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Logarithmic-Sobolev and Poincaré inequalities for the simple random walk on the Hanoi graph

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The random walk on the Tower of Hanoi

The tower of Hanoi puzzle is a single-player game where at each turn the player moves a disk to a tower that is different from the one it previously was. The game does not allow a disk above smaller disks and its aim is to move all disks from a tower to another one (see Figure 1).

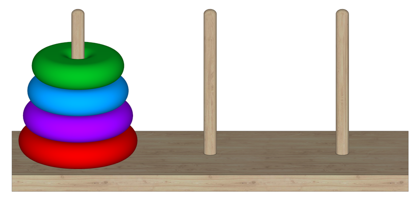

Figure 1: *Tower of Hanoi with* 4 *disks.*

The Hanoi graph $\mathcal{H}_m = (V_m, E_m)$ is the graph whose vertices represent the possible configurations of the tower of Hanoi puzzle with 3 towers and m disks. Its edges represent the moves between these configurations. Thus, $\mathcal{H}_1$ is isomorphic to a triangle and for each $m \geq 2$ we can construct $\mathcal{H}_m$ in the following way which is illustrated in Figure 2: we consider three isomorphic copies of $\mathcal{H}_{m-1}$ and we label them as $\mathcal{H}^i_{m-1} = (V^i_{m-1}, E^i_{m-1})$, $i \in \{1,2,3\}$. For each $i \in \{1,2,3\}$ let v^i_{top}, v^i_{lb} and v^i_{rb} be the vertices on the top, on the left, and on the right of the basis of the biggest triangle in $\mathcal{H}^i_{m-1}$. The graph $\mathcal{H}_m$ is the graph with vertex set $V_m = \cup_{i=1}^3 V^i_{m-1}$ and edge set $E_m = \left(\cup_{i=1}^3 E^i_{m-1}\right) \cup E^\star_m$, where $E^\star_m$ is defined as $E^\star_m := \{\{v^1_{lb}, v^2_{top}\}, \{v^1_{rb}, v^3_{top}\}, \{v^2_{rb}, v^3_{lb}\}\}$.

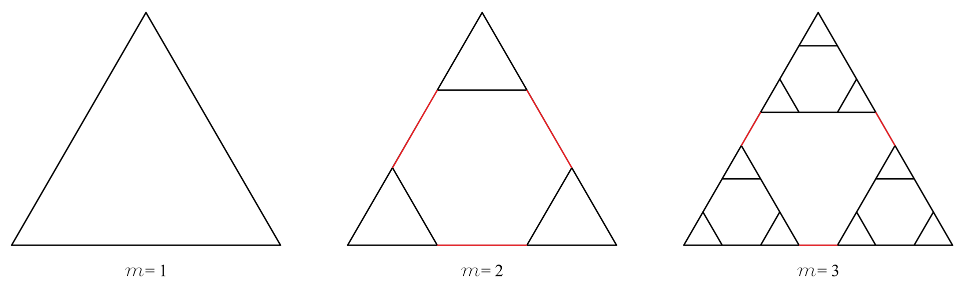

Figure 2: *Graphs* $\mathcal{H}_m$ *for* $m \in \{1,2,3\}$ *with edges of* $E^\star_m$ *coloured in red.*

The simple random walk on $\mathcal{H}_m$ is the process $\{X_t; t \geq 0\}$ described as follows: an exponential clock with rate one is attached to each edge of E_m. Whenever a clock rings, the edge associated with that clock is flipped, making the random walker jump if she was at one of the incident vertices to that edge. Its infinitesimal generator is the discrete Laplacian operator Δ_m given by

$$\Delta_m f(x) = \sum_{y \sim x} (f(y) - f(x)),$$

which says that if the random walker stands at a vertex x then it can jump to any of its adjacent vertices with rate 1. In the above formula, $x \sim y$ denotes that x and y share a common edge.

An interesting question to make is to ask how long the random walker takes to get completely lost. In order to answer this question, let $\mu^{x_0}_t(x)$ denote the probability that $X_t = x$ given that $X_0 = x_0$, and let U_m denote the uniform measure on V_m. The distance to equilibrium of the simple random walk on $\mathcal{H}_m$ is defined as

$$d_m(t) = \max_{x_0 \in V_m} \|\mu^{x_0}_t - U_m\|_{TV} = \max_{x_0 \in V_m} \left\{ \frac{1}{2} \sum_{x \in V_m} \left| \mu^{x_0}_t(x) - \frac{1}{3^m} \right| \right\}.$$

Not only the above function is decreasing, but it also takes values in the interval $[0,1]$. Thus, given a threshold $\varepsilon \in (0,1)$, it makes sense to define the ε-mixing time of the simple random walk as

$$t^m_{mix}(\varepsilon) = \inf\{t \geq 0;\ d_m(t) < \varepsilon\},$$

formalizing the answer to the aforementioned question.

Algebraic connectivity and Poincaré inequalities

The spectral gap γ_m of the simple random walk on $\mathcal{H}_m$ (also known as the *algebraic connectivity of the graph* $\mathcal{H}_m$) is defined as the symmetric of the second largest eigenvalue γ_m of the operator Δ_m. It also presents a variational formula [2]. Indeed, let $\mathcal{E}_m$ be the Dirichlet form of the simple random walk on $\mathcal{H}_m$ which is given by

$$\mathcal{E}_m(f,f) = \frac{1}{2} \sum_{x \in V_m} \sum_{y \sim x} |f(x) - f(y)|^2\, U_m(x).$$

Let $\mathrm{Var}(f\,;U_m)$ be the variance of a function $f : V_m \to \mathbb{R}$ for the simple random walk on $\mathcal{H}_m$, which is given by

$$\mathrm{Var}(f\,;U_m) = \frac{1}{2} \sum_{x,y} |f(x) - f(y)|^2\, U_m(y)\, U_m(x).$$

The spectral gap γ_m of the simple random walk on $\mathcal{H}_m$ can be defined as

$$\gamma_m := \inf_f \left\{ \frac{\mathcal{E}_m(f,f)}{\mathrm{Var}(f\,;U_m)};\ \mathrm{Var}(f\,;U_m) \neq 0 \right\}.$$

Namely, the relaxation time $t^m_{rel} := 1/\gamma_m$ of the simple random walk on $\mathcal{H}_m$ is the smallest constant that satisfies the *Poincaré inequality*

$$\mathrm{Var}(f\,;U_m) \leq C\, \mathcal{E}_m(f,f) \text{ for every function } f.$$

The spectral gap is strongly related to mixing because

$$d_m(t) \leq 3^{m/2}\, e^{-\gamma_m t} \text{ (see [5], for instance).}$$

Our result in this direction is the following:

Theorem 1: For every $m \geq 2$ we have

$$t^m_{rel} \leq \frac{1}{3\ (1/3\,;1/3)_{m-1}}, \text{ where } (a\,;q)_n := \prod_{k=0}^{n-1} (1 - a\,q^k)$$

is the q-Pochhammer symbol, also known as q-shifted factorial. Consequently,

$$t^m_{mix}(\varepsilon) \leq \frac{\log 3}{9} \left(\frac{3}{2}\right)^m m + \mathcal{O}_\varepsilon\left(\left(\frac{3}{2}\right)^m\right).$$

Logarithmic-Sobolev inequalities

The log-Sobolev constant α_m of the simple random walk on $\mathcal{H}_m$ presents a similar variational definition. Here the variance is replaced by the entropy-like quantity $\mathcal{L}_m$ given by

$$\mathcal{L}_m(f) = \sum_{x \in \mathcal{H}_m} |f(x)|^2 \log\left(\frac{|f(x)|^2}{\sum_{z \in \mathcal{H}_m} |f(z)|^2\, U_m(z)} \right) U_m(x).$$

Namely

$$\alpha_m := \inf_f \left\{ \frac{\mathcal{E}_m(f,f)}{\mathcal{L}_m(f)};\ \mathcal{L}_m(f) \neq 0 \right\},$$

that is, $1/\alpha_m$ is the smallest constant that satisfies the *logarithmic Sobolev inequality*

$$\mathcal{L}_m(f) \leq C\, \mathcal{E}_m(f,f) \text{ for every function } f.$$

The log-Sobolev constant is stronger than the spectral gap in the sense that

$$d_m(t) \leq \sqrt{m}\, e^{-\alpha_m t/2} \text{ (see [3], for instance).}$$

We prove the following result:

Theorem 2: There exists a constant $\alpha_1 \in (0.856, 1.500]$ such that for every $m \geq 1$ we have

$$\frac{1}{\alpha_m} \leq \frac{2}{\alpha_1} \left(\frac{3}{2}\right)^m. \text{ Consequently, } t^m_{mix}(\varepsilon) \leq \frac{2}{\alpha_1} \left(\frac{3}{2}\right)^m \log m + \mathcal{O}_\varepsilon\left(\left(\frac{3}{2}\right)^m\right).$$

Decomposing the graph

The main idea is to remove the edges of $E^\star_m$ and to decompose $\mathcal{H}_m$ into $\mathcal{H}^i_{m-1}$, $i \in \{1,2,3\}$, and then to do some analysis. Given a function $f : V_m \to \mathbb{R}$ and $i \in \{1,2,3\}$, denote the restriction of f to the domain V^i_{m-1} by $f|_{V^i_{m-1}}$, and define $U^i_{m-1} := U_m|_{V^i_{m-1}}$. Firstly, we prove that

$$\mathrm{Var}(f\,;U_m) = \frac{1}{3} \sum_{i=1}^3 \mathrm{Var}(f|_{V^i_{m-1}};\, U^i_{m-1}) + \mathrm{Var}(G\,;U_1) \leq \frac{1}{\gamma_{m-1}} \mathcal{E}_m(f\,,f) + \mathrm{Var}(G\,;U_1).$$

where $G(i) = \sum_{z \in \mathcal{H}^i_{m-1}} f(z)\, U^i_{m-1}(z)$ is the expectation of the function f, restricted to V^i_{m-1}, with respect to the measure U^i_{m-1}. Secondly, we show that $\mathrm{Var}(G\,;U_1) \leq 3^{1-m}\, \mathrm{Var}(f\,;U_m)$. By the definition of γ_m, we obtain

$$\frac{1}{\gamma_m} \leq \frac{1}{\gamma_{m-1}\,(1 - 3^{1-m})},$$

which after an induction argument, implies Theorem 1. Similarly, we prove that

$$\mathcal{L}_m(f) \leq \frac{1}{\alpha_{m-1}} \mathcal{E}_m(f,f) + \frac{1}{\alpha_1} \mathcal{E}_1(\sqrt{F}, \sqrt{F}),$$

where $F(i)$ is the square of the $\ell^2(U^i_{m-1})$ norm of the function f restricted to V^i_{m-1}. Then, we show that $\frac{1}{\alpha_1} \mathcal{E}_1(\sqrt{F}, \sqrt{F}) \leq \frac{\gamma_1}{\alpha_1 \gamma_m} \mathcal{E}_m(f\,,f)$. Theorem 2 follows from an induction argument.

Remark: Looking carefully at the lower bound on the log-Sobolev constant obtained in Theorem 2 with the above method, one can see that it strongly depends on the upper bound on the relaxation time. More precisely, if one can obtain a sharper exponential upper bound on t^m_{rel}, then, using our method, they can obtain a lower bound on the second parameter which has the same order.

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