The Strict Terminal Connection Problem on Chordal Bipartite Graphs

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Introduction

Let \( G \) be a graph and \( W \subseteq V(G) \) be a non-empty set, called terminal set. A strict connection tree of \( G \) for \( W \) is a tree subgraph of \( G \) whose leaf set is equal to \( W \). A non-terminal vertex of a strict connection tree \( T \) is called linker if its degree in \( T \) is exactly 2, and it is called router if its degree in \( T \) is at least 3. We remark that the vertex set of every connection tree can be partitioned into terminal vertices, linkers, and routers. For each connection tree \( T \), we let \( L(T) \) denote the linker set of \( T \) and \( R(T) \) denote the router set of \( T \). Figure 1 illustrates a graph \( G \), a terminal set \( W \), and a strict connection tree of \( G \) for \( W \).

Motivated by applications in information security, network routing and telecommunication, Dourado et al. [1] introduced the strict terminal connection problem, which is formally defined below.

**Strict Terminal Connection (S-TCP)**

**Input:** A graph \( G \), a non-empty terminal set \( W \subseteq V(G) \) and two non-negative integers \( \ell \) and \( r \).

**Question:** Does there exist a strict connection tree \( T \) of \( G \) for \( W \), such that \( |L(T)| \leq \ell \) and \( |R(T)| \leq r \)?

Table 1 summarises the complexity of S-TCP with respect to the parameters \( \ell, r, \Delta(G) \), and the classes of split graphs and cographs. In addition to these results, it is known that S-TCP is \( \text{NP} \)-complete even if \( \Delta(G) = 4 \) and \( \ell \geq 0 \) is fixed, or \( \Delta(G) = 3 \) and \( \ell \) is arbitrarily large [3]; on the other hand, if \( \Delta(G) = 3 \), the problem can be solved in time \( \text{pO}(|G|) \) [3].

**S-TCP on Chordal Bipartite Graphs**

A graph \( G \) is called chordal bipartite if every induced cycle of \( G \) has length 4. Equivalently, a graph \( G \) is chordal bipartite if \( G \) is bipartite and every cycle of \( G \) of length at least 6 has a chord, i.e. an edge between two non-consecutive vertices of the cycle.

To prove that S-TCP is \( \text{NP} \)-complete on chordal bipartite graphs, we present a polynomial-time reduction from VERTEX-COVER, which is formally defined below. The proposed reduction is based on the polynomial-time algorithm by Muller and Brandstadt [4] as to prove that Steiner tree is also \( \text{NP} \)-complete on chordal bipartite graphs.

**CONCLUDING REMARKS**

We conclude this work by posing some open questions.

- Is S-TCP parameterized by \( r \geq 2 \) in \( \text{XP} \)?
- Is S-TCP parameterized by \( r \geq 2 \) in FPT when restricted to chordal bipartite graphs? If not, is it in \( \text{XP} \)?
- Is S-TCP parameterized by \( \ell \) in FPT when restricted to graphs of maximum degree 3?
- In addition to cographs, on which graph classes is S-TCP in \( \text{P} \)?

**REFERENCES**


**Figure 1:** (a) Graph \( G \) and terminal set \( W \) (blue squared vertices). (b) Strict terminal connection tree \( T \) of \( G \) for \( W \), such that \( |L(T)| = 3 \) and \( |R(T)| = 3 \).

**Figure 2:** Gadget \( H_i \).

**Figure 3:** Gadgets \( H_1 \) and \( H_2 \), respectively.