Mutually Included Biclique Graphs of Interval Containment Bigraphs and Interval Bigraphs

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Abstract

The recognition of biclique graphs in general is still open. Recently, Groshaus and Guedes introduced the mutually included biclique graph as an intermediate operator to define the biclique graph. Also, we previously studied the biclique graph of interval bigraphs and proper interval bigraphs. In this work, we extend the results to a superclass, the interval containment bigraphs, in the context of the mutually included biclique graphs.

Introduction

Given a graph G, its ***biclique graph*** ($KB(G)$) is the intersection graph of the bicliques of G. It was introduced by Groshaus and Szwarcfiter in 2010 [5]. They presented a characterization of biclique graphs and a characterization of biclique graphs of bipartite graphs, but the time complexity of the problem of recognizing biclique graphs remains open.

Bicliques in graphs have applications in various fields, for example, biology: protein-protein interaction networks, social networks: web community discovery, genetics, medicine, information theory. More applications (including some of these) can be found in the work of Liu, Sim, and Li [9].

In 2018 Groshaus and Guedes introduced the ***mutually included biclique graph*** ($KB_m(G)$) [3, 4] as a spanning subgraph of $KB(G)$. They proved that $KB(G) = (KB_m(G))^2$ for any K_3-free graph G, and that KB_m(bipartite) $\subset$ comparability graphs.

In this work, we present some results about biclique graphs and mutually included biclique graphs of interval bigraph, interval containment bigraph, and bipartite graphs in general.

Classes Studied

- CGI: Containment graph of intervals [2]
  A graph G is a ***containment graph of intervals*** if its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if one of the corresponding intervals contains the other. Call that family of intervals a ***interval containment model*** of G.
- ICB: Interval containment bigraphs [8]
  A bipartite graph G is an ***interval containment bigraph*** if its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if they are of different parts and one of the corresponding intervals contains the other. Call that family of intervals a ***bipartite interval containment model*** of G.
- IBG: Interval bigraphs ($IBG \subseteq ICB$) [6]
  A bipartite graph G is an ***interval bigraph*** if its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if they are of different parts and the corresponding intervals intersect. Call that family of intervals a ***bipartite interval model*** of G.
- PG: Permutation Graphs = CGI = comparability $\cap$ co-comparability [1]

Definitions

- Two bicliques B_1, B_2 are ***vertex-intersecting*** if they intersect and $G[B_1 \cap B_2]$ has no edges.
- Two bicliques B_1, B_2 are ***edge-intersecting*** if $G[B_1 \cap B_2]$ has at least an edge.
- Two bicliques B_1, B_2 are ***mutually included*** if one part of B_1 is properly include in one part of B_2, and the other part of B_2 is properly included in the other part of B_2 [3, 4]. See Figure 1.
- The ***mutually included biclique graph*** of a graph G (denoted $KB_m(G)$) is the graph which each vertex corresponds to a biclique of G and two vertices are adjacent if the corresponding bicliques are mutually included [3, 4]. Note that the binary relation of being mutually include is not transitive.

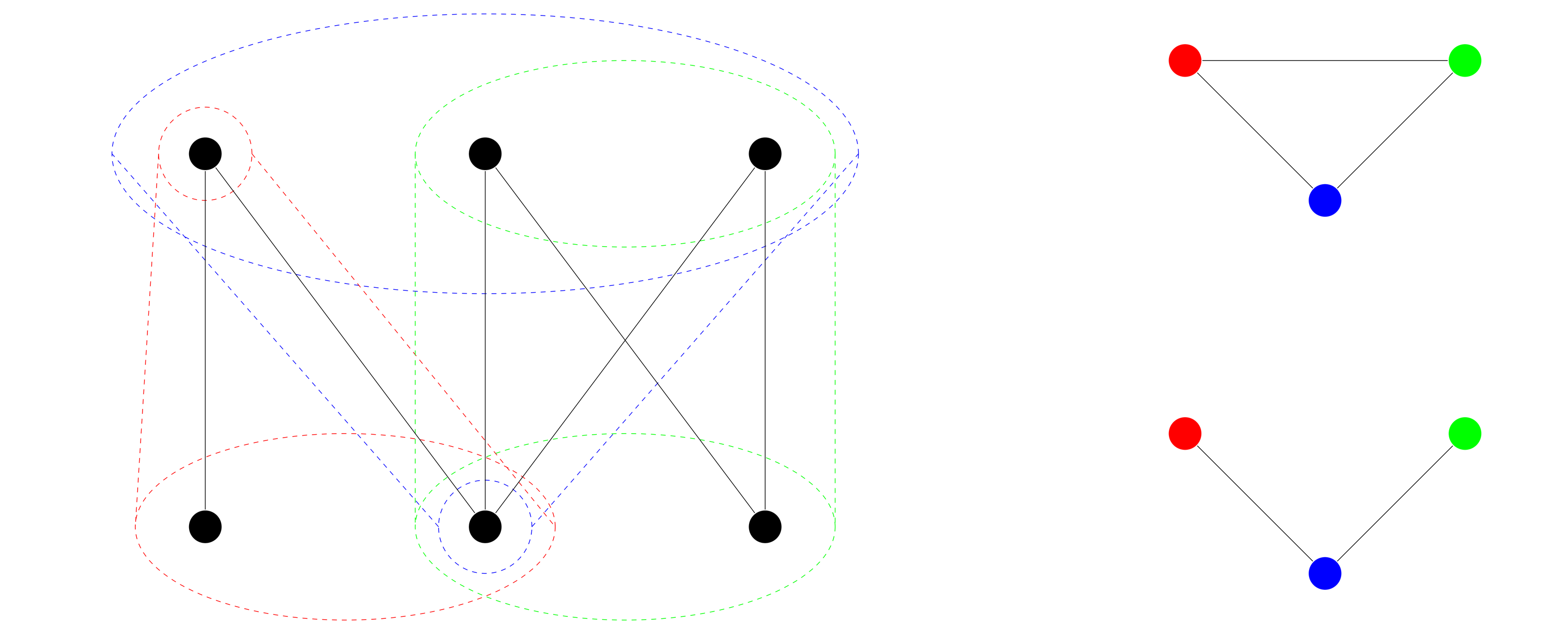

Figure 1: Graph G with 3 bicliques, red, blue and green. The blue biclique is mutually included with both red and green, but the red and green bicliques are not mutually included with each other. On the right, the graph above is $KB(G)$ and the one below is $KB_m(G)$.

- An ***asteroidal triple*** (AT) is an independent set with 3 vertices such that for every pair of vertices there is a path connecting them while avoiding the neighbors of the third vertex [7]. See Figure 2 (right) for an example of an AT.
- A ***bi-asteroidal triple*** ($biAT$) is an asteroidal triple such that the path between each pair of vertices is not adjacent to the neighborhood of the third vertex [6]. See Figure 2 (left) for an example of an $biAT$.

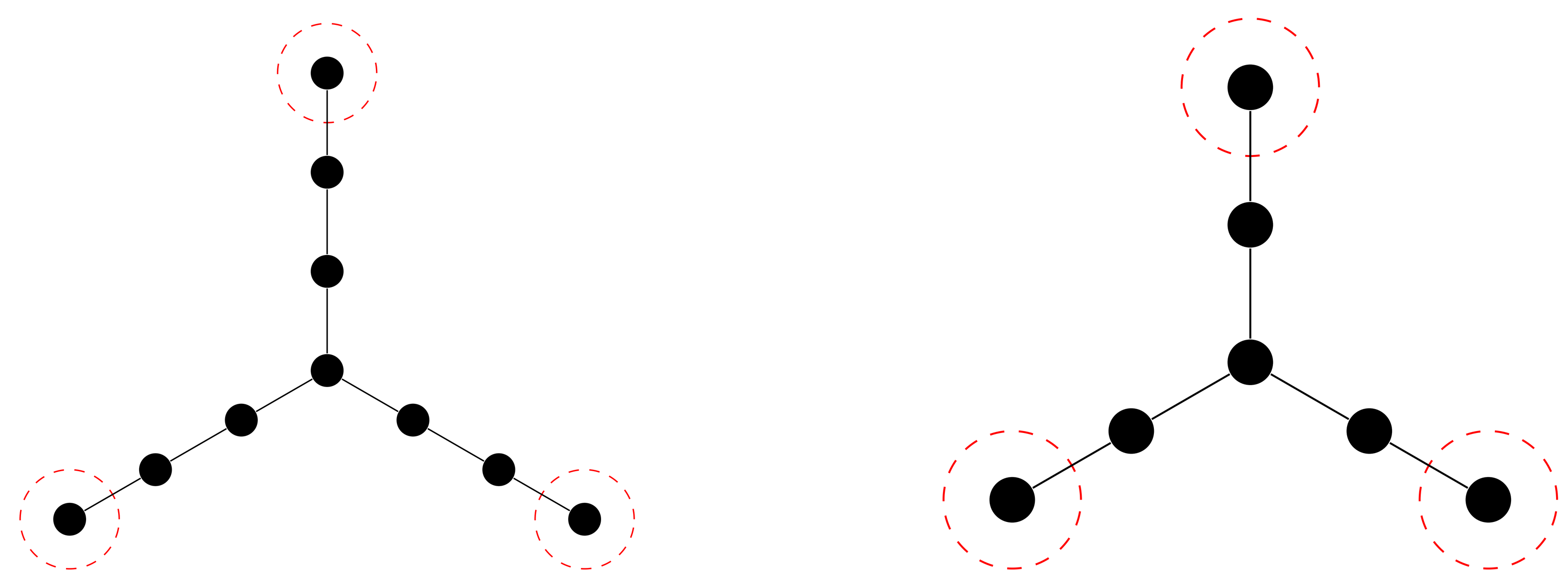

Figure 2: Example of a bi-asteroidal triple (left) and an asteroidal-triple (right). Note that the graph on the right is the same as the KB_m graph of the graph on the left.

Results

KB and KB_m of ICB and IBG

- $\mathbf{KB_m(ICB) \subset PG}$.
  Proof idea: Find a partial order $\leq_1$ such that, for $B_1 \neq B_2$, $\{B_1, B_2\} \in E(KB_m(G))$ if and only if $B_1 \leq_1 B_2$ or $B_2 \leq_1 B_1$, and a partial order $\leq_2$ such that $\{B_1, B_2\} \notin E(KB_m(G))$ if and only if $B_1 \leq_2 B_2$ or $B_2 \leq_2 B_1$. That is, prove that $KB_m(G)$ is a comparability and a co-comparability graph.
- $\mathbf{KB(ICB) \subseteq PG^2}$.
  Proof idea: Corollary of previous item and the fact that $KB(G) = (KB_m(G))^2$ [4].
- **For every $\mathbf{H \in PG}$, there is a $\mathbf{G \in IBG}$ such that $\mathbf{H \subseteq KB_m(G)}$.**
  Proof idea: Construct an interval bigraph G (constructing a bipartite interval model) from an interval containment model C of H (as H is also an CGI) such that $H \subseteq KB_m(G)$.

Bipartite Graphs

Let G be a bipartite graph, then:

- **If $\mathbf{KB_m(G)}$ is AT-free then G is $biAT$-free.**
  Consequently, for some AT-free graph class $\mathcal{A}$, $KB_m^{-1}(\mathcal{A}) \cap$ bipartite is $biAT$-free.
  Proof idea: By construction, proving that if $\mathbf{G}$ has a bi-asteroidal triple then $\mathbf{KB_m(G)}$ has an asteroidal triple.
- **G is $\mathbf{P_4}$-free if and only if $\mathbf{E(KB_m(G)) = \emptyset}$.**
  Proof idea: By inspection of the possibilities of mutually included bicliques. If there is a pair of mutually included bicliques then there is a P_4 in G. See Figure 3 (left).
- **If G is $\mathbf{P_5}$-free then there is no pair of vertex-intersecting bicliques.**
  Proof idea: By inspection of the possibilities of vertex-intersecting bicliques. If there is a pair of vertex-intersecting bicliques then there is a P_5 in G. See Figure 3 (middle).
- **G is domino-free if and only if every pair of edge-intersecting bicliques are mutually included.**
  Proof idea: By inspection of the possibilities of edge-intersecting bicliques that are not mutually included. If there is a pair of not mutually included edge-intersecting bicliques then there is a domino in G. See Figure 3 (right).

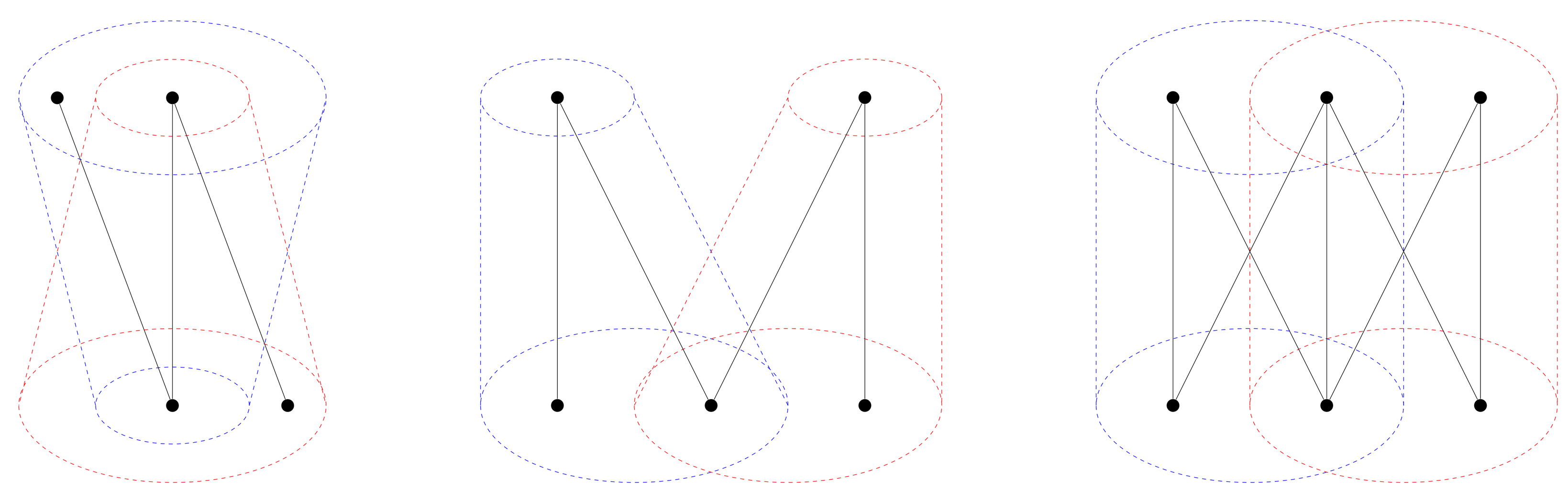

Figure 3: Edge types of bicliques that are: mutually included and a P_4 (left), vertex-intersecting and a P_5 (middle), and edge-intersecting not mutually included and a domino (right).

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