The Strict Terminal Connection Problem on Chordal Bipartite Graphs

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Introduction

Let G be a graph and $W \subseteq V(G)$ be a non-empty set, called *terminal* set. A *strict connection tree* of G for W is a tree subgraph of G whose leaf set is equal to W. A non-terminal vertex of a strict connection tree T is called *linker* if its degree in T is exactly 2, and it is called *router* if its degree in T is at least 3. We remark that the vertex set of every connection tree can be partitioned into terminal vertices, linkers and routers. For each connection tree T, we let $\mathsf{L}(T)$ denote the linker set of T and $\mathsf{R}(T)$ denote the router set of T. Figure 1 illustrates a graph G, a terminal set W and a strict connection tree of G for W.

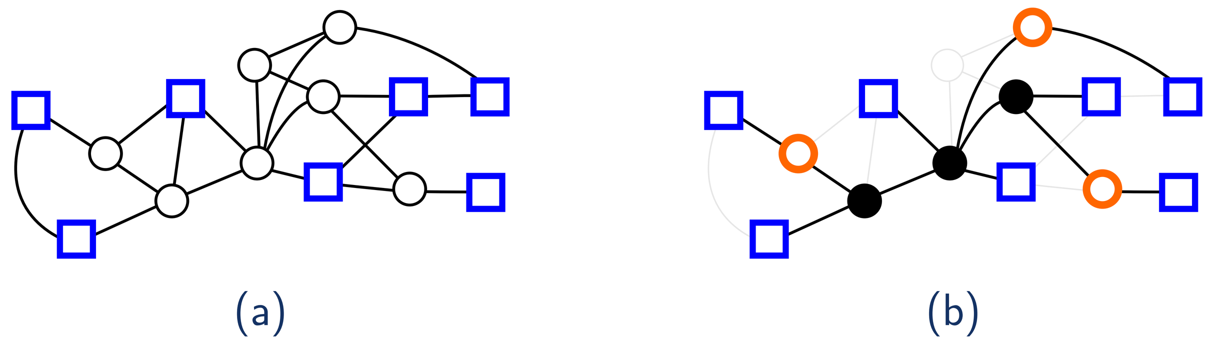

Figure 1: (a) Graph G and terminal set W (blue squared vertices). (b) Strict terminal connection tree T of G for W, such that $|\mathsf{L}(T)| = 3$ and $|\mathsf{R}(T)| = 3$.

Motivated by applications in information security, network routing and telecommunication, Dourado et al. [1] introduced the Strict Terminal connection problem, which is formally defined below.

Strict Terminal Connection (S-TCP)

Input: A graph G, a non-empty terminal set $W \subseteq V(G)$ and two non-negative integers ℓ and r.

Question: Does there exist a strict connection tree T of G for W, such that $|\mathsf{L}(T)| \leq \ell$ and $|\mathsf{R}(T)| \leq r$?

Table 1 summarises the complexity of S-TCP with respect to the parameters $\ell, r, \Delta(G)$, and the classes of split graphs and cographs. In addition to these results, it is known that S-TCP is NP-complete even if $\Delta(G) = 4$ and $\ell \geq 0$ is fixed, or $\Delta(G) = 3$ and ℓ is arbitrarily large [3]; on the other hand, if $\Delta(G) = 3$, the problem can be solved in time $n^{\mathcal{O}(\ell)}$ [3].

	Parameters				
Graph class	–	ℓ	r	ℓ, r	$\ell, r, \Delta(G)$
General	NPC [1]	NPC [1]	P for $r \in \{0, 1\}$ [2] but W[2]h [3]	XP [1] but W[2]h [3]	FPT [1, 3] but No-poly kernel [3]
Split	NPC [3]	NPC [3]	XP [3] but W[2]h [3]	XP [1, 3] but W[2]h [3]	FPT [1, 3]
Cographs	P [3]	P [3]	P [3]	P [3]	P [3]

Table 1: Computational complexity of S-TCP. (Adapted from [3].)

Contribution

In this work, we prove that S-TCP remains NP-complete when restricted to **chordal bipartite graphs**, even if $\ell \geq 0$ is fixed.

S-TCP on Chordal Bipartite Graphs

A graph G is called *chordal bipartite* if every induced cycle of G has length 4. Equivalently, a graph G is chordal bipartite if G is bipartite and every cycle of G of length at least 6 has a *chord*, i.e. an edge between two non-consecutive vertices of the cycle.

To prove that S-TCP is NP-complete on chordal bipartite graphs, we present a polynomial-time reduction from Vertex-cover, which is formally defined below. The proposed reduction is based on the polynomial-time reduction given by Müller and Brandstädt [4] so as to prove that Steiner tree is also NP-complete on chordal bipartite graphs.

Vertex cover

Input: A graph G and a positive integer k.

Question: Does there exist a subset $S \subseteq V(G)$ such that $|S| \leq k$ and every edge of G has an endpoint in S?

Construction. Let $I = (G, k)$ be an instance of Vertex cover and $c \geq 0$ be a constant. Assume that $V(G) = \{v_1, \ldots, v_n\}$ for some positive integer $n \geq 2$. Moreover, assume that G has at least one edge, i.e. $m = |E(G)| \geq 1$. We let $f(I, c) = (H, W, \ell = c, r)$ be the instance of S-TCP defined as follows.

- For each $v_i \in V(G)$, create the gadget H_i as illustrated in Figure 2.

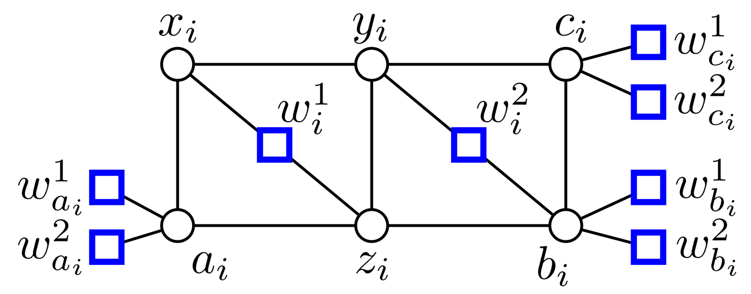

Figure 2: Gadget H_i.

- Subdivide the edge $w_{a_1}^1 a_1$ of H_1 into ℓ new vertices $u_1, u_2, \ldots, u_\ell$, creating the induced path $\langle w_{a_1}^1, u_1, \ldots, u_\ell, a_1 \rangle$.
- For each pair $v_i, v_j \in V(G)$, with $i \neq j$, add the edges $x_i y_j$ and $z_i y_j$, making the subgraph of H induced by $X \cup Y \cup Z$ a complete bipartite graph with bipartition $(X \cup Z, Y)$, where $X = \{x_i \mid v_i \in V(G)\}$, $Y = \{y_i \mid v_i \in V(G)\}$ and $Z = \{z_i \mid v_i \in V(G)\}$.
- For each $v_i v_j \in E(G)$, create the gadgets H_{ij} and H_{ji} as illustrated in Figure 3.

Figure 3: Gadgets H_{ij} and H_{ji}, respectively.

- Finally, define $W = W_1 \cup W_2 \cup W_3$ and $r = k + 4n + 4m$, where
  $W_1 = \{w_i^1, w_i^2 \mid v_i \in V(G)\}$,
  $W_2 = \{w_{a_i}^1, w_{a_i}^2, w_{b_i}^1, w_{b_i}^2, w_{c_i}^1, w_{c_i}^2 \mid v_i \in V(G)\}$, and
  $W_3 = \{w_{p_{ij}}^1, w_{p_{ij}}^2, w_{q_{ij}}^1, w_{q_{ij}}^2 \mid v_i v_j \in E(G)\}$.

Theorem. Let $I = (G, k)$ be an instance of Vertex-cover, such that G has at least one edge, and let $c \geq 0$ be a constant. The graph H of $f(I, c)$ is chordal bipartite. Moreover, I is a yes-instance of Vertex-cover if and only if $f(I, c)$ is a yes-instance of S-TCP.

Concluding remarks

We conclude this work by posing some open questions.

- Is S-TCP parameterized by $r \geq 2$ in XP?
- Is S-TCP parameterized by $r \geq 2$ in FPT when restricted to chordal bipartite graphs? If not, is it in XP?
- Is S-TCP parameterized by ℓ in FPT when restricted to graphs of maximum degree 3?
- In addition to cographs, on which graph classes is S-TCP in P?

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