Subclass Hierarchy on Circular Arc Bigraphs

A study of graph class containments

Marina E. Groshaus & André L. P. Guedes & Fabricio S. Kolberg
Universidade Tecnológica Federal do Paraná - Departamento Acadêmico de Informática
Universidade Federal do Paraná - Departamento de Informática
E-mails: marinagroshaus@utfpr.edu.br, andre@inf.ufpr.br gretgor42@gmail.com

remote
9th LAWCG
and
MDA
November 25th, 2020

Abstract

A bipartite graph is a circular arc bigraph if there exists a bijection between its vertices and a family of arcs on a circle such that vertices of opposing partite sets are neighbors precisely if their corresponding arcs intersect. The class is a relatively unexplored subject, with most results on it and its subclasses being quite recent. In our work, we provide a full exploration of the containment relations and intersections between seven subclasses of circular arc bigraphs.

Introduction

The class of ***Circular arc bigraphs*** is a bipartite variation of the class of circular arc graphs. A bipartite graph $G = (V, W, E)$ is a circular arc bigraph if there exists a bijection $b : V \cup W \to \mathbb{A}$ such that $\mathbb{A}$ is a family of arcs on a circle and two vertices $v \in V, w \in W$ are neighbors if and only if $b(v) \cap b(w) \neq \emptyset$. Unlike its non-bipartite counterpart, circular arc bigraphs are a mostly unexplored topic of research, with most results on it being relatively recent.

In 2013, Basu et al. [1] published matrix-based characterizations for the class of circular arc bigraphs, as well as for the subclasses of ***proper circular arc bigraphs*** and ***unit circular arc bigraphs***. In 2015, Das and Chakraborty [2] presented a vertex order based characterization of proper circular arc bigraphs and ***proper interval bigraphs***. More recently, in 2019, Safe [4] presented a forbidden structure characterization and efficient recognition algorithm for proper circular arc bigraphs.

In our work, we study seven different subclasses of circular arc bigraphs, and provide comprehensive results on their pairwise comparability and containment relations.

Classes studied

The classes studied in our work are the following, defined in detail in the Definitions section.

- ***Circular Convex Bipartite (CCB) graphs***, including its subclass of ***Doubly Circular Convex Bipartite (doubly-CCB) graphs***.
- ***Helly circular arc bigraphs***, including its subclasses of ***non-bichordal Helly circular arc bigraphs*** and ***Helly interval bigraphs***.
- ***Proper circular arc bigraphs*** including its subclass of ***proper interval bigraphs***.

Definitions

Assume a bipartite graph $G = (V, W, E)$. We use $N(v)$ and $N[v]$ to denote the open and closed neighborhoods of vertex v, respectively.

Proper family: no two elements in it are properly contained in one another.

***Biclique*:** a maximal subset of $V(G)$ that induces a bipartite-complete subgraph.

***Bichordal*:** bipartite graph that does not admit any induced cycles of length greater than 4.

***Non-bichordal*:** is not bichordal.

***Twins*:** two vertices $v, w \in V \cup W$ such that $N(v) = N(w)$.

Circular arc bigraph (CAB)*:** bipartite graph that admits a bijection $b : V \cup W \to \mathbb{A}$ where $\mathbb{A}$ is a family of arcs on a circle such that $v \in V, w \in W$ are neighbors if and only if $b(v) \cap b(w) \neq \emptyset$. Call such a bijection a ***bi-circular-arc model of G.

Interval bigraph*:** a bipartite graph that admits a bijection $b : V \cup W \to \mathbb{A}$ such that $\mathbb{A}$ is a family of intervals on the number line, and $v \in V, w \in W$ are neighbors if and only if $b(v) \cap b(w) \neq \emptyset$. Call such a bijection a ***bi-interval model of G.

Circular convex bipartite (CCB)*:** a bipartite graph that V can be circularly ordered such that for every $w \in W$, $N(w)$ is an interval in the order. Call such an order a ***CCB order of V. Graph G is ***doubly-CCB (D-CCB)*** if both V and W partite sets admit such an order.

***Proper circular arc bigraph (P)* (resp. *proper interval bigraph (PI)*):** a bipartite graph that it admits a bi-circular-arc (resp. bi-interval) model b such that $b(V)$ and $b(W)$ are proper families.

Helly circular arc bigraph (H)* (resp. *Helly interval bigraph (HI)*):** a bipartite graph that it admits a bi-circular-arc (resp. bi-interval) model b such that, for every biclique $K \subset V \cup W$, there exists a point p on the circle (on the number line) such that $p \in X$ for all $X \in b(K)$. Call such a model a ***Helly bi-circular-arc (resp. ***Helly bi-interval***) ***model*** of G.

***Non-bichordal Helly circular arc bigraph (NBH)*:** a non-bichordal Helly circular arc bigraph.

Findings

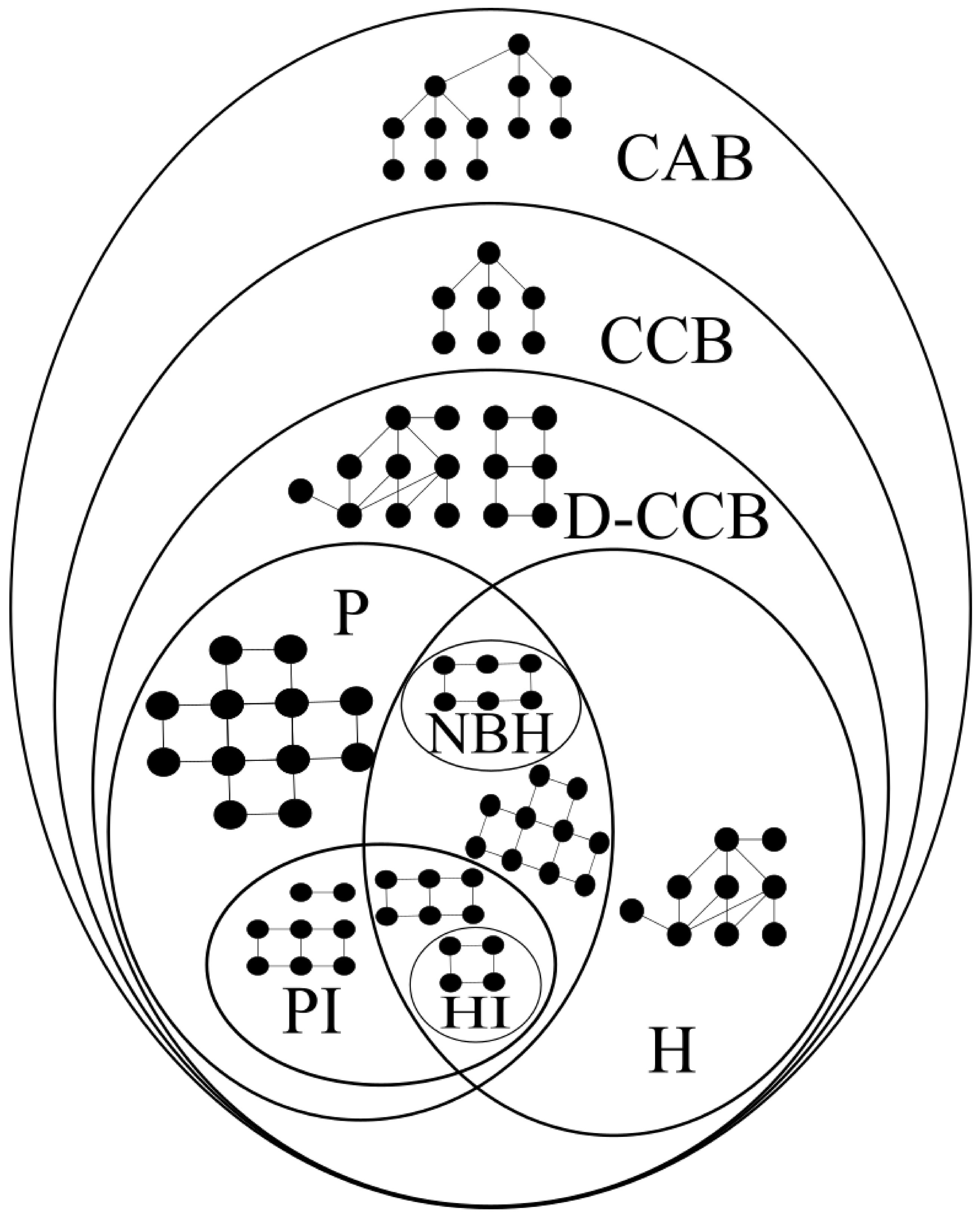

Figure 1: The Venn diagram of the classes studied, with an example graph in each region.

- **CCB ⊂ CAB**

  Proof idea. In a bi-circular-arc model b, attribute to the vertices of V pairwise disjoint arcs on the circle ordered according to the circular order of the vertices. For every $w \in W$, there is an arc that intersects every arc in $b(N(w))$ and no arcs in $b(V) - b(N(w))$. □

- **P ⊂ D-CCB**

  Proof idea. Let b be a bi-circular-arc model of G such that $b(V)$ and $b(W)$ are proper families. The clockwise order of the beginning points of the arcs in $b(V)$ (resp. $b(W)$) is a CCB order. □

- **H ⊂ D-CCB**

  Proof idea. Let b be a Helly bi-circular-arc model of G. For every $v \in V$ (every $w \in W$) there is a biclique K_v (K_w) that contains $N[v]$ ($N[w]$). For each of those, there is a point p_v (p_w) on the circle that every arc in $b(K_v)$ ($b(K_w)$) contains. The clockwise order of the points in $\{p_v | v \in V\}$ ($\{p_w | w \in W\}$) is a CCB order. □

- **NBH ⊂ P**

  Proof idea. In [3], we show that every twin-free **NBH** graph is an induced subgraph of a restrictive set of graphs. It is possible to show that every graph of that set is **P**. □

- **HI ⊂ PI**

  Proof idea. In [3], we show that every twin-free **HI** is an induced subgraph of a restrictive set of graphs. It is possible to show that every graph of that set is **PI**. □

- **H and P are uncomparable.**

  Proof idea. The Venn diagram has examples of a **P** graph that is not **H**, and vice-versa. □

Conclusion

We provided a comprehensive study of the relationship between seven different subclasses of circular arc bigraphs. We showed that doubly CCB graphs, a proper subclass of circular convex bipartite graphs, are a proper superclass of both Helly and proper circular arc bigraphs. We also showed that non-bichordal Helly circular arc bigraphs are a proper subclass of proper circular arc bigraphs, and that Helly interval bigraphs are a proper subclass of proper interval bigraphs. We also showed that proper and Helly circular arc bigraphs, which contain a non-empty intersection, are not comparable.

The results provide a full understanding of the containment hierarchies of the classes mentioned, allowing us to present a comprehensive Venn diagram of them.

Future Research

Future research includes looking into relationships between other subclasses of circular arc bigraphs, such as unit circular arc bigraphs (graphs that admit a bi-circular-arc model such that all arcs are of the same length), cross-proper circular arc bigraphs (graphs that admit a bi-circular-arc model where no two arcs corresponding to vertices of opposing partite sets are comparable), and normal circular arc bigraphs (graphs that admit a bi-circular-arc model where no union of two arcs equals the entire circle).

It also includes looking into the recognition problems of circular arc bigraphs, and any important subclasses of circular arc bigraphs for which no efficient recognition algorithm is known.

References

[1] Asim Basu, Sandip Das, Shamik Ghosh, and Malay Sen. Circular-arc bigraphs and its subclasses. *Journal of Graph Theory*, 73(4):361–376, 2013.

[2] Ashok Kumar Das and Ritapa Chakraborty. New characterizations of proper interval bigraphs and proper circular arc bigraphs. In Sumit Ganguly and Ramesh Krishnamurti, editors, *Algorithms and Discrete Applied Mathematics*, pages 117–125, Cham, 2015. Springer International Publishing.

[3] M. Groshaus, A. L. P. Guedes, and F. S. Kolberg. On the helly subclasses of interval bigraphs and circular arc bigraphs. In *Proceedings of the 14th Latin American Theoretical Informatics Symposium*, São Paulo - SP, Brazil, 2020.

[4] Martín D. Safe. Circularly compatible ones, d-circularity, and proper circular-arc bigraphs. *ArXiv*, abs/1906.00321, 2019.