Hamiltonicity of Token Graphs of Some Fan Graphs

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Abstract

In this poster we present some recent results about the Hamiltonicity of the 2-token graph $F_2(G)$ and the 2-multiset graph $M_2(G)$ of some fan graphs G. In particular, we exhibit an infinite family of graphs for which $F_2(G)$ and $M_2(G)$ are Hamiltonian.

Introduction

As far as we know, token graphs have been defined, independently, at least four times since 1988. Since then, several combinatorial parameters of token graphs have been studied, such as connectivity, regularity, planarity, Hamiltonicity, Eulerianicity, and, chromatic, clique, independence and packing numbers; as well as their automorphism group and spectrum. Also, several connections between token graphs and other research areas have been discovered, such as Quantum Mechanics and Coding Theory. For example, token graphs model the following system in Quantum Mechanics: consider a cluster of n interacting qubits (two-level atoms) represented by a graph G (where the qubits are interacting via an (excitation)-exchange Hamiltonian), in which, at each moment, exactly k qubits are in the *excited* state and the remaining in the *ground* state; this system corresponds to the k-token graph of G. In Coding Theory, the packing number of the k-token graph of P_n corresponds to the largest code of length n and constant weight k that can correct a single adjacent transposition; also the k-token graph of the Complete graph K_n is isomorphic to the Johnson graph $J(n,k)$, which have several applications in Coding Theory. Besides, token graphs have been used to study the Isomorphism Problem of Graphs.

Motivation

Besides the possible applications of token graphs, one of our motivations to study the Hamiltonicity of token graphs was to extend our result of 2018 [5]:

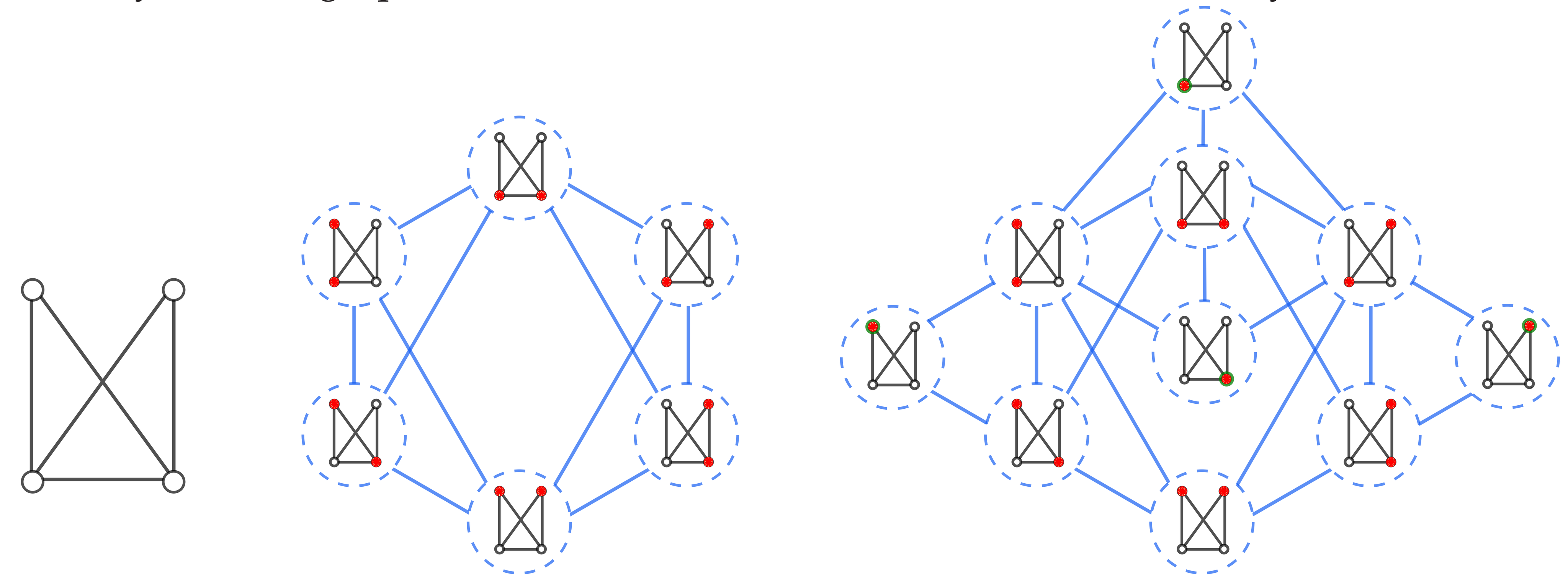
Theorem 1 *If $n \geq 3$, and $1 \leq k \leq n-1$, then the k-token graph of the fan graph $F_{1,n-1}$ is Hamiltonian.*

Definitions

For two disjoint graphs G_1 and G_2, the *join graph* $G = G_1 + G_2$ of graphs G_1 and G_2 is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and its edge set is $E(G_1) \cup E(G_2) \cup \{uv : u \in G_1 \text{ and } v \in G_2\}$, a simple example is the *fan graph* $F_{m,n} = E_m + P_n$, where E_m denotes the graph of m isolated vertices and P_n denotes the path graph of n vertices.

Let G be a simple graph of order n. The *k-token graph* $F_k(G)$ of G is the graph whose vertices are the k-subsets of $V(G)$, where two of such vertices are adjacent if their symmetric difference is a pair of adjacent vertices in G. The *k-multiset graph* $M_k(G)$ of G is the graph whose vertices are the k-multisubsets of $V(G)$, and two of such vertices are adjacent if their symmetric difference (as multisets) is a pair of adjacent vertices in G. See an example of these constructions in the figure below. The 2-token graph is usually called the *double vertex graph* and the 2-multiset graph is called the *complete double vertex graph*.

A *Hamiltonian path* (resp. a *Hamiltonian cycle*) of a graph G is a path (resp. cycle) containing each vertex of G exactly once. A graph G is *Hamiltonian* if it contains a Hamiltonian cycle.



Fan graph $F_{2,2}$ 2-token graph $F_2(F_{2,2})$ 2-multiset graph $M_2(F_{2,2})$

Previous results

It is well known that the Hamiltonicity of G does not imply the Hamiltonicity of $F_k(G)$. For example it is know that if $n = 4$ or $n \geq 6$, then $F_2(C_n)$ is not Hamiltonian. On the other hand, there exist non-Hamiltonian graphs for which its double vertex graph is Hamiltonian, for example $F_2(K_{1,3})$ is Hamiltonian. Next, we list the known results about the Hamiltonicity of $F_k(G)$ or the existence of a Hamiltonian path in $F_k(G)$, when k may be greater than two.

- If $n \geq 3$ and $1 \leq k \leq n-1$, then $F_k(K_n)$ is Hamiltonian, see for example [3].
- If $m \geq 2$, then $F_k(K_{m,m})$ has a Hamiltonian path if and only if k is odd [4].
- If G is a graph containing a Hamiltonian path and n is even and k is odd, then $F_k(G)$ has a Hamiltonian path [4].
- If $n \geq 3$ and $1 \leq k \leq n-1$, then $F_k(F_{1,n-1})$ is Hamiltonian [5].

In addition to these results, the following are some known results for the double vertex graph ($k = 2$).

- $F_2(C_n)$ is non-Hamiltonian [2].
- If G is a cycle with an odd chord, then $F_2(G)$ is Hamiltonian [2].
- $F_2(K_{m,n})$ is Hamiltonian if and only if $(m-n)^2 = m+n$ [2].

More results about the Hamiltonicity of double vertex graphs can be found in the survey of Alavi et. al. [1].

Results

These results were obtained by Luis Adame and the authors of this poster.

Theorem 2 *Let $m \geq 1$ and $n \geq 2$. Then, $F_2(F_{m,n})$ is Hamiltonian if and only if $m \leq 2n$, and $M_2(F_{m,n})$ is Hamiltonian if and only if $m \leq 2(n-1)$.*

This theorem implies the following result.

Corollary 3 *Let G_1 and G_2 be two graphs of order $m \geq 1$ and $n \geq 2$, respectively, such that G_2 has a Hamiltonian path. Let $G = G_1 + G_2$. If $m \leq 2n$ then $F_2(G)$ is Hamiltonian, and if $m \leq 2(n-1)$ then $M_2(G)$ is Hamiltonian.*

Open Questions

1. To study the Hamiltonicity of $F_k(F_{m,n})$ and $M_k(F_{m,n})$, for $k > 2$.
2. Given two graphs G and H, to study the Hamiltonicity of $F_k(G \square H)$ and $M_k(G \square H)$.
3. To find other families of non-Hamiltonian graphs for which their k-token graph and k-multiset graph are Hamiltonian.
4. What is the smallest Hamiltonian graph G for which $F_k(G)$ and $M_k(G)$ are Hamiltonian?

References

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