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On total coloring of circulant graphs

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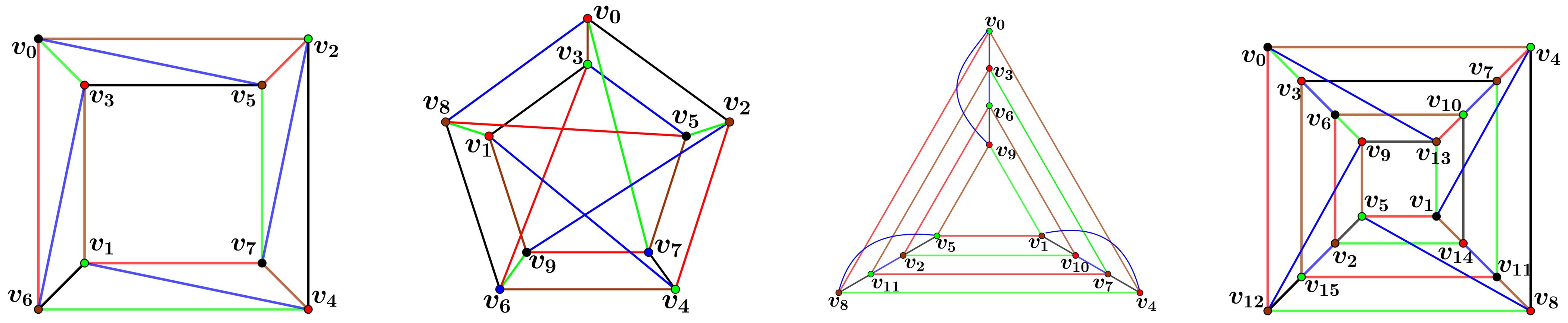
Introduction

A k-total coloring of a graph G is an assignment of k colors to the elements of G such that adjacent elements have different colors. The total chromatic number $\chi''(G)$ is the smallest integer k for which G has a k-total coloring. Clearly, $\chi''(G) \geq \Delta+1$, and the Total Coloring Conjecture (TCC) states that for any simple graph G, $\chi''(G) \leq \Delta+2$, where Δ is the maximum degree of G [2, 8]. Graphs with $\chi''(G) = \Delta(G)+1$ are called Type 1, and graphs with $\chi''(G) = \Delta(G)+2$ are called Type 2. A circulant graph $C_n(d_1, d_2, \cdots, d_l)$ with $1 \leq d_1 < \cdots < d_l \leq \lfloor \frac{n}{2} \rfloor$ has vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ and edge set $E = \bigcup_{i=1}^{l} E_i$ where $E_i = \{e_0^i, e_1^i, \cdots, e_{n-1}^i\}$ and $e_j^i = (v_j, v_{j+d_i})$ where the indexes of the vertices are considered modulo n. An edge of E_i is called edge of length d_i. In this work, we determine the Type of an infinite family of 4−regular circulant graphs, that is, $C_n(a, b)$. When a divide n (or b divide n), we will have a Prism graph $G(\frac{n}{a}, 1)$ as subgraph of $C_n(a, b)$. A Prism graph $G(n, 1)$ is defined by $V(G(n, 1)) = \{u_i, v_i \mid 0 \leq i < n\}$ and $E(G(n, 1)) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i \mid 0 \leq i < n\}$. See some examples of $C_n(a, b)$ with $G(\frac{n}{a}, 1)$ as a subgraph in Figure 1.



(a) $\chi''(C_8(2, 3)) = 5$ (b) $\chi''(C_{10}(2, 3)) = 5$ (c) $\chi''(C_{12}(3, 4)) = 5$ (d) $\chi''(C_{16}(3, 4)) = 5$

Figura 1:Examples of $C_n(a, b)$ with $G(\frac{n}{a}, 1)$ as a subgraph.

General results

In the table below, we present some results already known about the total coloring of circulant graphs.

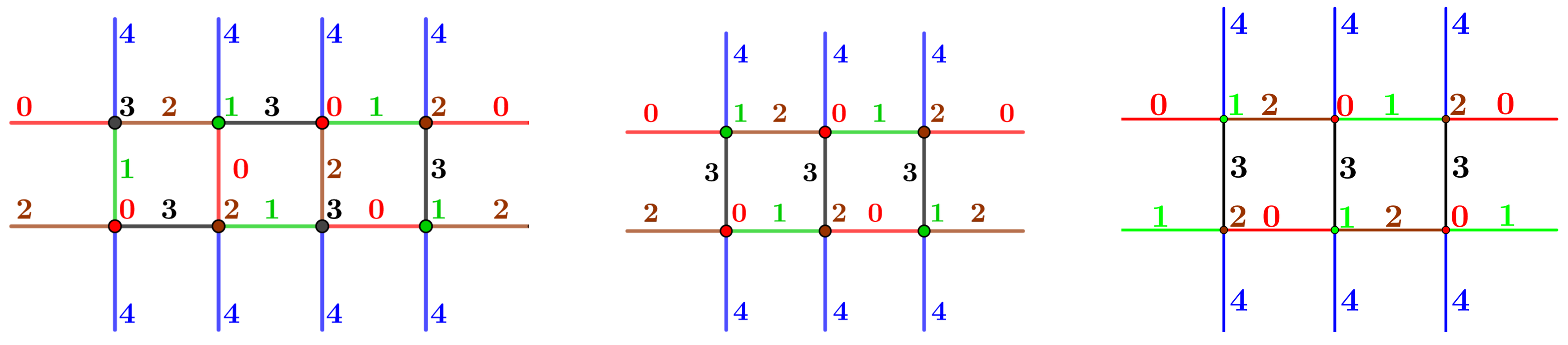
Circulant graph	Type 1	Type 2
$C_n(1)$ [9]	$n \equiv 0 \mod 3$	otherwise
$C_n(1, 2, .., \lfloor \frac{n}{2} \rfloor)$ [9]	n is odd	otherwise
$C_{2n}(d, n)$ [5]	$l = \mathrm{gdc}(d, n)$ with $d = lm$, m is even and $C_{2n}(d, n) \not\simeq l$ copies of $C_{10}(2, 5)$	otherwise
$C_n(1, 2)$ [3]	$n \neq 7$	otherwise
$C_{5p}(1, k)$ [6]	$k \equiv 2 \mod 5$ or $k \equiv 3 \mod 5$	
$C_{6p}(1, k)$ [6]	$k \equiv 1 \mod 3$ or $k \equiv 2 \mod 3$	
$C_n(1, 3)$ [9]		$tn = 8$

Tabela 1:State of the art

Our results

It is known that the Prism graphs $G(n, 1)$ are Type 1, except $G(5, 1)$ [7, 4]. The 4−total coloring for this family will be useful in the proof of the following theorem about 4−regular circulant graphs in which $G(n, 1)$ is a subgraph.

Theorem 1. *Let $C_n(2k, 3)$ be a 4−regular circulant graph. The graph $C_n(2k, 3)$ is Type 1 for $n = (8\mu + 6\lambda)k$, with $k \geq 1$ and non-negative integers μ and λ.*

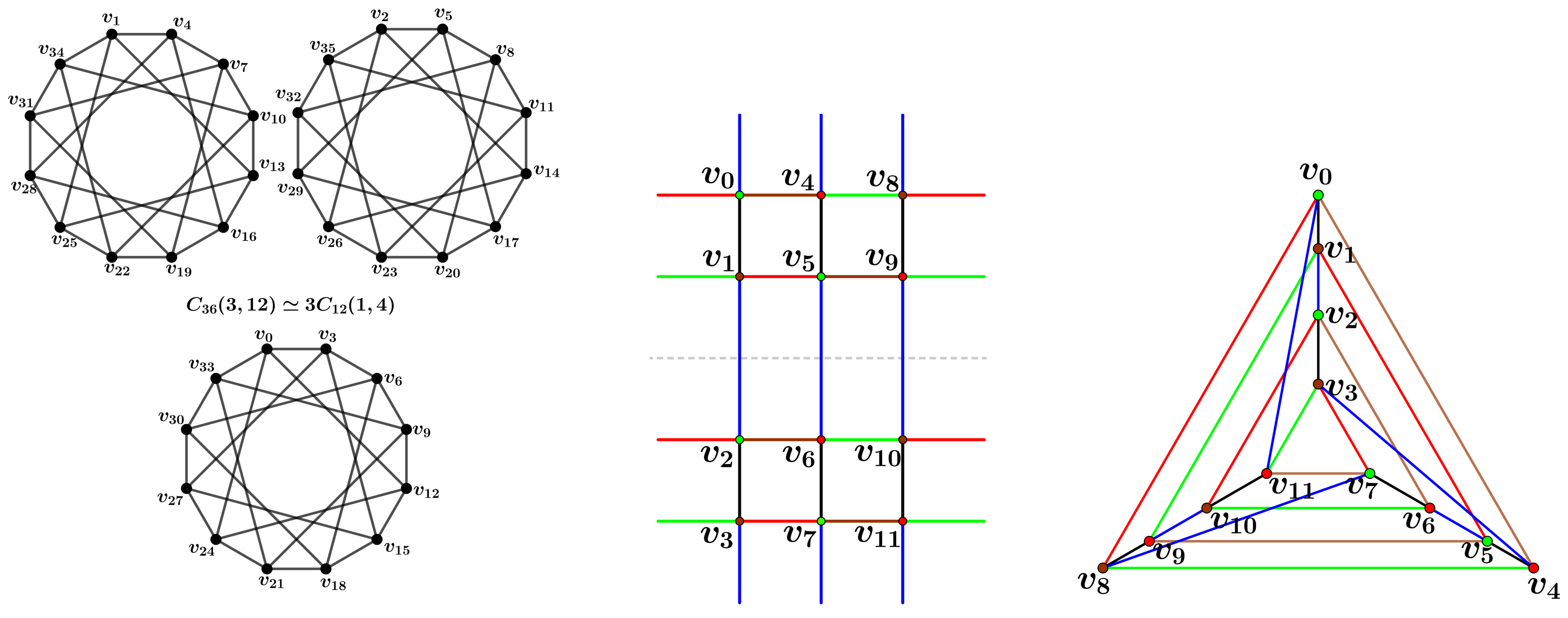


(a) $B(8, 2)$ colored (b) $B(6, 2)$ colored (c) $B(6, 2)$ with other coloring

Figura 2:Semigraph $B(n, a)$

A semigraph is a triple $B = (V, E, S)$, where V is the set of vertices of B,E is a set of edges having two distinct endpoints in V, and S is a set of semiedges having one endpoint in V. In this work we consider 4−regular semigraphs. Notice that a k−total coloring of a semigraph B is an assignment of k colors to the edges, semiedges and vertices of B such that adjacent elements have different colors.

Sketch of the proof. The result was proved in [1] when $C_n(2k, 3)$ is connected, using the Figure 2(a) . Hence, suppose that $C_{(8\mu+6\lambda)k}(2k, 3)$ is disconnected, that is $k = 3\alpha$. In this case, note that $C_{(8\mu+6\lambda)3\alpha}(3, 6\alpha)$ is isomorphic to three copies of $C_{(8\mu+6\lambda)\alpha}(1, 2\alpha)$. To construct the colorings of these graphs, we consider two cases: $\mu = 0$ and $\mu \neq 0$. When $\mu = 0$, we construct the desired coloring by making the junction of λ copies of the semigraph $B(6, 2)$ (Figure 2(c)) vertically and horizontally, recursively. When $\mu \neq 0$, we make the junction of μ copies of the semigraph $B(8, 2)$ with λ copies of $B(6, 2)$ (Figure 2 (b)) vertically and horizontally, recursively (the same for the case when $C_n(2k, 3)$ is connected). However the process of joining its semiedges to construct the desired graph is different. See an example in Figure 3.



(a) $C_{36}(3, 12)$ (b) $B(12, 4)$ (c) $\chi''(C_{12}(1, 4)) = 5$

Figura 3:The graph $C_{36}(3, 12)$ with a total coloring with 5 colors.

Conclusion

The total chromatic number of several circulant graphs has been determined, including the total chromatic number of the cubic circulant graphs $C_{2n}(d, n)$. As a future work, we would like to determine the total chromatic number of all 4−regular circulant graphs $C_n(a, b)$.

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