EQUITABLE TOTAL COLORING OF BLOWUP SNARKS

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INTRODUCTION

Let G be a *simple graph*. A *k-total-coloring* of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by $\chi''(G)$ is is the least k for which G has a k-total-coloring. Evidently, $\chi''(G) \geq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. The Total Coloring Conjecture [1] afirms that $\chi''(G) \leq \Delta(G)+2$. This conjecture has been proved for cubic graphs [2], so the total chromatic number of a cubic graph is 4 or 5. Graphs with $\chi''(G) \geq \Delta(G)+1$ are said to be *Type 1* and graphs with $\chi''(G) \leq \Delta(G)+2$ are said to be *Type 2*. Deciding whether a graph is Type 1 has been shown NP-complete [3].

A k-total-coloring is *equitable* if the cardinalities of any two color classes differ by at most one. The least k for which G has an equitable k-total-coloring is the *equitable total chromatic number* of G and its denoted by $\chi''_e(G)$.

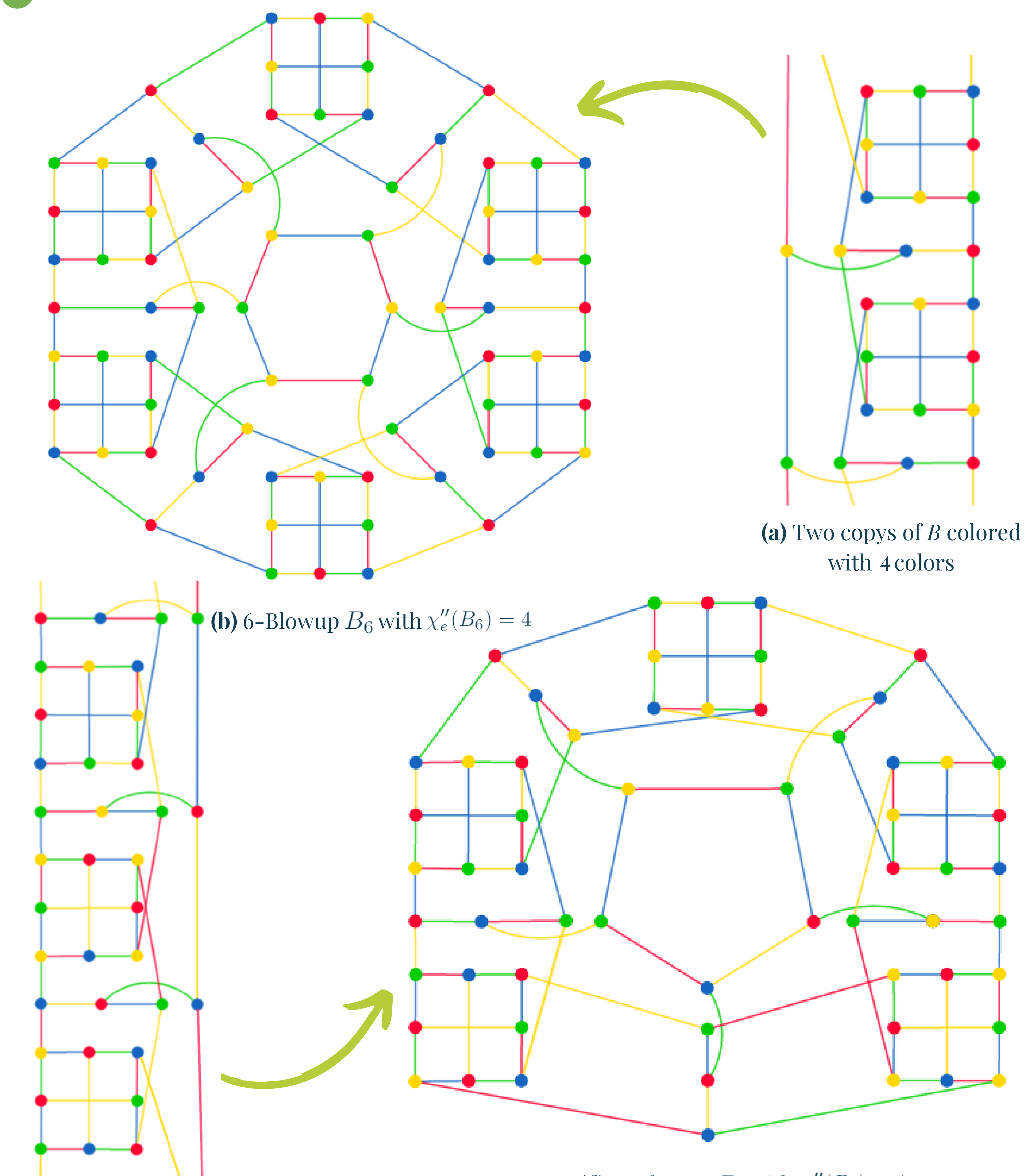
Figure 1: Lewis Carrol book cover

The search for connected, bridgeless, 3-regular graphs with chromatic index equals 4, was motivated by the Four Color Problem. Due the difficult to find them, they were named *Snarks* after Lewis Carrol poem "*The hunting of the Snark*", by M. Gardner [4]. Snarks were fictional animal species described by Carrol as unimaginable creatures.

The *girth* of G is the length of the shortest cycle contained in G. One condition often imposed on snarks is that they must have girth at least 5, to avoid graphs that can be reduced to a smaller graph by replacing a subgraph for structures that do not affect the edge colorability.

In this study we investigate the k-total-coloring of Blowup infinite family of snarks, recently defined by Hägglund [5], with girth at least 5.

RESULTS



(a) Two copys of B colored with 4 colors

(b) 6-Blowup B_6 with $\chi''_e(B_6) = 4$

(c) Three copys of B colored with 4 colors

(d) 5-Blowup B_5 with $\chi''_e(B_5) = 4$

Figure 3: Construction of 5-Blowup and 6-Blowup with $\chi''_e(B) = 4$

Let B be the cubic semigraph with 6 semi-edges, illustrated in Figure 2. Blowup graphs are constructed by connecting copies of B as in the examples of Figures 3.

An *n-Blowup* is a graph build with n copies of B.

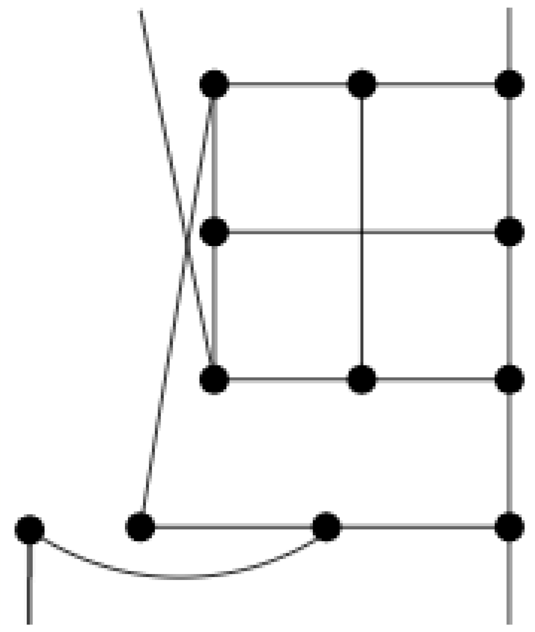


Figure 2: cubic semigraph B

Theorem: *All n-Blowups with n≥5 have equitable total chromatic number equals 4.*

The sketch of the proof is by construction and two different equitable 4-total-colorings were necessary to obtain the result.

We represent 1 for blue, 2 for green, 3 for red and 4 for yellow.

First coloring is showed in Figure 3(a). It's composed by two copies of B colored with 4 colors. More specifically, in this figure, $\phi(1) = \phi(2) = \phi(3) = \phi(4) = 15$ (semiedges counts 0.5).

When $n \equiv 0 \pmod 2$ we repeat this coloring $\frac{n}{2}$ times. Evidently, $\phi(1) = \phi(2) = \phi(3) = \phi(4) = \frac{n}{2} \cdot 15$. Figure 3(b) shows 6-Blowup colored following this rule.

The second coloring is showed in Figure 3(c) and its composed by 3 copies of B colored with 4 colors.

In this coloring, $\phi(1) = \phi(4) = 22$ and $\phi(2) = \phi(3) = 23$.

When $n \equiv 1 \pmod 2$ we use this coloring once and for the remaining n-3 copies of B we repeat $\frac{n-3}{2}$ times the coloring showed in 3(a).

Thus, $\phi(1) = \phi(4) = 22 + \frac{n-3}{2} \cdot 15$ and $\phi(2) = \phi(3) = 23 + \frac{n-3}{2} \cdot 15$. Evidently, $\phi(1)$, $\phi(2)$, $\phi(3)$ and $\phi(4)$ differ at most one. Figure 3(d) shows 5-Blowup colored following this rule.

REFERENCES

[1] VIZING, V. G. On an estimate of the chromatic class of a p-graph. Diskret. Analiz., 3, p. 25-30, 1964.

[2] ROSENFELD, M. On the total coloring of certain graphs. Israel J. Math., 9, p. 396-402, 1971.

[3] MCDIARMID, C. J. H., SÁNCHEZ-ARROYO, A. Total coloring regular bipartide graphs is NP-hard. Discrete Math., 124, p. 155-162, 1994.

[4] GARDNER, M. Mathematical games: Snarks, Boojums and other conjectures related to the four-color-map theorem, Scientific American., 234, p. 126–130, 1976.

[5] HÄGGLUND, J. On Snarks that are far from beeing 3-edge-colorable., The Electronic Journal of Combinatorics., 23, p.1-10, 2016.

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