph classes with *few P_4's*, such as *P_4-sparse graphs* and *P_4-tidy graphs*. Formally, a graph is P_4-sparse if every set of five vertices contains at most one induced path on four vertices.

Spider graphs and P_4-sparse graphs

A *spider* is a graph $G = (V, E)$ such that V is partitioned into sets S, C and H, where $S = \{s_j : j \in [n]\}$ is a stable set, $C = \{c_j : j \in [n]\}$ is a clique, $n \geq 2$, and the *head* H is allowed to be empty. Moreover, all vertices in H are adjacent to all vertices in C and no vertex in S. Besides, in a *thin spider* graph, s_i is adjacent to c_j if and only if $i = j$, and in a *thick spider* graph, s_i is adjacent to c_j if and only if $i \neq j$. We denote a spider by (C, S, H).



Lemma 2. *Let $G = (C, S, H)$ be a thin spider graph. Then*

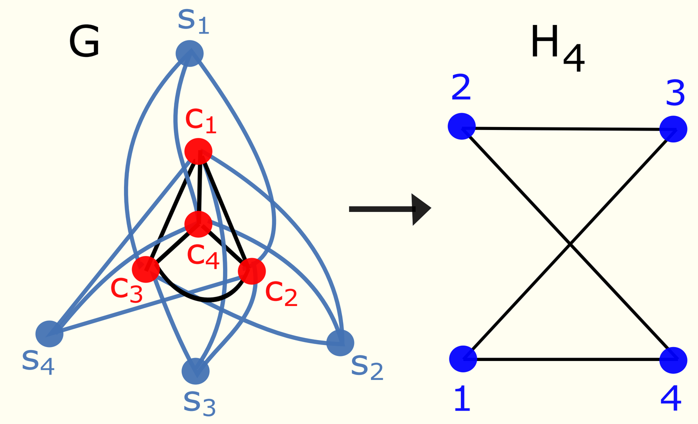
$$L_{\mathbf{k},\mathbf{u}}(G) = u(S) + L_{\tilde{\mathbf{k}},\tilde{\mathbf{u}}}(G[C] \vee G[H]).$$

where $\tilde{k}(h) = k(h)$ and $\tilde{u}(h) = u(h)$ for all $h \in H$, $\tilde{k}(c_i) = k(c_i) - u(s_i)$ and $\tilde{u}(c_i) = min\{u(c_i), k(s_i) - u(s_i)\}$.

The approach to study the problem in thick spiders is based on technical lemmas. They allow us to reduce the general problem to a particular instance $(\mathbf{k} = \mathbf{u})$ in thick spiders with empty head. Then, we apply a transformation from a thick spider with empty head to a particular graph H_n, which has even order and the edges missing form a perfect matching as is shown in the next example.

For a spider graph G and $\mathbf{k} = (6,3,5,4,6,7,5,3)$ we have

$$\tilde{\mathbf{k}} = (7, 6, 3, 5),\ \mathbf{u} = (6, 3, 5, 4) \Rightarrow\ L_{\mathbf{k},\mathbf{k}}(G) = L_{\tilde{\mathbf{k}},\mathbf{u}}(H_4)$$



Finally, this process derives the following result.

Proposition 1. *If the PFP is polynomial (linear) time solvable on a graph family $\mathcal{F}$, then the PFP can be solved in polynomial (linear) time on spider graphs such that the graph induced by the head is in $\mathcal{F}$.*

From the decomposition results [1, 2], if $\mathcal{F}$ is the class of P_4-sparse graphs, we know that the graphs in $M(\mathcal{F})$ are the trivial graph and spider graphs such that the graph induced by the head is P_4-sparse.
Lastly, considering the previous results and Lemma 1, we obtain the next theorem.

Theorem 1. *The PFP is linear time solvable for P_4-sparse graphs.*

Particular case in P_4-tidy graphs

A partner of a path P on four vertices in a graph G is a vertex $v \in V(G) \setminus V(P)$ such that the subgraph induced by $V(P) \cup \{v\}$ has at least two paths on four vertices. A graph G is a P_4-tidy if every path on four vertices has at most one partner.

Considering the problem, a particular case of the PFP is obtained when $\mathbf{u}(v) = \mathbf{k}(v) = k\ \forall v \in V$, for $k \in \mathbb{Z}_+$ fixed. In this case the problem is denoted $\{k\}$-PFP.

Concerning this restricted problem, the linearity result can be settled in P_4-tidy graphs, a graph class larger than P_4-sparse. Regarding modular decomposition, it is known that a P_4-tidy graph G is modular if and only if G is the trivial graph, C_5, P_5, $\overline{P_5}$, or a quasi-spider graph such that the graph induced by the head is P_4-tidy. A quasi-spider graph is a graph obtained from a spider by replacing at most one vertex in $S \cup C$ by a K_2 or a $\overline{S_2}$.
Based on modular decomposition and applying the results obtained for the mentioned P_4-tidy modular graphs, we derive the following.

Theorem 2. *The $\{k\}$-PFP is linear time solvable for P_4-tidy graphs.*

References

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