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Total coloring of some unitary Cayley graphs

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Unitary Cayley graphs

For a positive integer n, the *unitary Cayley* graph $X_n = \text{Cay}(\mathbb{Z}_n, \mathbb{U}_n)$ is defined by the additive group of the ring $\mathbb{Z}_n$ of integers modulo n and the multiplicative group $\mathbb{U}_n$ of its units, where $\mathbb{U}_n = \{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}$. The vertex set of X_n is the set $V(X_n) = \mathbb{Z}_n = \{0, 1, \cdots, n-1\}$ and its edge set is $E(X_n) = \{ab : a, b \in \mathbb{Z}_n$ and $\gcd(a-b, n) = 1\}$. The unitary Cayley graphs X_n are regular of degree $|\mathbb{U}_n| = \phi(n)$, where $\phi(n)$ is the Euler function.

Total coloring

A k-total coloring of G is an assignment of k colors to the edges and vertices of G, such that no adjacent elements (vertices and edges) receive the same color. The total chromatic number of G, denoted by $\chi_T(G)$, is the least k for which G has a k-total coloring. Let $\Delta(G)$ be the maximum degree of G, clearly, $\chi_T(G) \geqslant \Delta(G) + 1$ and the Total Coloring Conjecture (TCC) [1, 6] states that $\chi_T(G) \leqslant \Delta(G) + 2$. This conjecture has been verified for some classes but the general statement has remained open for more than fifty years and has not been settled even for regular graphs. If $\chi_T(G) = \Delta(G) + 1$, then G is said to be Type 1, and if $\chi_T(G) = \Delta(G) + 2$, then G is said to be Type 2. The problem of deciding if a graph is Type 1 has been shown NP-complete [5].

For more information, we refer to [3], which is the first PhD thesis on total coloring developed in Brazil.

Total coloring of unitary Cayley graphs

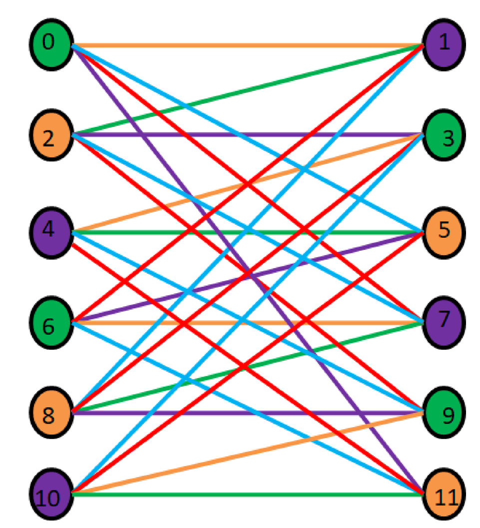
Prajnanaswaroopa et al. [4] established the TCC for all unitary Cayley graphs. Some unitary Cayley graphs are already known to be Type 1 or Type 2. If $n = p^r$ is a prime power, then X_{p^r} is a complete p-partite graph and the total chromatic number is well known: if p is odd, then X_{p^r} is Type 1, and if p is even, then X_{p^r} is Type 2 [3].

We determine the total chromatic number of all members of two families of unitary Cayley graphs X_n: when $n = 6s$, for a positive integer s, and when $n = 3p$, for prime $p \geq 5$.

Boggess et al. [2] proved that for $n \geq 3$, graph X_n can be decomposed into $\frac{\phi(n)}{2}$ edge-disjoint Hamiltonian cycles, denoted by H_n^j, with $j \in \mathbb{U}_n$; and this result is used to prove the following theorems. Consider directed edges $\{\langle i, i+j \mod n\rangle : 0 \leq i \leq n-1\}$ to indicate the direction used to construct the cycles H_n^j, as H_n^j and H_n^{n-j} are the same cycle.

Theorem 1. *For positive integer s, the graph X_{6s} is Type 1.*

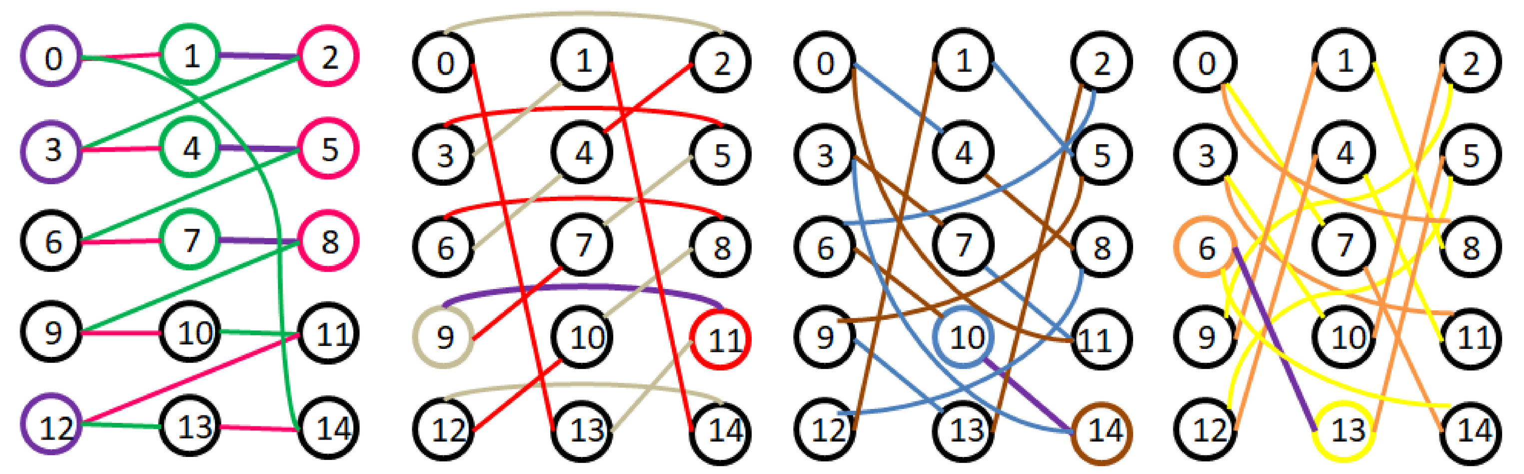
Proof. Graph X_{6s} is bipartite with parts $A = \{2i : 0 \leq i \leq \frac{6s-2}{2}\}$ and $B = \{2i+1 : 0 \leq i \leq \frac{6s-2}{2}\}$. Consider the Hamiltonian cycle H_{6s}^1, since it has $6s$ vertices, it is well known that admits a 3-total coloring T such that vertices i, with $i \equiv 0 \mod 3$ (resp. $i \equiv 1 \mod 3$ and $i \equiv 2 \mod 3$) receive the same color. As $3 \notin \mathbb{U}_{6s}$, the adjacent vertices in X_{6s} do not have the same color assigned by T. Now, remove from X_{6s} all the edges in H_{6s}^1. Clearly, the resulting bipartite graph is $(\Delta(X_{6s}) - 2)$-regular and, by Hall's theorem, it can be edge colored with $\Delta(X_{6s}) - 2$ colors. Therefore, X_{6s} is Type 1. The following figure presents a 5-total coloring of X_{12}. □



Theorem 2. *For prime $p \geq 5$, the graph X_{3p} is Type 1.*

Idea of the proof. Graph X_{3p} is a 3-partite graph with parts $A = \{3i : 0 \leq i \leq p-1\}$, $B = \{3i+1 : 0 \leq i \leq p-1\}$ and $C = \{3i+2 : 0 \leq i \leq p-1\}$. By Vizing's theorem, each Hamiltonian cycle H_{3p}^j admits a 3-edge coloring. For $j > 1$, assign 3 colors to the edges of every H_{3p}^j such that a special color c_0 is used in all cycles in a particular directed edge $\langle a, a+j \mod 3p\rangle$, and the endpoints $\{a, a+j \mod 3p\}$ receive 2 different colors already used in the respective cycle. For $j = 1 \in \mathbb{U}_{3p}$, assign 3 colors to the edges of H_{3p}^1 so that the special color c_0 is assigned to exactly 3 directed edges: $\langle 1, 2\rangle, \langle 4, 5\rangle, \langle 7, 8\rangle$; and the endpoints $\{1, 4, 7\} \in B$ and $\{2, 5, 8\} \in C$ receive the 2 colors already used in the respective cycle, one color to each part. The remaining vertices not colored in X_{3p} are in part A, and we assign color c_0 to these vertices.

Notice that the assignment of colors does not have conflict. We used 2 colors for the elements of each of the $p-1$ Hamiltonian cycles and used color c_0 in all cycles. Thus, we obtain a $2(p-1) + 1 = \Delta(X_{3p}) + 1$-total coloring. The figure below presents the four edge-disjoint Hamiltonian cycles $H_{3p}^1, H_{3p}^2, H_{3p}^4$ and H_{3p}^7 of X_{15} with a 9-total coloring such that the color c_0 is represented by purple color. □



References

[1] M. Behzad, G. Chartrand, and J. K. Cooper Jr. The colour numbers of complete graphs. *J. Lond. Math. Soc.*, 42:226–228, 1967.

[2] M. Boggess, T. Jackson-Henderson, I. Jimnez, and R. Karpman. The structure of unitary Cayley graphs. *SUMSRI Journal*, pages 1–23, 2008.

[3] C. N. Campos. O problema da coloração total em classes de grafos. *PhD thesis supervised by Professor Célia Picinin de Mello, Unicamp*, 2006.

[4] S. Prajnanaswaroopa, J. Geetha, K. Somasundaram, H. Lin-Fu, and Narayanan N. On total coloring of some classes of regular graphs. 2020. ArXiv:1910.03922v4.

[5] A. Sánchez-Arroyo. Determining the total colouring number is NP-hard. *Discrete Math.*, 78:315–319, 1989.

[6] V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Metody Diskret. Analiz.*, 3:25–30, 1964.

Paths On Hosts: B_1-EPG, EPT and VPT Graphs

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Introduction

This research contains as a main result the proof that every Chordal B1-EPG graph is simultaneously in the VPT and EPT graph classes. In addition, we describe a set of graphs that defines Helly-B_1-EPG families. In particular, this work presents some features of non-trivial families of graphs properly contained in Helly-B_1 EPG, namely Bipartite, Block, Cactus and Line of Bipartite graphs.

Objective

In this work we will mainly explore the EPG graphs, in particular B_1-EPG graphs. However, other classes of intersection graphs will be studied such as EPT and VPT graph classes.

Definitions and Technical Results

- A graph is a B_k-EPG graph if it admits an EPG representation in which each path has at most k bends;
- When k = 1 we say that this is a **single bend EPG representation** or simply a B_1-EPG representation;
- In a B_1-EPG representation, a clique K can be **edge-clique** or **claw-clique** [3].

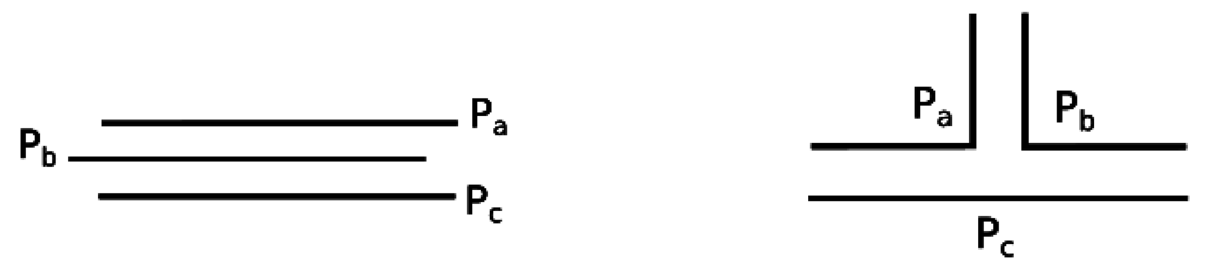

Figure 1: Representation of a clique as edge-clique and as claw-clique.

- A collection of sets satisfies the **Helly property** when every pairwise intersecting sub-collection has at least one common element;
- When this property is satisfied by the set of paths used in a representation, we get a Helly representation;
- Helly-B_1-EPG graphs were studied in [2];
- EPG, EPT and VPT representations arise in circuit layout problems and layout opt imization [4];
- VPT and EPT graphs are the vertex-intersection and edge-intersection graphs of paths on trees, respectively;
- VPT and EPT graphs are incomparable families of graphs.

Subclasses of Helly-B1-EPG Graphs

Theorem 1: Let G be a B_1-EPG graph. If G is $\{S_3, S_{3'}, S_{3''}, C_4\}$-free then G is a Helly-$B_1$-EPG graph.

(a) Claw with paths. **(b) Subgraph induced by paths.**

Figure 2: Reconstruction of intersection model.

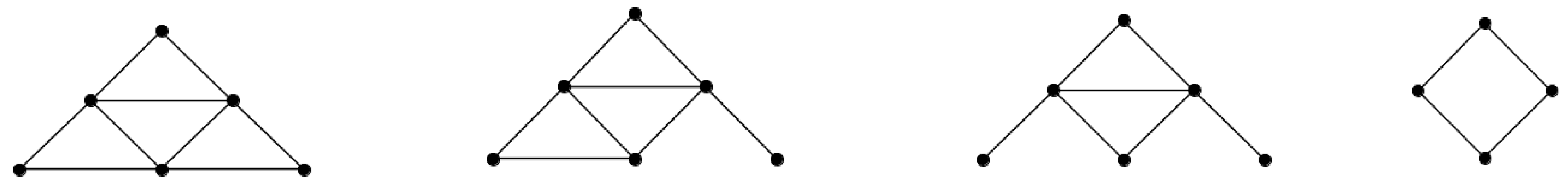

Figure 3: Graphs on statement of Theorem 1: S_3, $S_{3'}$, $S_{3''}$, C_4.

- Bull-free graphs are $\{S_3, S_{3'}, S_{3''}\}$-free, so these results implies in results of [1].

Theorem 2: If the graph G is B_1-EPG and diamond-free then G is Helly-B_1-EPG.
Corollary: Bipartite, Block, Cactus and Line of Bipartite graphs are Helly-B1-EPG.

Relationship among Chordal B1-EPG, VPT and EPT graphs

Theorem 3: Chordal B_1-EPG $\subsetneq$ VPT. *Theorem 4: Chordal B_1-EPG $\subsetneq$ EPT.*

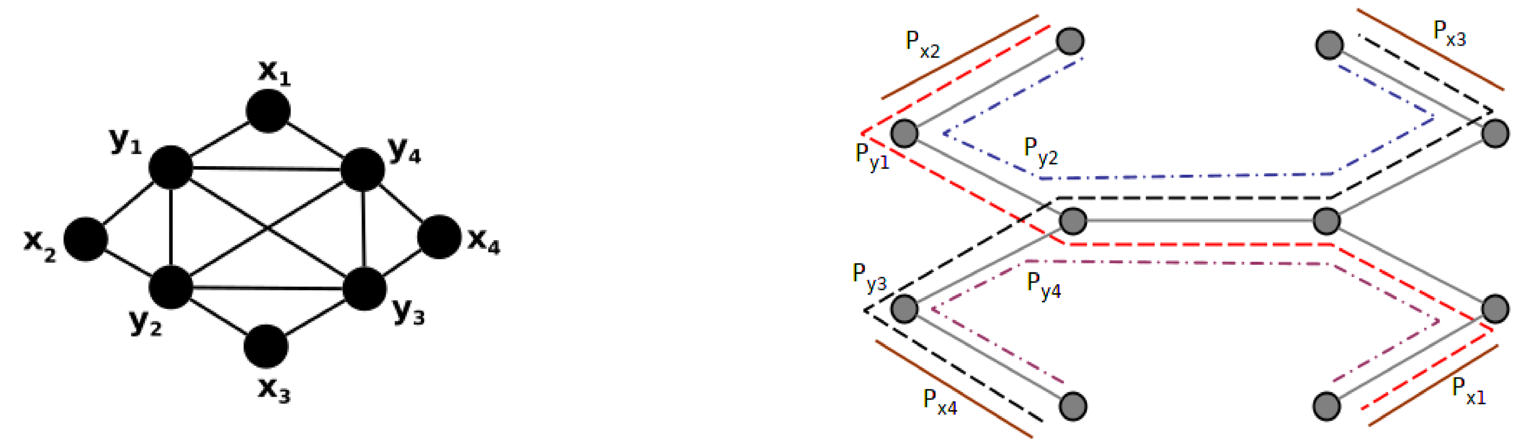

Figure 4: Graph S_4 and one of its possible VPT and EPT representations.

References

[1] A. Asinowski and B. Ries. Some properties of edge intersection graphs of single bend paths on a grid. Electronic Notes in Discrete Mathematics, 312 (2012), pp. 427-440.
[2] C.F. Bornstein, M.C. Golumbic, T.D. Santos, U.S. Souza, and J.L. Szwarcfiter. The Complexity of Helly-B_1-EPG Graph Recognition. Discrete Mathematics & Theoretical Computer Science, 22 (2020), https://dmtcs.episciences.org/6506/pdf.
[3] M.C. Golumbic, M. Lipshteyn and M. Stern. Edge intersection graphs ofsingle bend paths on a grid. Networks, 54 (2009), pp. 130-138.
[4] F.W. Sinden. "Topology of thin film RC circuits",Bell System Technical Journal, 45 (1966), n. 9, pp. 1639-1662.

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A Matheuristic Approach for the Weighted Just-in-Time Scheduling Problem

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INTRODUCTION

This work presents a hybrid exact-heuristic algorithmic approach, based on an arc-time indexed mixed-integer programming formulation and a generalized evolutionary based on a strong local search, in order to better solve the problem $P||\sum \alpha_j E\_j + \beta_j T_j$ (WET). The selected arcs from local optimal solutions generated by a Genetic Algorithm based on a strong Local Search (GLS), are given as input to an IP Arc-time indexed formulation, which is solved to produce better solutions at CPLEX. The proposed Hybrid Matheuristic method is capable to produce better results when compared with the previous best results in the literature.

OBJECTIVE

The objective of this work is to develop an exact-heuristic method to solve large instances of the the identical parallel machine Weighted Just-in-Time Scheduling Problem.

JUST-IN-TIME SCHEDULING PROBLEM

Considering the classical NP-hard parallel-machine weighted earliness-tardiness scheduling problem, $\boldsymbol{P||\sum \alpha_j E_j + \sum \beta_j T_j}$ **(WET)**, in 3-field notation [1], where $\boldsymbol{j = \{1, \ldots, n\}}$ is the set of independent jobs to be processed without preemption, in $\boldsymbol{m}$ **identical parallel machines**, where each one can process at least one job on a given time. Every job j has a positive processing time p_j, a due date d_j and a positive earliness (α_j) and tardiness (β_j) weights. The earliness of a job is defined as $E_j = \max\{0; d_j - C_j\}$ and the tardiness of a job is defined as $T_j = \max\{0; C_j - d_j\}$, where C_j is the completion time of the job [2]. Figure 1 (a) presents an example of 8 jobs for the problem followed by a solution representation for single machine scheduling in Figure 1 (b) and its corresponding representation for identical parallel machines in Figure 1 (c), considering three identical parallel machines.

J	p_j	d_j	α_j	β_j	C_j	E_j	T_j	$\alpha_j E_j$	$\beta_j T_j$
j_1	3	3	3	5	3	0	0	0	0
j_2	2	6	4	5	2	4	0	16	0
j_3	3	6	8	8	3	3	0	24	0
j_4	4	4	8	10	6	0	2	0	12
j_5	6	6	6	4	9	0	3	0	27
j_6	7	10	7	3	10	0	0	0	0
j_7	3	11	4	2	10	1	0	4	0
j_8	3	8	5	8	12	0	4	0	48

$\sum \alpha_j E_j = 44$ and $\sum \beta_j T_j = 87$

(a)

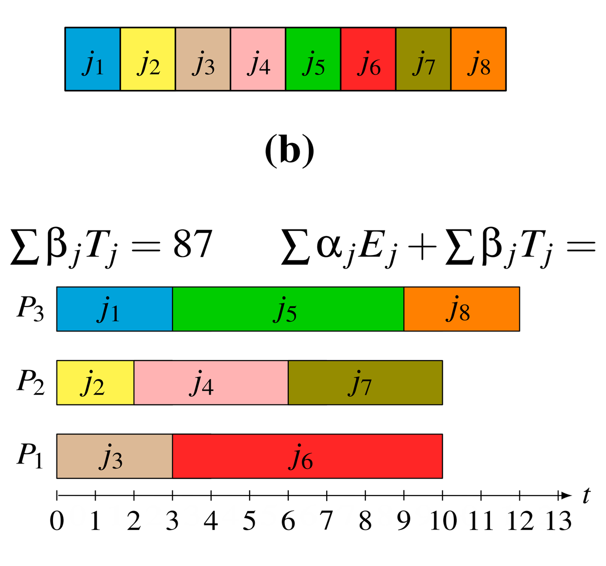

Figure 1: (a) An instance example of 8 jobs for the weighted tardiness and earliness-tardiness scheduling problem. Scheduling examples for the (b) identical parallel machines using machine-oriented Gantt chart and (c) the single sequence representation.

THE HYBRID MATHEURISTIC

The Hybrid Matheuristic (MathGLS-IP) is based on two steps:

- STEP 1: Heuristic approach (GLS)

The best local optimal solution generated by the GLS (Figure 2) is kept in a Hash Table on every generation, which will be used as a selected set of arcs to the IP Arc-time formulation. A solution representation of the Arc-time is presented in Figure 3.

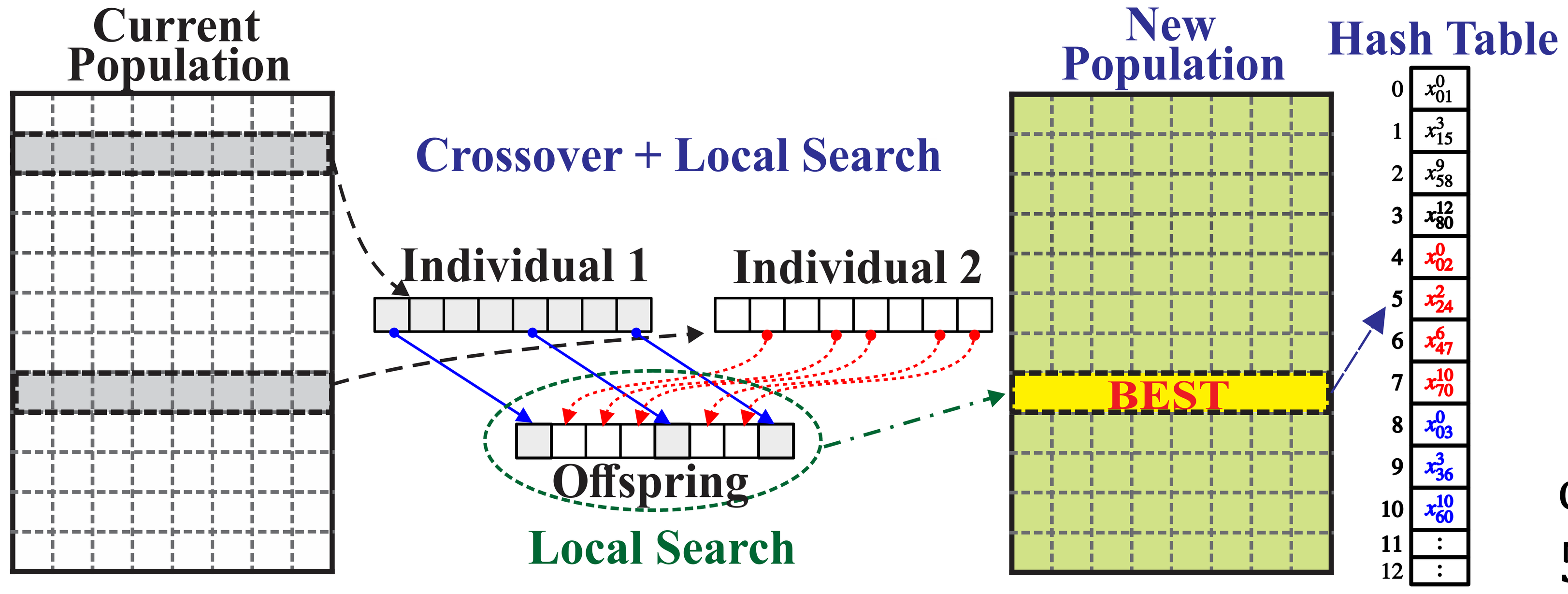

Figure 2: GLS - Crossover operation followed by local search, where the best local optimal solution of every generation is kept in a Hash Table.

- STEP 2: Exact approach (Solving the Arc-time)

When GLS procedure finishes, the selected arcs kept in the Hash Table are used to build the Arc-time, and then, solve it in CPLEX to get better convergence or improve the solution for a given instance of the problem. The Arc-time indexed formulation, proposed by Pessoa et al. [3], is presented bellow. The MathGLS-IP method eliminates the Constraints (4), in order to decrease the number of binary variables of idle time at the end of a scheduling.

$\sum \alpha_j E\_j + \sum \beta_j T_j$

Every job must be visited by exactly one path

Define the network flow of m units over an acyclic layered graph $G = (V,A)$

Eliminated Constraints

Binary variables

$$\textbf{Minimize} \quad \sum_{j \in J_+} \sum_{j \in J \setminus \{i\}} \sum_{t=p_i}^{T-p_j} f_j(t+p_j)\, x_{ij}^t \tag{1}$$

$$\textbf{s.t.} \quad \sum_{j \in J_+} \sum_{t=p_i}^{T-p_j} x_{ij}^t = 1 \qquad (\forall j \in J) \tag{2}$$

$$\sum_{\substack{j \in J_+ \setminus \{i\},\\ t-p_j \geq 0}} x_{ji}^t - \sum_{\substack{j \in J_+ \setminus \{i\},\\ t+p_i+p_j \leq T}} x_{ij}^{t+p_i} = 0 \qquad (\forall i \in J;\, t = 0, \ldots, T - p_i) \tag{3}$$

$$\sum_{\substack{j \in J_+,\\ t-p_j \geq 0}} x_{j0}^t - \sum_{\substack{j \in J_+,\\ t+p_j+1 \leq T}} x_{0j}^{t+1} = 0 \qquad (t = 0, \ldots, T-1) \tag{4}$$

$$\sum_{j \in J_+} x_{0j}^0 = m \tag{5}$$

$$x_{ij}^t \in Z_+ \quad (\forall i \in J_+; \forall j \in J_+\{i\}; t = p_i, \ldots, T - p_j) \tag{6}$$

$$x_{00}^t \in Z_+ \quad (t = 0, \ldots, T-1) \tag{7}$$

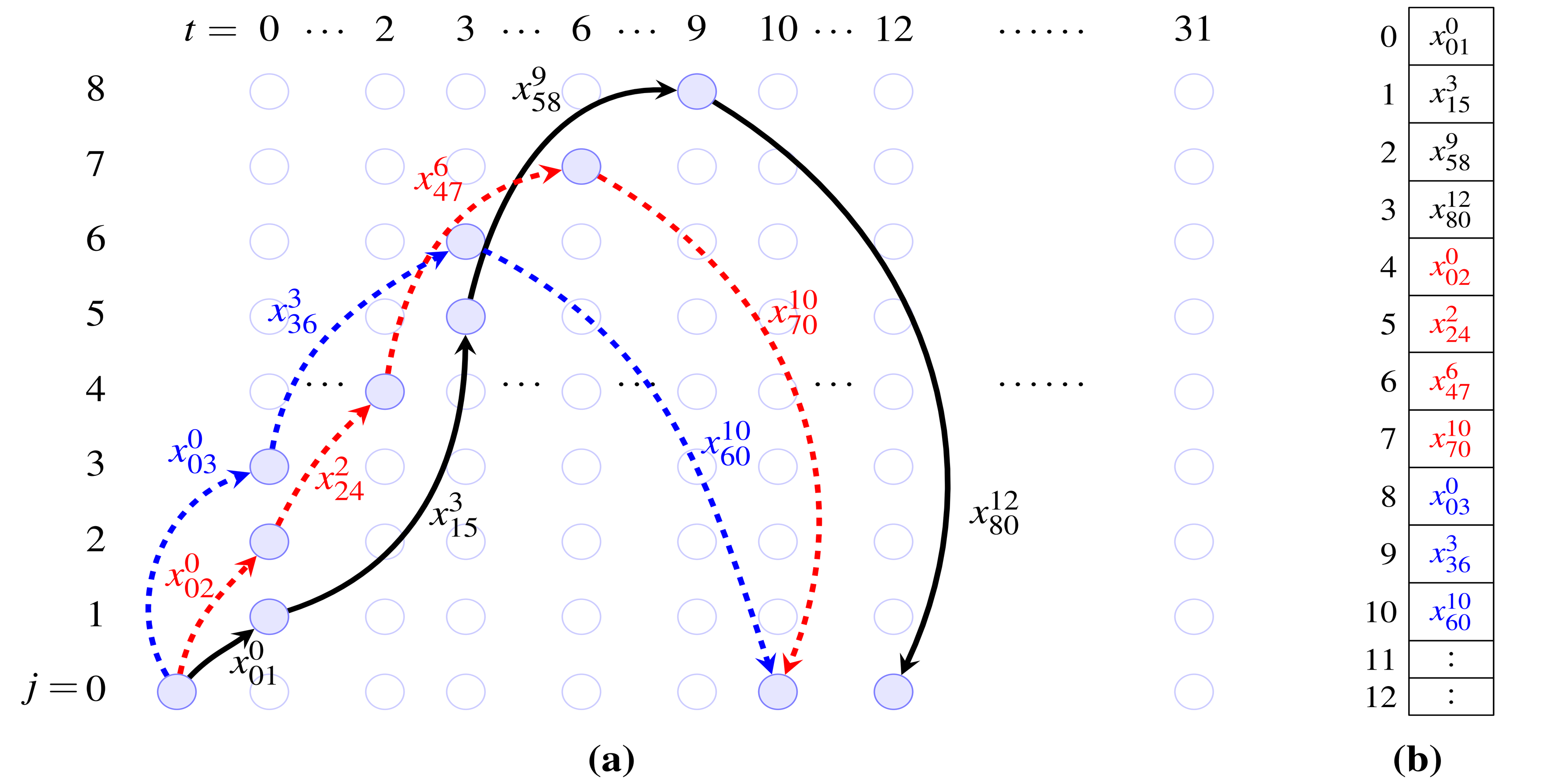

Figure 3: (a) Parallel machine network flow representation for the solution in Figure 1 (c) and the stored arcs from this solution in a hash table presented in (b) (we keep a set of stored solutions - not only one solution).

COMPUTATIONAL EXPERIMENTS

In Table 1 we present a resume of the computational experiments, compared with the literature. MathGLS-IP solves large instances up to 500 jobs and 2, 4 and 10 identical parallel machines. Our method also presents results for 200 instances, not yet known in the literature, and improved 4. Detailed results can be observed at Amorim [4].

Table 1: Computational Experiments compared with the Literature for the problem $P||\sum \alpha_j E_j + \sum \beta_j T_j$ with 40, 50 and 100 jobs on 2-10 machines.

Instance group	Kramer and Subramanian [5]				MathGLS-IP				
	Best run		Average		Best run			Average	
	GAP (%)	BKS	GAP (%)	Time (s)	GAP (%)	#	*	GAP (%)	Time (s)
wet40-2m	0,000	12	0,000	5,592	0,000	24	0	0,001	6,420
wet40-4m	0,000	12	0,001	6,258	0,000	24	0	0,001	9,261
wet40-10m	0,000	5	0,000	4,080	0,000	25	0	0,002	8,882
wet50-2m	0,000	11	0,001	12,617	0,000	23	0	0,000	13,623
wet50-4m	0,000	12	0,306	14,145	0,000	25	0	0,004	15,939
wet50-10m	0,000	5	0,014	9,320	0,000	24	0	0,038	13,686
wet100-2m	0,000	12	0,008	168,483	0,000	24	0	0,004	166,087
wet100-4m	0,790	6	0,858	190,309	0,000	23	4	0,055	114,163
wet100-10m	0,161	0	0,227	140,380	0,089	8	0	0,332	102,153
Total	---	**75**	---	---	---	**200**	**4**	---	---
Average	0,106	---	0,157	61,243	0,010	---	---	0,049	50,024

BKS – Amount of Best Known Solutions in the literature * – Amount of improved solutions
\# – Amount of solutions equal to BKS

REFERENCES

[1] GRAHAM, R. L.; LAWLER, E. L.; LENSTRA, J. K.; and RINNOOY KAN, A. H. G. Optimization and approximation in deterministic sequencing and scheduling: a survey. Annals of Discrete Mathematics, 5:287-326, 1979.
[2] PINEDO, M. L. Scheduling: Theory, algorithms, and systems. Springer Publishing Company, Incorporated, 4a ed.:1-104, 2012.
[3] PESSOA, A.; UCHOA, E.; ARAGÃO, M. P. de; and DE FREITAS, R. Exact algorithm over an arc-time-indexed formulation for parallel machine scheduling problems. Mathematical Programming Computation, 2(3-4):259-290, 2010.
[4] AMORIM, R. Estratégias Algorítmicas Exatas e Híbridas para Problemas de Escalonamento em Máquinas Paralelas com Penalidades de Antecipação e Atraso. Tese (Doutorado em Informática), 2017.
[5] KRAMER, A.; SUBRAMANIAN, A. A unified heuristic and an annotated bibliography for a large class of earliness–tardiness scheduling problems. Journal of Scheduling, 22(1): 21-57, 2019.

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Layout de Redes de Sensores Sem Fio com Múltiplas Origens e Destinos: Uma Abordagem Combinatória

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Introdução

Há diversas aplicações para as Redes de Sensores sem Fio(RSSF): monitoramento de sinais ambientais[1], aplicações militares[2], entre outras. Neste trabalho, investiga-se o problema de Planejamento de Redes de Sensores Sem Fio (PRSSF-MOD), onde a rede é formada por múltiplas origens (sensores) e múltiplos destinos (sorvedouros). A topologia da rede é representada através de um grafo e na resolução do problema proposto, iremos definir um modelo de Programação Linear Inteira (PLI) e uma grafo auxiliar que será utilizado junto ao modelo.

Objetivo

O objetivo deste trabalho é minimizar o número de sensores da topologia da rede em uma dada região de interesse, de modo a atender as conexões entre múltiplas origens e destinos.

Definição do problema

Dado um conjunto S de sensores, onde para cada $s \in S$ é associado um conjunto $\{si\}i=1..k$, um raio de comunicação r, um custo de alocação c e um conjunto P de origens e destinos $p=\{op,dp\}$. Utilizando essas informações pretende-se construir uma topologia $T \subseteq S$ que conecte todos os pares de origens destinos $p=\{op,dp\}$, de forma direta ou por múltiplos saltos entre sensores intermediários, de modo a minimizar o custo de instalação da rede.

Modelo PLI e grafo auxiliar

Para resolver o problema foram definidos um modelo PLI[3] (Figura 1) e um grafo auxiliar $G=(V,E)$ (Figura 2), onde V é definido pelas possíveis posições de S e vértices artificiais A que representam origens e destinos e E compreende as arestas definidas pela intersecção entre os raios de comunicação dos sensores em diferentes grupos. Para finalizar a aplicação são adicionadas arestas entre os nós artificiais de origem e destino e seus respectivos grupos de sensores.

$$\min \sum_{i \in V} c_i y_i$$

$$\text{s.t.} \sum_{(i,j) \in \delta^+(i)} x_{ij}^p - \sum_{(j,i) \in \delta^-(i)} x_{ji}^p = b_i^p, \qquad \forall i \in V, p \in P, \qquad (1)$$

$$\sum_{\alpha \in C(i)} y_\alpha \leq 1, \qquad \forall i \in S \qquad (2)$$

$$x_{ij}^p \leq y_i, \qquad \forall (i,j) \in E, p \in P \qquad (3)$$

$$x_{ij}^p \leq y_j, \qquad \forall (i,j) \in E, p \in P \qquad (4)$$

$$y_i = 1, \qquad \forall i \in A \qquad (5)$$

$$x_{ij}^p \in \{0,1\}, \qquad \forall (i,j) \in E, p \in P \qquad (6)$$

$$y_i \in \{0,1\}, \qquad \forall i \in V \qquad (7)$$

where

$$b_i^p = \begin{cases} -1 & \text{if } i = d_p, \\ 1 & \text{if } i = o_p, \\ 0 & \text{otherwise.} \end{cases} \qquad (8)$$

Figura 1: Modelo PRSSF-MOD

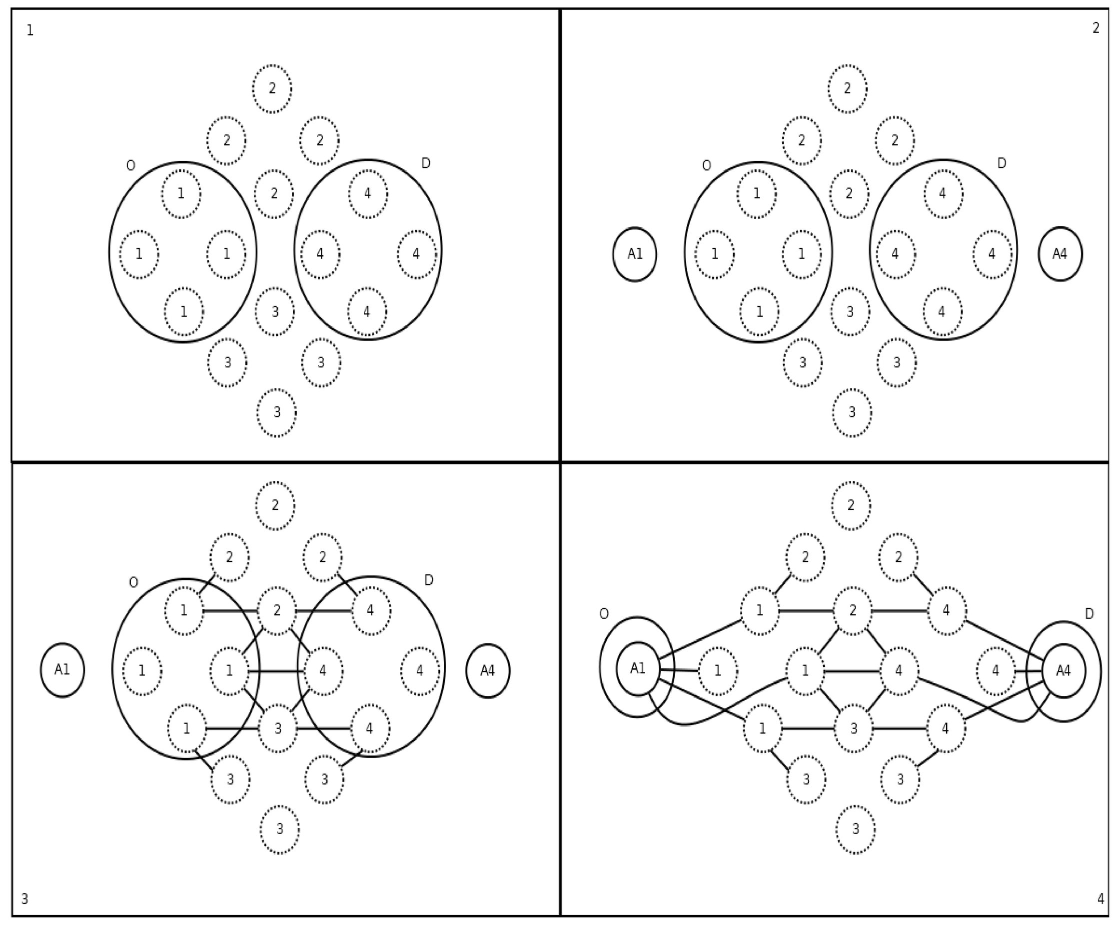

Figura 2: Grafo auxiliar

Restrição (1) garante a existência de um caminho entre origens e destinos, Restrição (2) garante que apenas uma posição dentre as candidatas sera escolhida, (3) e (4) representam que uma aresta só pode ser usada se existe um sensor naquela posição e a Restrição (5) define que todo nó artificial está na solução.

Experimentos computacionais

Os experimentos computacionais se baseiam na instância real do intel lab data[4], que possui 54 sensores de monitoramento ambiental. Os grupos de possíveis posições e as origens e destinos de cada experimento estão apresentadas abaixo.

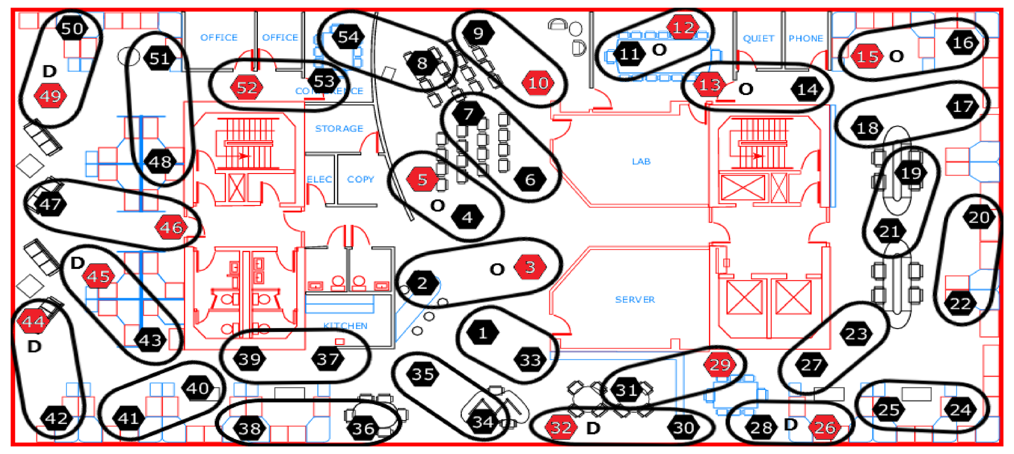

Figura 3(a): Instância com 27 possíveis Sensores e 5 Origens e Destinos

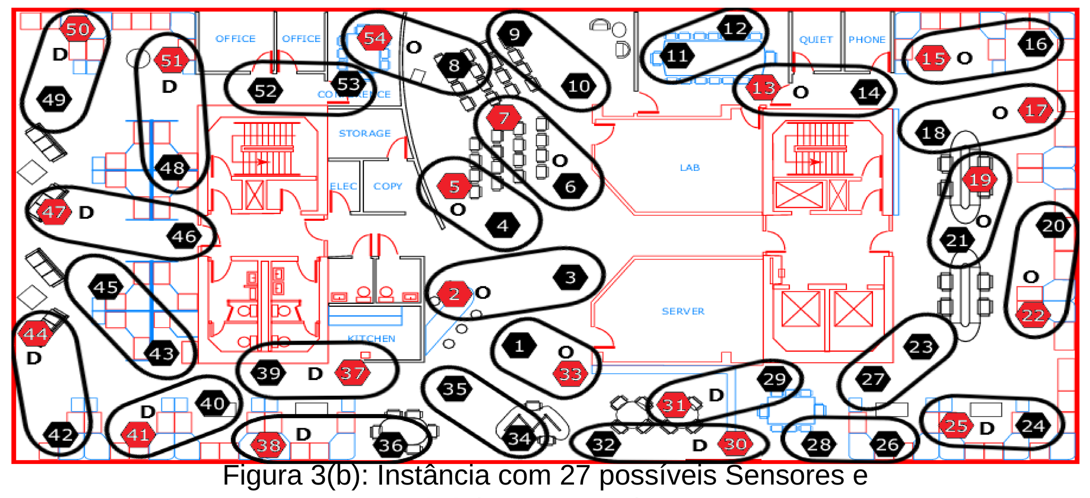

Figura 3(b): Instância com 27 possíveis Sensores e 10 Origens e Destinos

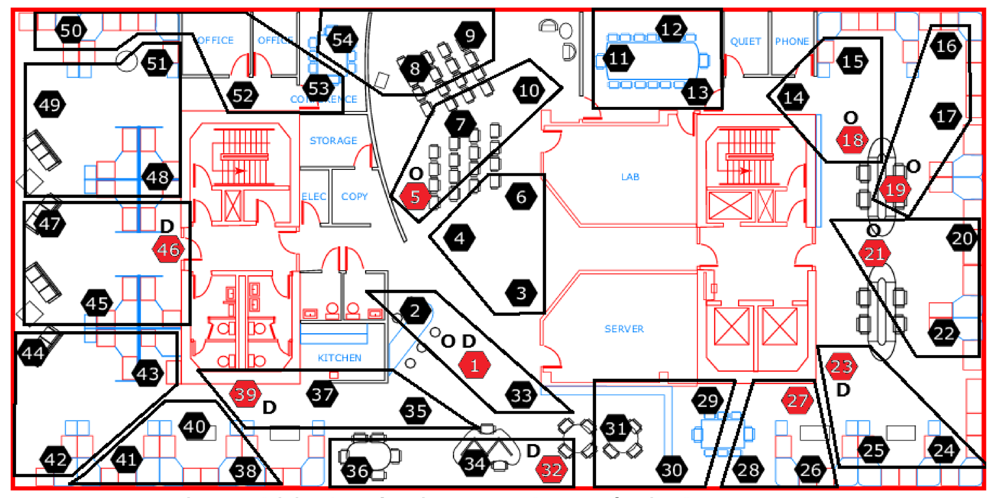

Figura 3(c): Instância com 18 possíveis Sensores e 5 Origens e Destinos

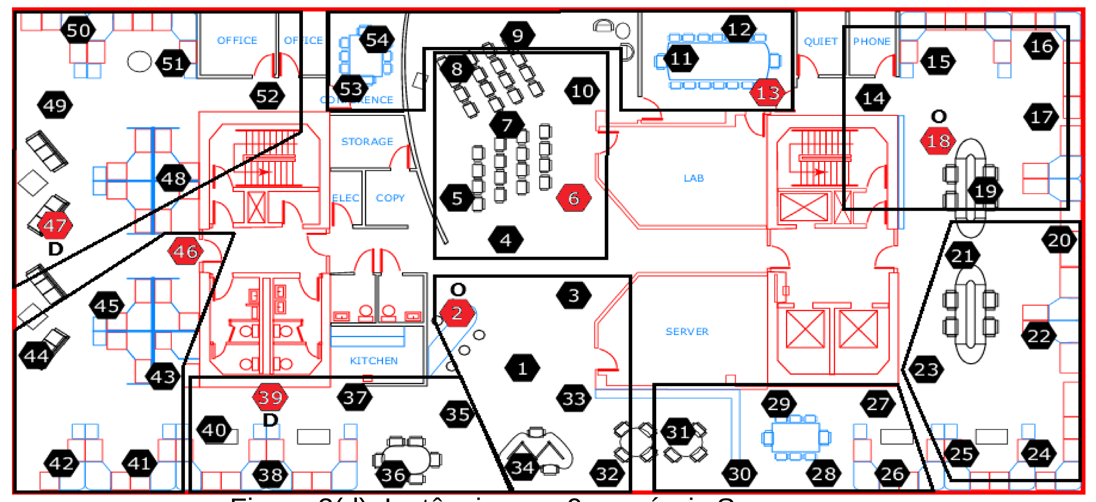

Figura 3(d): Instância com 9 possíveis Sensores e 2 Origens e Destinos

Os experimentos foram executados em um Core i5 2.3 GHz, 16 GB de RAM, implementados em C++ usando CPLEX e compilados no gcc 9.3.0. Foram necessários, no maximo, 14 segundos para a execução das instâncias, o que demonstra a viabilidade da solução proposta.

Conclusões

Este trabalho apresentou uma versão modificada do PRSSF que considera múltiplas origens e destinos. O modelo foi avaliado em uma instância real e obteve uma redução na quantidade de sensores de 25% utilizados na topologia.

Referências

[1] Henriques, F. R., Lovisolo, L., and Rubinstein, M. G. . DECA: distributed energy conservation algorithm for process reconstruction with bounded relative error in wireless sensor networks. EURASIP Journal on Wireless Com. and Net., v. 163, p. 1–18, 2016.

[2] Azzabi, T., Farhat, H., and Sahli, N. (2017). A survey on wireless sensor networks secu-rity issues and military specificities. In2017 International Conference on AdvancedSystems and Electric Technologies (IC˙ASET), pages 66–72.

[3] Wolsey, L. A. (1998).Integer programming, volume 52. John Wiley & Sons.

[4] lab WSN, I. B. (2004). http://db.csail.mit.edu/labdata/labdata.html.

Logarithmic-Sobolev and Poincaré inequalities for the simple random walk on the Hanoi graph

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The random walk on the Tower of Hanoi

The tower of Hanoi puzzle is a single-player game where at each turn the player moves a disk to a tower that is different from the one it previously was. The game does not allow a disk above smaller disks and its aim is to move all disks from a tower to another one (see Figure 1).

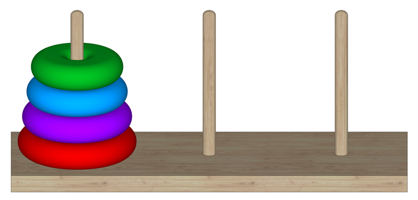

Figure 1: *Tower of Hanoi with* 4 *disks.*

The Hanoi graph $\mathcal{H}_m = (V_m, E_m)$ is the graph whose vertices represent the possible configurations of the tower of Hanoi puzzle with 3 towers and m disks. Its edges represent the moves between these configurations. Thus, $\mathcal{H}_1$ is isomorphic to a triangle and for each $m \geq 2$ we can construct $\mathcal{H}_m$ in the following way which is illustrated in Figure 2: we consider three isomorphic copies of $\mathcal{H}_{m-1}$ and we label them as $\mathcal{H}^i_{m-1} = (V^i_{m-1}, E^i_{m-1})$, $i \in \{1,2,3\}$. For each $i \in \{1,2,3\}$ let v^i_{top}, v^i_{lb} and v^i_{rb} be the vertices on the top, on the left, and on the right of the basis of the biggest triangle in $\mathcal{H}^i_{m-1}$. The graph $\mathcal{H}_m$ is the graph with vertex set $V_m = \cup^3_{i=1} V^i_{m-1}$ and edge set $E_m = \left(\cup^3_{i=1} E^i_{m-1}\right) \cup E^\star_m$, where $E^\star_m$ is defined as $E^\star_m := \{\{v^1_{lb}, v^2_{top}\}, \{v^1_{rb}, v^3_{top}\}, \{v^2_{rb}, v^3_{lb}\}\}$.

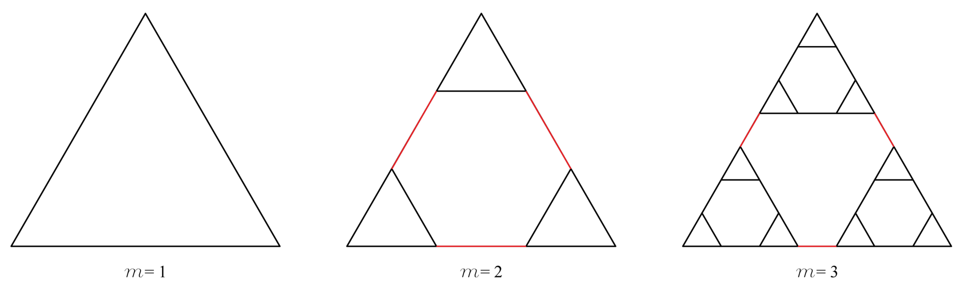

Figure 2: *Graphs* $\mathcal{H}_m$ *for* $m \in \{1,2,3\}$ *with edges of* $E^\star_m$ *coloured in red.*

The simple random walk on $\mathcal{H}_m$ is the process $\{X_t; t \geq 0\}$ described as follows: an exponential clock with rate one is attached to each edge of E_m. Whenever a clock rings, the edge associated with that clock is flipped, making the random walker jump if she was at one of the incident vertices to that edge. Its infinitesimal generator is the discrete Laplacian operator Δ_m given by

$$\Delta_m f(x) = \sum_{y \sim x} (f(y) - f(x)),$$

which says that if the random walker stands at a vertex x then it can jump to any of its adjacent vertices with rate 1. In the above formula, $x \sim y$ denotes that x and y share a common edge.

An interesting question to make is to ask how long the random walker takes to get completely lost. In order to answer this question, let $\mu^{x_0}_t(x)$ denote the probability that $X_t = x$ given that $X_0 = x_0$, and let U_m denote the uniform measure on V_m. The distance to equilibrium of the simple random walk on $\mathcal{H}_m$ is defined as

$$d_m(t) = \max_{x_0 \in V_m} \|\mu^{x_0}_t - U_m\|_{TV} = \max_{x_0 \in V_m} \left\{ \frac{1}{2} \sum_{x \in V_m} \left| \mu^{x_0}_t(x) - \frac{1}{3^m} \right| \right\}.$$

Not only the above function is decreasing, but it also takes values in the interval $[0,1]$. Thus, given a threshold $\varepsilon \in (0,1)$, it makes sense to define the ε-mixing time of the simple random walk as

$$t^m_{mix}(\varepsilon) = \inf\{t \geq 0;\ d_m(t) < \varepsilon\},$$

formalizing the answer to the aforementioned question.

Algebraic connectivity and Poincaré inequalities

The spectral gap γ_m of the simple random walk on $\mathcal{H}_m$ (also known as the *algebraic connectivity of the graph* $\mathcal{H}_m$) is defined as the symmetric of the second largest eigenvalue γ_m of the operator Δ_m. It also presents a variational formula [2]. Indeed, let $\mathcal{E}_m$ be the Dirichlet form of the simple random walk on $\mathcal{H}_m$ which is given by

$$\mathcal{E}_m(f,f) = \frac{1}{2} \sum_{x \in V_m} \sum_{y \sim x} |f(x) - f(y)|^2 U_m(x).$$

Let $\mathrm{Var}(f\,;U_m)$ be the variance of a function $f : V_m \to \mathbb{R}$ for the simple random walk on $\mathcal{H}_m$, which is given by

$$\mathrm{Var}(f\,;U_m) = \frac{1}{2} \sum_{x,y} |f(x) - f(y)|^2 U_m(y)\, U_m(x).$$

The spectral gap γ_m of the simple random walk on $\mathcal{H}_m$ can be defined as

$$\gamma_m := \inf_f \left\{ \frac{\mathcal{E}_m(f,f)}{\mathrm{Var}(f\,;U_m)};\ \mathrm{Var}(f\,;U_m) \neq 0 \right\}.$$

Namely, the relaxation time $t^m_{rel} := 1/\gamma_m$ of the simple random walk on $\mathcal{H}_m$ is the smallest constant that satisfies the *Poincaré inequality*

$$\mathrm{Var}(f\,;U_m) \leq C\,\mathcal{E}_m(f,f) \text{ for every function } f.$$

The spectral gap is strongly related to mixing because

$$d_m(t) \leq 3^{m/2} e^{-\gamma_m t} \text{ (see [5], for instance).}$$

Our result in this direction is the following:

Theorem 1: For every $m \geq 2$ we have

$$t^m_{rel} \leq \frac{1}{3\ (1/3;\,1/3)_{m-1}}, \text{ where } (a\,;q)_n := \prod_{k=0}^{n-1} (1 - a\,q^k)$$

is the q-Pochhammer symbol, also known as q-shifted factorial. Consequently,

$$t^m_{mix}(\varepsilon) \leq \frac{\log 3}{9} \left(\frac{3}{2}\right)^m m + \mathcal{O}_\varepsilon\left(\left(\frac{3}{2}\right)^m\right).$$

Logarithmic-Sobolev inequalities

The log-Sobolev constant α_m of the simple random walk on $\mathcal{H}_m$ presents a similar variational definition. Here the variance is replaced by the entropy-like quantity $\mathcal{L}_m$ given by

$$\mathcal{L}_m(f) = \sum_{x \in \mathcal{H}_m} |f(x)|^2 \log\left(\frac{|f(x)|^2}{\sum_{z \in \mathcal{H}_m} |f(z)|^2 U_m(z)} \right) U_m(x).$$

Namely

$$\alpha_m := \inf_f \left\{ \frac{\mathcal{E}_m(f,f)}{\mathcal{L}_m(f)};\ \mathcal{L}_m(f) \neq 0 \right\},$$

that is, $1/\alpha_m$ is the smallest constant that satisfies the *logarithmic Sobolev inequality*

$$\mathcal{L}_m(f) \leq C\,\mathcal{E}_m(f,f) \text{ for every function } f.$$

The log-Sobolev constant is stronger than the spectral gap in the sense that

$$d_m(t) \leq \sqrt{m}\, e^{-\alpha_m t/2} \text{ (see [3], for instance).}$$

We prove the following result:

Theorem 2: There exists a constant $\alpha_1 \in (0.856, 1.500]$ such that for every $m \geq 1$ we have $\frac{1}{\alpha_m} \leq \frac{2}{\alpha_1}\left(\frac{3}{2}\right)^m$. Consequently, $t^m_{mix}(\varepsilon) \leq \frac{2}{\alpha_1}\left(\frac{3}{2}\right)^m \log m + \mathcal{O}_\varepsilon\left(\left(\frac{3}{2}\right)^m\right)$.

Decomposing the graph

The main idea is to remove the edges of $E^\star_m$ and to decompose $\mathcal{H}_m$ into $\mathcal{H}^i_{m-1}$, $i \in \{1,2,3\}$, and then to do some analysis. Given a function $f : V_m \to \mathbb{R}$ and $i \in \{1,2,3\}$, denote the restriction of f to the domain V^i_{m-1} by $f|_{V^i_{m-1}}$, and define $U^i_{m-1} := U_m|_{V^i_{m-1}}$. Firstly, we prove that

$$\mathrm{Var}(f\,;U_m) = \frac{1}{3} \sum_{i=1}^3 \mathrm{Var}(f|_{V^i_{m-1}};\,U^i_{m-1}) + \mathrm{Var}(G\,;U_1) \leq \frac{1}{\gamma_{m-1}} \mathcal{E}_m(f\,,f) + \mathrm{Var}(G\,;U_1).$$

where $G(i) = \sum_{z \in \mathcal{H}^i_{m-1}} f(z)\, U^i_{m-1}(z)$ is the expectation of the function f, restricted to V^i_{m-1}, with respect to the measure U^i_{m-1}. Secondly, we show that $\mathrm{Var}(G\,;U_1) \leq 3^{1-m}\,\mathrm{Var}(f\,;U_m)$. By the definition of γ_m, we obtain

$$\frac{1}{\gamma_m} \leq \frac{1}{\gamma_{m-1}\,(1 - 3^{1-m})},$$

which after an induction argument, implies Theorem 1. Similarly, we prove that

$$\mathcal{L}_m(f) \leq \frac{1}{\alpha_{m-1}} \mathcal{E}_m(f,f) + \frac{1}{\alpha_1} \mathcal{E}_1(\sqrt{F}, \sqrt{F}),$$

where $F(i)$ is the square of the $\ell^2(U^i_{m-1})$ norm of the function f restricted to V^i_{m-1}. Then, we show that $\frac{1}{\alpha_1}\mathcal{E}_1(\sqrt{F}, \sqrt{F}) \leq \frac{\gamma_1}{\alpha_1 \gamma_m} \mathcal{E}_m(f\,,f)$. Theorem 2 follows from an induction argument.

Remark: Looking carefully at the lower bound on the log-Sobolev constant obtained in Theorem 2 with the above method, one can see that it strongly depends on the upper bound on the relaxation time. More precisely, if one can obtain a sharper exponential upper bound on t^m_{rel}, then, using our method, they can obtain a lower bound on the second parameter which has the same order.

Acknowledgments

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References

[1] Diaconis, P., and Saloff-Coste, L. *Comparison techniques for random walk on finite groups.* The Annals of Probability (1993): 2131–2156.

[2] Diaconis, P., and Saloff-Coste, L. *Comparison theorems for reversible Markov chains.* The Annals of Applied Probability (1993): 696–730.

[3] Diaconis, P., Saloff-Coste, L. *Logarithmic Sobolev inequalities for finite Markov chains.* The Annals of Applied Probability 6.3 (1996): 695–750.

[4] Egler, S. *Graphs, Random Walks, and the Tower of Hanoi.* Rose–Hulman Undergraduate Mathematics Journal 20.1 (2019): 6.

[5] Levin, D., Peres, Y., and Wilmer, E. *Markov Chains and Mixing Times.* American Mathematical Society, Providence, RI, (2009).

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On total coloring of circulant graphs

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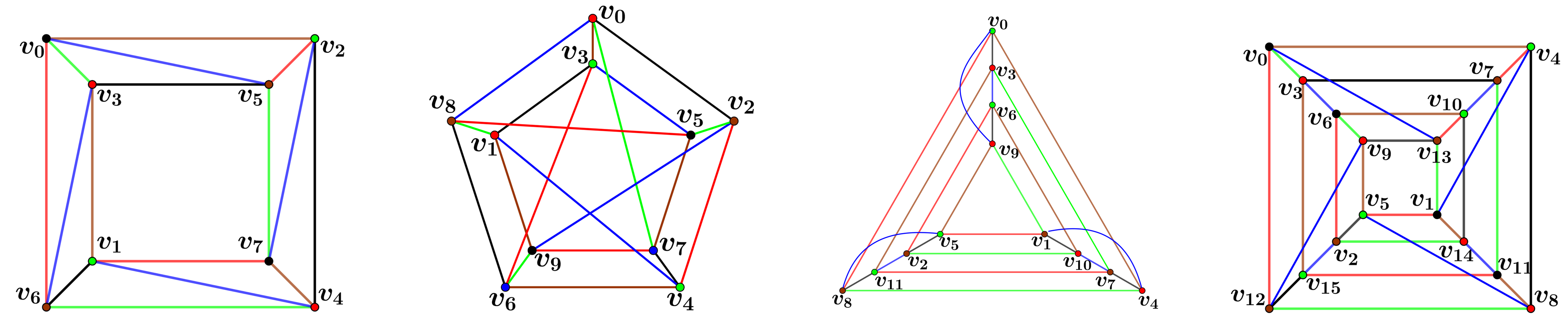
Introduction

A k-total coloring of a graph G is an assignment of k colors to the elements of G such that adjacent elements have different colors. The total chromatic number $\chi''(G)$ is the smallest integer k for which G has a k-total coloring. Clearly, $\chi''(G) \geq \Delta+1$, and the Total Coloring Conjecture (TCC) states that for any simple graph G, $\chi''(G) \leq \Delta+2$, where Δ is the maximum degree of G [2, 8]. Graphs with $\chi''(G) = \Delta(G)+1$ are called Type 1, and graphs with $\chi''(G) = \Delta(G)+2$ are called Type 2. A circulant graph $C_n(d_1, d_2, \cdots, d_l)$ with $1 \leq d_1 < \cdots < d_l \leq \lfloor \frac{n}{2} \rfloor$ has vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ and edge set $E = \bigcup_{i=1}^{l} E_i$ where $E_i = \{e_0^i, e_1^i, \cdots, e_{n-1}^i\}$ and $e_j^i = (v_j, v_{j+d_i})$ where the indexes of the vertices are considered modulo n. An edge of E_i is called edge of length d_i. In this work, we determine the Type of an infinite family of 4−regular circulant graphs, that is, $C_n(a, b)$. When a divide n (or b divide n), we will have a Prism graph $G(\frac{n}{a}, 1)$ as subgraph of $C_n(a, b)$. A Prism graph $G(n, 1)$ is defined by $V(G(n, 1) = \{u_i, v_i \mid 0 \leq i < n\}$ and $E(G(n, 1)) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i \mid 0 \leq i < n\}$. See some examples of $C_n(a, b)$ with $G(\frac{n}{a}, 1)$ as a subgraph in Figure 1.



(a) $\chi''(C_8(2, 3)) = 5$ (b) $\chi''(C_{10}(2, 3)) = 5$ (c) $\chi''(C_{12}(3, 4)) = 5$ (d) $\chi''(C_{16}(3, 4)) = 5$

Figura 1:Examples of $C_n(a, b)$ with $G(\frac{n}{a}, 1)$ as a subgraph.

General results

In the table below, we present some results already known about the total coloring of circulant graphs.

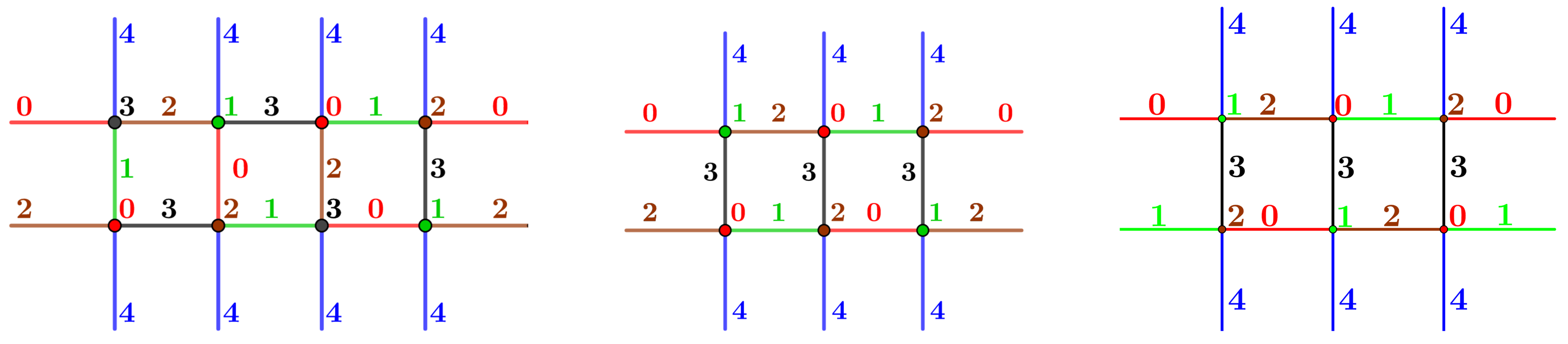
Circulant graph	Type 1	Type 2
$C_n(1)$ [9]	$n \equiv 0 \mod 3$	otherwise
$C_n(1, 2, .., \lfloor \frac{n}{2} \rfloor)$ [9]	n is odd	otherwise
$C_{2n}(d, n)$ [5]	$l = \gcd(d, n)$ with $d = lm$, m is even and $C_{2n}(d, n) \not\simeq l$ copies of $C_{10}(2, 5)$	otherwise
$C_n(1, 2)$ [3]	$n \neq 7$	otherwise
$C_{5p}(1, k)$ [6]	$k \equiv 2 \mod 5$ or $k \equiv 3 \mod 5$	
$C_{6p}(1, k)$ [6]	$k \equiv 1 \mod 3$ or $k \equiv 2 \mod 3$	
$C_n(1, 3)$ [9]		$tn = 8$

Tabela 1:State of the art

Our results

It is known that the Prism graphs $G(n, 1)$ are Type 1, except $G(5, 1)$ [7, 4]. The 4−total coloring for this family will be useful in the proof of the following theorem about 4−regular circulant graphs in which $G(n, 1)$ is a subgraph.

Theorem 1. *Let $C_n(2k, 3)$ be a 4−regular circulant graph. The graph $C_n(2k, 3)$ is Type 1 for $n = (8\mu + 6\lambda)k$, with $k \geq 1$ and non-negative integers μ and λ.*

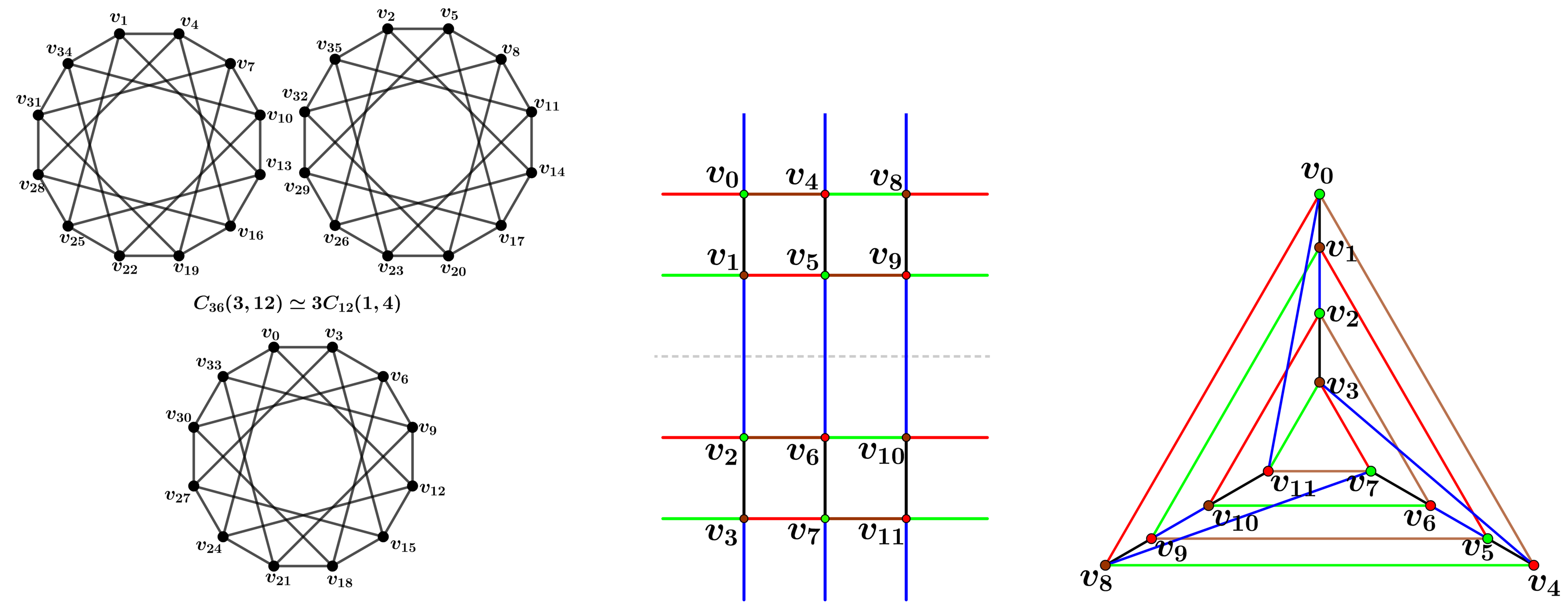


(a) $B(8, 2)$ colored (b) $B(6, 2)$ colored (c) $B(6, 2)$ with other coloring

Figura 2:Semigraph $B(n, a)$

A semigraph is a triple $B = (V, E, S)$, where V is the set of vertices of B,E is a set of edges having two distinct endpoints in V, and S is a set of semiedges having one endpoint in V. In this work we consider 4−regular semigraphs. Notice that a k−total coloring of a semigraph B is an assignment of k colors to the edges, semiedges and vertices of B such that adjacent elements have different colors.

Sketch of the proof. The result was proved in [1] when $C_n(2k, 3)$ is connected, using the Figure 2(a) . Hence, suppose that $C_{(8\mu+6\lambda)k}(2k, 3)$ is disconnected, that is $k = 3\alpha$. In this case, note that $C_{(8\mu+6\lambda)3\alpha}(3, 6\alpha)$ is isomorphic to three copies of $C_{(8\mu+6\lambda)\alpha}(1, 2\alpha)$. To construct the colorings of these graphs, we consider two cases: $\mu = 0$ and $\mu \neq 0$. When $\mu = 0$, we construct the desired coloring by making the junction of λ copies of the semigraph $B(6, 2)$ (Figure 2(c)) vertically and horizontally, recursively. When $\mu \neq 0$, we make the junction of μ copies of the semigraph $B(8, 2)$ with λ copies of $B(6, 2)$ (Figure 2 (b)) vertically and horizontally, recursively (the same for the case when $C_n(2k, 3)$ is connected). However the process of joining its semiedges to construct the desired graph is different. See an example in Figure 3.



(a) $C_{36}(3, 12)$ (b) $B(12, 4)$ (c) $\chi''(C_{12}(1, 4)) = 5$

Figura 3:The graph $C_{36}(3, 12)$ with a total coloring with 5 colors.

Conclusion

The total chromatic number of several circulant graphs has been determined, including the total chromatic number of the cubic circulant graphs $C_{2n}(d, n)$. As a future work, we would like to determine the total chromatic number of all 4−regular circulant graphs $C_n(a, b)$.

References

[1] M. N. Alves Junior and D. Sasaki, *A result on total coloring of circulant graphs*, in ENCONTRO DE TEORIA DA COMPUTAÇÃO (ETC), vol. 5, Porto Alegre, 2020, pp. 81–84.
[2] M. Behzad, *Graphs and Their Chromatic Numbers*, PhD thesis, Michigan University, 1965.
[3] C. Campos, *O problema da coloração total em classes de grafos*, PhD thesis, State University of Campinas, São Paulo, 2006.
[4] A. G. Chetwynd and A. J. W. Hilton, *Some refinements of the total chromatic number conjecture*, Congr Numer., (1988), pp. 195–216.
[5] A. Hackmann and A. Kemnitz, *Circular total colorings of cubic circulant graphs*, J. Combin. Math. Combin. Comput., (2004), pp. 65–72.
[6] R. Khennoufa and O. Togni, *Total and fractional total colourings of circulant grap*, Discrete Math., (2008), pp. 6316–3329.
[7] D. Sasaki et al., *On the total coloring of generalized Petersen graphs*, Discrete Math., (2014), pp. 1471–1475.
[8] V. Vizing, *Some unsolved problems in graph theory*, Russian Mathematical Surveys, 23 (1968), pp. 125–141.
[9] H. Yap, *Total colourings of graphs*, Springer, Berlin, 1996.

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CAPES

EQUITABLE TOTAL COLORING OF BLOWUP SNARKS

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INTRODUCTION

Let G be a *simple graph*. A *k-total-coloring* of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by $\chi''(G)$ is is the least k for which G has a k-total-coloring. Evidently, $\chi''(G) \geq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. The Total Coloring Conjecture [1] afirms that $\chi''(G) \leq \Delta(G)+2$. This conjecture has been proved for cubic graphs [2], so the total chromatic number of a cubic graph is 4 or 5. Graphs with $\chi''(G) \geq \Delta(G)+1$ are said to be *Type 1* and graphs with $\chi''(G) \leq \Delta(G)+2$ are said to be *Type 2*. Deciding whether a graph is Type 1 has been shown NP-complete [3].

A k-total-coloring is *equitable* if the cardinalities of any two color classes differ by at most one. The least k for which G has an equitable k-total-coloring is the *equitable total chromatic number* of G and its denoted by $\chi''_e(G)$.

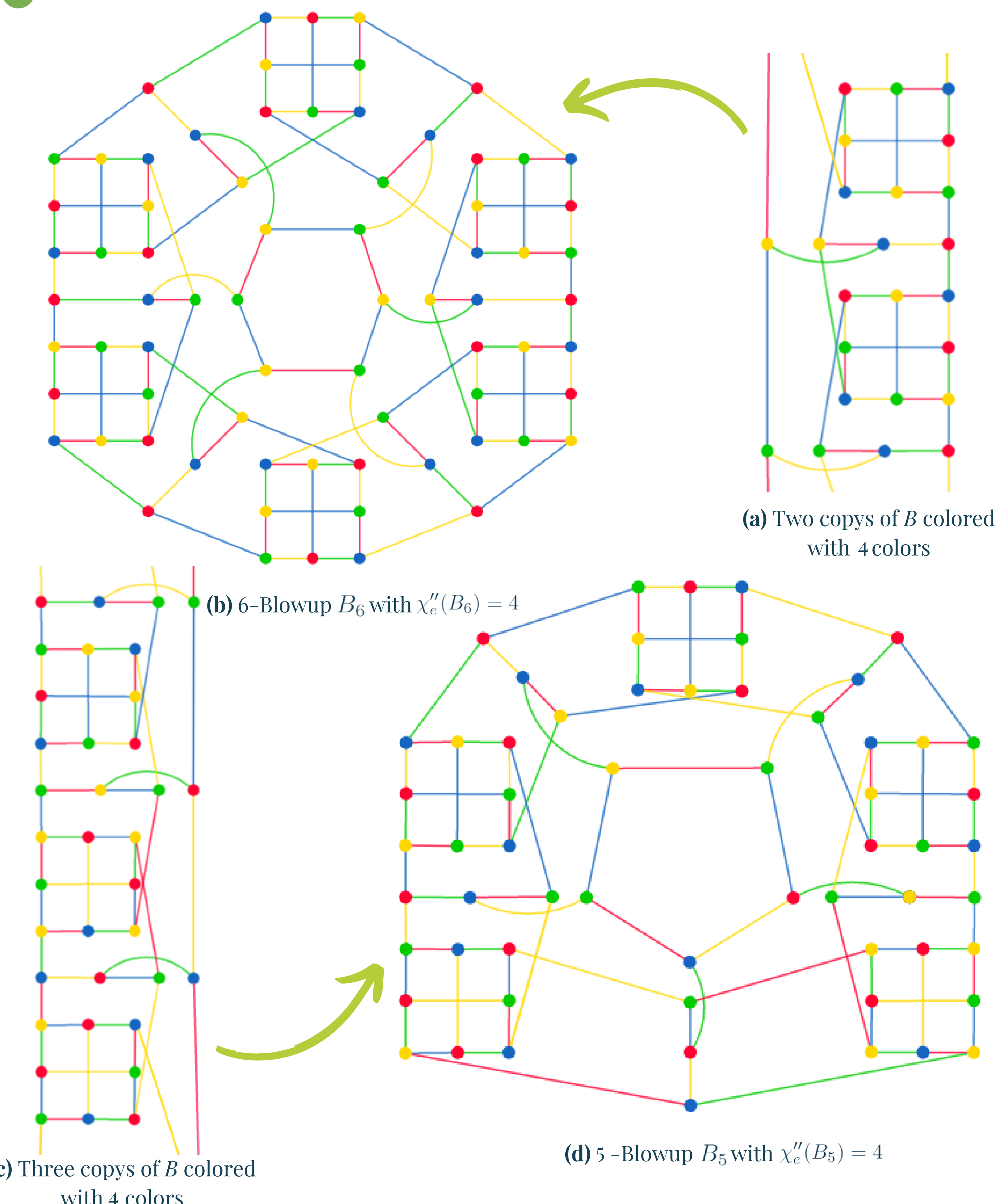
The search for connected, bridgeless, 3-regular graphs with chromatic index equals 4, was motivated by the Four Color Problem. Due the difficult to find them, they were named *Snarks* after Lewis Carrol poem "*The hunting of the Snark*", by M. Gardner [4]. Snarks were fictional animal species described by Carrol as unimaginable creatures.

Figure 1: Lewis Carrol book cover

The *girth* of G is the length of the shortest cycle contained in G. One condition often imposed on snarks is that they must have girth at least 5, to avoid graphs that can be reduced to a smaller graph by replacing a subgraph for structures that do not affect the edge colorability.

In this study we investigate the k-total-coloring of Blowup infinite family of snarks, recently defined by Hägglund [5], with girth at least 5.

RESULTS



(a) Two copys of B colored with 4 colors

(b) 6-Blowup B_6 with $\chi''_e(B_6)=4$

(c) Three copys of B colored with 4 colors

(d) 5-Blowup B_5 with $\chi''_e(B_5)=4$

Figure 3: Construction of 5-Blowup and 6-Blowup with $\chi''_e(B)=4$

Let B be the cubic semigraph with 6 semi-edges, illustrated in Figure 2. Blowup graphs are constructed by connecting copies of B as in the examples of Figures 3.

An *n-Blowup* is a graph build with n copies of B.

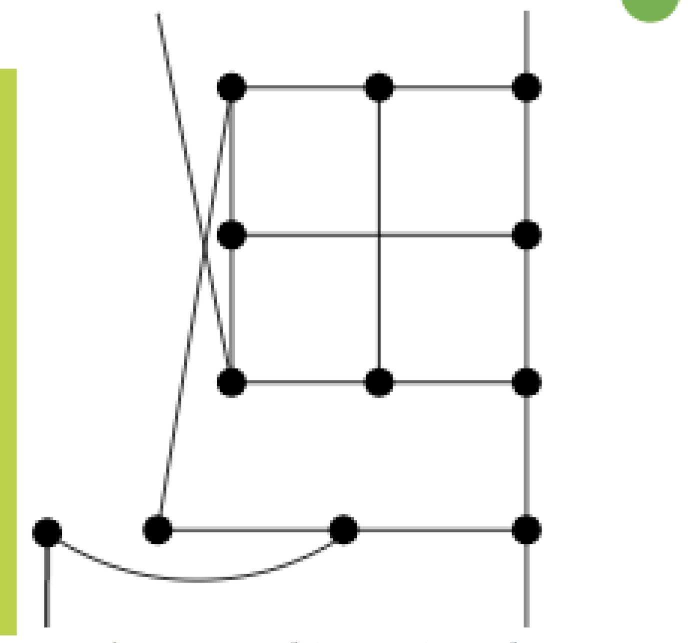


Figure 2: cubic semigraph B

Theorem: *All n-Blowups with n≥5 have equitable total chromatic number equals 4.*

The sketch of the proof is by construction and two different equitable 4-total-colorings were necessary to obtain the result.

We represent 1 for blue, 2 for green, 3 for red and 4 for yellow.

First coloring is showed in Figure 3(a). It's composed by two copies of B colored with 4 colors. More specifically, in this figure, $\phi(1)=\phi(2)=\phi(3)=\phi(4)=15$ (semiedges counts 0.5).

When $n \equiv 0 \pmod 2$ we repeat this coloring $\frac{n}{2}$ times. Evidently, $\phi(1)=\phi(2)=\phi(3)=\phi(4)=\frac{n}{2}\cdot 15$. Figure 3(b) shows 6-Blowup colored following this rule.

The second coloring is showed in Figure 3(c) and its composed by 3 copies of B colored with 4 colors.

In this coloring, $\phi(1)=\phi(4)=22$ and $\phi(2)=\phi(3)=23$.

When $n \equiv 1 \pmod 2$ we use this coloring once and for the remaining n-3 copies of B we repeat $\frac{n-3}{2}$ times the coloring showed in 3(a).

Thus, $\phi(1)=\phi(4)=22+\frac{n-3}{2}\cdot 15$ and $\phi(2)=\phi(3)=23+\frac{n-3}{2}\cdot 15$. Evidently, $\phi(1)$, $\phi(2)$, $\phi(3)$ and $\phi(4)$ differ at most one. Figure 3(d) shows 5-Blowup colored following this rule.

REFERENCES

[1] VIZING, V. G. On an estimate of the chromatic class of a p-graph. Diskret. Analiz., 3, p. 25-30, 1964.

[2] ROSENFELD, M. On the total coloring of certain graphs. Israel J. Math., 9, p. 396-402, 1971.

[3] MCDIARMID, C. J. H., SÁNCHEZ-ARROYO, A. Total coloring regular bipartide graphs is NP-hard. Discrete Math., 124, p. 155-162, 1994.

[4] GARDNER, M. Mathematical games: Snarks, Boojums and other conjectures related to the four-color-map theorem, Scientific American., 234, p. 126–130, 1976.

[5] HÄGGLUND, J. On Snarks that are far from beeing 3-edge-colorable., The Electronic Journal of Combinatorics., 23, p.1-10, 2016.

ACKNOWLEDGMENT

The Strict Terminal Connection Problem on Chordal Bipartite Graphs

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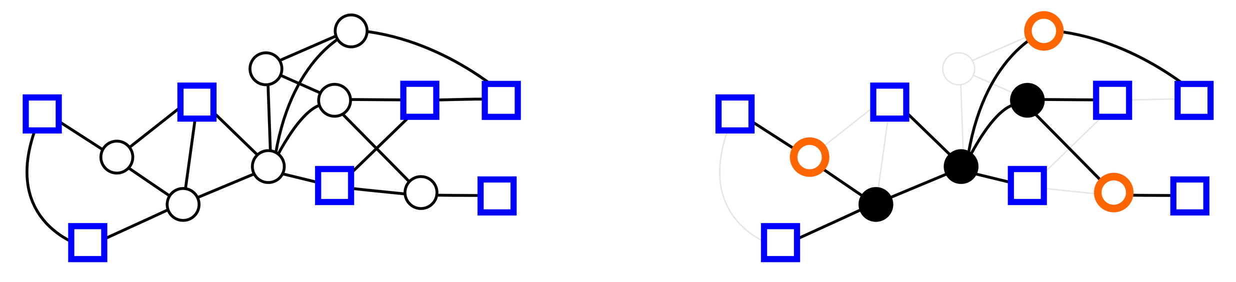
Introduction

Let G be a graph and $W \subseteq V(G)$ be a non-empty set, called *terminal* set. A *strict connection tree* of G for W is a tree subgraph of G whose leaf set is equal to W. A non-terminal vertex of a strict connection tree T is called *linker* if its degree in T is exactly 2, and it is called *router* if its degree in T is at least 3. We remark that the vertex set of every connection tree can be partitioned into terminal vertices, linkers and routers. For each connection tree T, we let $\mathsf{L}(T)$ denote the linker set of T and $\mathsf{R}(T)$ denote the router set of T. Figure 1 illustrates a graph G, a terminal set W and a strict connection tree of G for W.



(a) (b)

Figure 1: (a) Graph G and terminal set W (blue squared vertices). (b) Strict terminal connection tree T of G for W, such that $|\mathsf{L}(T)| = 3$ and $|\mathsf{R}(T)| = 3$.

Motivated by applications in information security, network routing and telecommunication, Dourado et al. [1] introduced the Strict Terminal connection problem, which is formally defined below.

Strict Terminal Connection (S-TCP)

Input: A graph G, a non-empty terminal set $W \subseteq V(G)$ and two non-negative integers ℓ and r.

Question: Does there exist a strict connection tree T of G for W, such that $|\mathsf{L}(T)| \leq \ell$ and $|\mathsf{R}(T)| \leq r$?

Table 1 summarises the complexity of S-TCP with respect to the parameters $\ell, r, \Delta(G)$, and the classes of split graphs and cographs. In addition to these results, it is known that S-TCP is NP-complete even if $\Delta(G) = 4$ and $\ell \geq 0$ is fixed, or $\Delta(G) = 3$ and ℓ is arbitrarily large [3]; on the other hand, if $\Delta(G) = 3$, the problem can be solved in time $n^{\mathcal{O}(\ell)}$ [3].

	Parameters				
Graph class	–	ℓ	r	ℓ, r	$\ell, r, \Delta(\mathbf{G})$
General	NPC [1]	NPC [1]	P for $r \in \{0, 1\}$ [2] but W[2]h [3]	XP [1] but W[2]h [3]	FPT [1, 3] but No-poly kernel [3]
Split	NPC [3]	NPC [3]	XP [3] but W[2]h [3]	XP [1, 3] but W[2]h [3]	FPT [1, 3]
Cographs	P [3]	P [3]	P [3]	P [3]	P [3]

Table 1: Computational complexity of S-TCP. (Adapted from [3].)

Contribution

In this work, we prove that S-TCP remains NP-complete when restricted to **chordal bipartite graphs**, even if $\ell \geq 0$ is fixed.

S-TCP on Chordal Bipartite Graphs

A graph G is called *chordal bipartite* if every induced cycle of G has length 4. Equivalently, a graph G is chordal bipartite if G is bipartite and every cycle of G of length at least 6 has a *chord*, i.e. an edge between two non-consecutive vertices of the cycle.

To prove that S-TCP is NP-complete on chordal bipartite graphs, we present a polynomial-time reduction from Vertex-Cover, which is formally defined below. The proposed reduction is based on the polynomial-time reduction given by Müller and Brandstädt [4] so as to prove that Steiner tree is also NP-complete on chordal bipartite graphs.

Vertex cover

Input: A graph G and a positive integer k.

Question: Does there exist a subset $S \subseteq V(G)$ such that $|S| \leq k$ and every edge of G has an endpoint in S?

Construction. Let $I = (G, k)$ be an instance of Vertex cover and $c \geq 0$ be a constant. Assume that $V(G) = \{v_1, \ldots, v_n\}$ for some positive integer $n \geq 2$. Moreover, assume that G has at least one edge, i.e. $m = |E(G)| \geq 1$. We let $f(I, c) = (H, W, \ell = c, r)$ be the instance of S-TCP defined as follows.

- For each $v_i \in V(G)$, create the gadget H_i as illustrated in Figure 2.

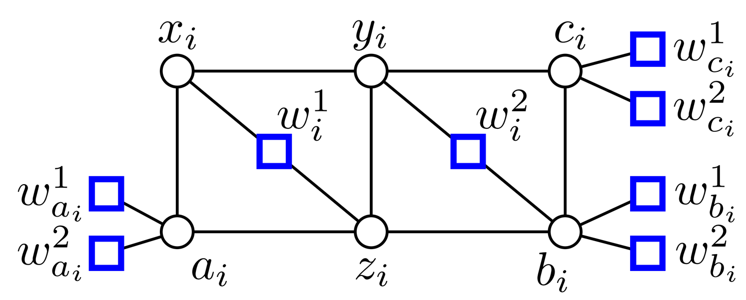


Figure 2: Gadget H_i.

- Subdivide the edge $w^1_{a_1} a_1$ of H_1 into ℓ new vertices $u_1, u_2, \ldots, u_\ell$, creating the induced path $\langle w^1_{a_1}, u_1, \ldots, u_\ell, a_1 \rangle$.
- For each pair $v_i, v_j \in V(G)$, with $i \neq j$, add the edges $x_i y_j$ and $z_i y_j$, making the subgraph of H induced by $X \cup Y \cup Z$ a complete bipartite graph with bipartition $(X \cup Z, Y)$, where $X = \{x_i \mid v_i \in V(G)\}$, $Y = \{y_i \mid v_i \in V(G)\}$ and $Z = \{z_i \mid v_i \in V(G)\}$.
- For each $v_i v_j \in E(G)$, create the gadgets H_{ij} and H_{ji} as illustrated in Figure 3.

Figure 3: Gadgets H_{ij} and H_{ji}, respectively.

- Finally, define $W = W_1 \cup W_2 \cup W_3$ and $r = k + 4n + 4m$, where $W_1 = \{w^1_i, w^2_i \mid v_i \in V(G)\}$, $W_2 = \{w^1_{a_i}, w^2_{a_i}, w^1_{b_i}, w^2_{b_i}, w^1_{c_i}, w^2_{c_i} \mid v_i \in V(G)\}$, and $W_3 = \{w^1_{p_{ij}}, w^2_{p_{ij}}, w^1_{q_{ij}}, w^2_{q_{ij}} \mid v_i v_j \in E(G)\}$.

Theorem. Let $I = (G, k)$ be an instance of Vertex-cover, such that G has at least one edge, and let $c \geq 0$ be a constant. The graph H of $f(I, c)$ is chordal bipartite. Moreover, I is a yes-instance of Vertex-cover if and only if $f(I, c)$ is a yes-instance of S-TCP.

Concluding remarks

We conclude this work by posing some open questions.

- Is S-TCP parameterized by $r \geq 2$ in XP?
- Is S-TCP parameterized by $r \geq 2$ in FPT when restricted to chordal bipartite graphs? If not, is it in XP?
- Is S-TCP parameterized by ℓ in FPT when restricted to graphs of maximum degree 3?
- In addition to cographs, on which graph classes is S-TCP in P?

References

[1] Mitre C. Dourado, Rodolfo A. Oliveira, Fábio Protti, and Uéverton S. Souza. Conexão de terminais com número restrito de roteadores e elos. In *proceedings of XLVI Simpósio Brasileiro de Pesquisa Operacional*, pages 2965–2976, 2014.

[2] Alexsander A. Melo, Celina M. H. Figueiredo, and Uéverton S. Souza. Connecting terminals using at most one router. In *proceedings of VII Latin-American Workshop on Cliques in Graphs*, volume 45 of *Matemática Contemporânea*, pages 49–57. SBM, 2017.

[3] Alexsander A. Melo, Celina M. H. Figueiredo, and Uéverton S. Souza. A multivariate analysis of the strict terminal connection problem. *Journal of Computer and System Sciences*, 111:22–41, 2020.

[4] Haiko Müller and Andreas Brandstädt. The NP-completeness of Steiner tree and dominating set for chordal bipartite graphs. *Theoretical Computer Science*, 53(2-3):257–265, 1987.

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BOUNDS FOR RANGE-RELAXED GRACEFUL GAME

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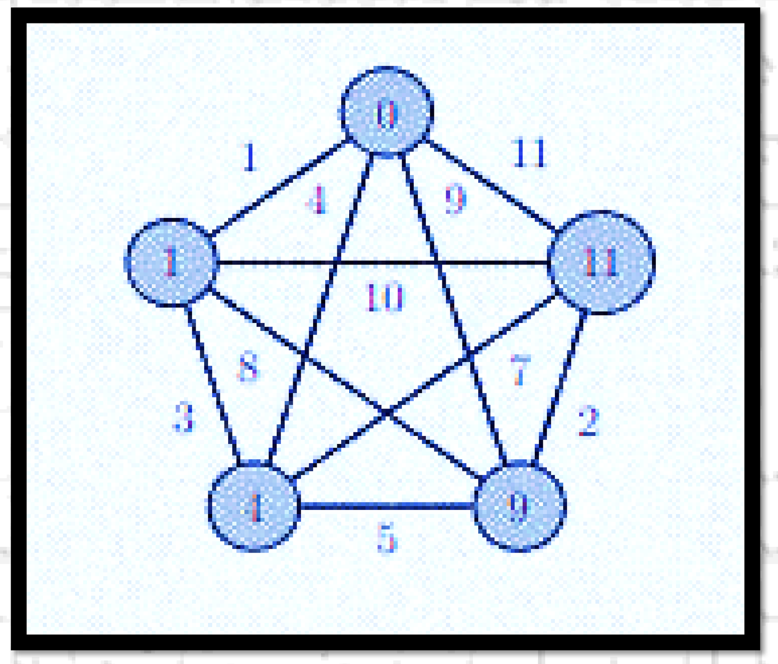
INTRODUCTION

Tuza [1] contributed to the area of graph labeling presenting many results in his seminal paper and proposing new labeling games. We investigate the Range-Relaxed Graceful game (RRG game) and present a lower bound for the number of available labels for which Alice has a winning strategy in the RRG game on a simple graph G, on a cycle and on a path graph.

RANGE RELAXED GRACEFUL LABELING

Given a graph G and the set of consecutive integer labels $\mathrm{L} = \{0, \dots, k\}$, $k \geq |E(G)|$, a labeling f: V(G) → L is said to be a Range-Relaxed Graceful Labeling if: (i) f is injective; (ii) each edge $uv \in E(G)$ is assigned the (induced) label $g(uv) = |f(u) - f(v)|$, then all induced edge labels are distinct.



RRG Labeling of K_5

RRG GAME

Two players, called Alice and Bob, alternately assign a previously unused label $f(v) \in L = \{0, \dots, k\}$, k ≥ $|E(G)|$ to an unlabeled vertex v of a given graph G. If both ends of uv $\in E(G)$ are already labeled, then the label of the edge is defined as $|f(u) - f(v)|$. A move is said legal if, after it, all edge labels are distinct. Alice's goal is to end up with a vertex labeling of the whole G where all of its edges have distinct labels and Bob's goal is to prevent it from happening.

OBJECTIVE

To investigate the Range-Relaxed Graceful game, present a lower bound on the number of consecutive nonnegative integer labels necessary for Alice to win the RRG game on a simple graph G and contribute to the study of the question posed by Tuza [1]:

TUZA'S QUESTION

Given a simple graph G and a set of consecutive nonnegative integer labels f(v) $\in$ $\mathrm{L} = \{0, \dots, k\}$, for which values of k can Alice win the range-relaxed graceful game?

RESULTS

THEOREM 1

Let G be a simple graph on n vertices and maximum degree Δ. Alice wins the RRG game on G for any set of integer labels $\mathrm{L} = \{0, \dots, k\}$, with

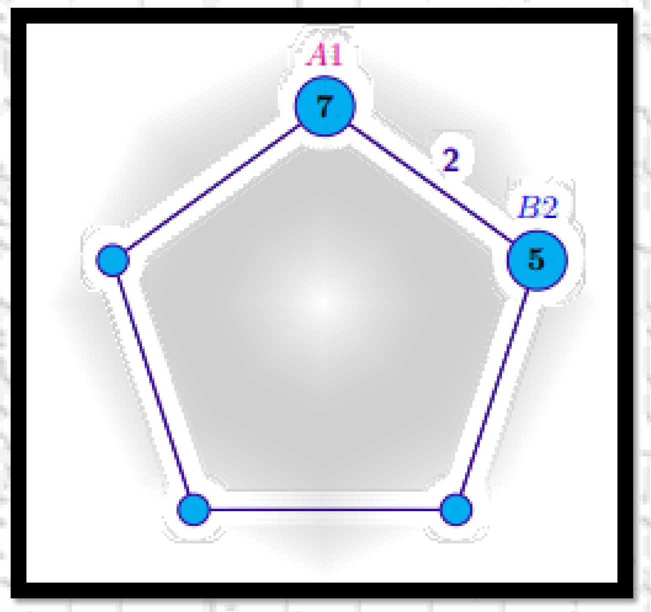
$$k \geq (2\Delta^2 + 1)(n-1) + (2\Delta + 1)\binom{n-1}{2}.$$

SKETCH OF THE PROOF

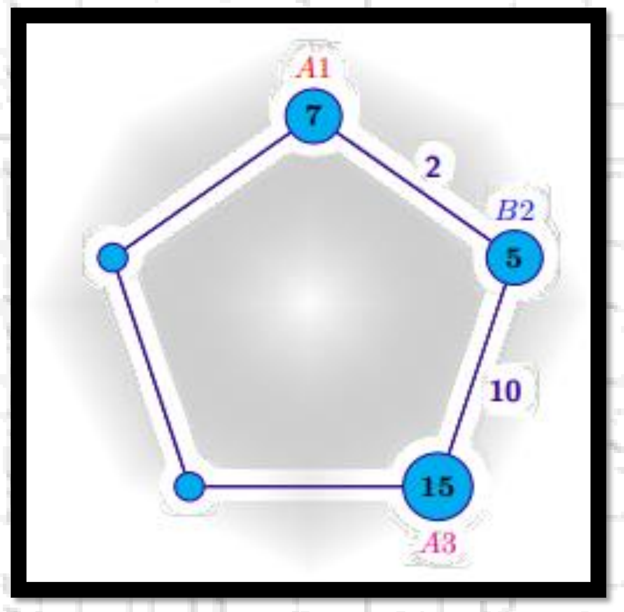
For each vertex v $\in$ V(G), we define a set of available labels L_v. When the game starts, L_v= L, for every v $\in$ V(G). At each iteration, a player assigns a label to an unlabeled vertex u from its set L_u and, then, the set of available labels of each remaining vertex is updated. Only vertex labels that can not generate repeated edge labels in future iterations can last at each set. We consider four cases that can give rise to repeated edge labels and, for each one, we count how many labels are deleted, throughout the game, from each set of available labels. From our analysis, we conclude that at most $(2\Delta^2 + 1)(n-1) + (2\Delta + 1)\binom{n-1}{2}$ labels are deleted from each set of available labels. Since |L| is greater than this value, there is always an available label at each set that can be assigned to a vertex.

EXAMPLE

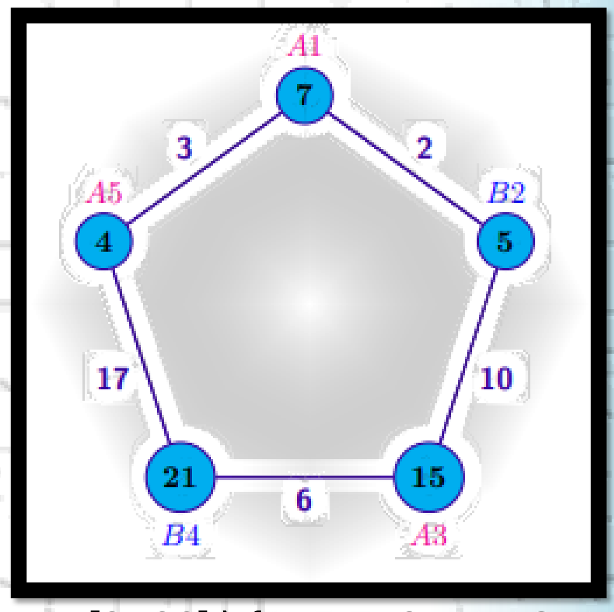
Consider C_5 and the set L = {0, 1, 2, 3, ..., 66}. Suppose that Alice starts the game by assigning label 7 to a vertex v_1. Below, we present the first three iterations, where the players play at v_1, v_2, v_3 consecutively, and we show the last iteration.



L_{v_2} = [0, 66] \ {7}
L_{v_3} = [0, 66] \ { 3, 5,7}



L_{v_4} = [0,66] \ {5,7,13,15,17,25}



L_{v_5} = [0, 66] \ {1, 5,7, 9, 11,13, 15, 17,19,21,23,27,31}

A similar proof is obtained for the following result.

THEOREM 2

Given any integer n ≥ 4, Alice wins the RRG game on the path P_n and on the cycle C_n for any set of integer labels $\mathrm{L} = \{0, \dots, k\}$, with $k \geq 9n - 17$.

REFERENCES

[1] Z. Tuza, Graph labeling games, Eletronic Notes in Discrete Mathematics 60 (2017), 61-68.

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The sandpile group of outerplanar graphs

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Introduction

We compute the sandpile groups of outerplanar planar graphs. The method can be used to determine the algebraic structure of the sandpile groups of other planar graph families.

Sandpile groups

The sandpile group was originated in statistical physics. It was the first model of a dynamical system exhibiting self-organized criticality.

The dynamics of the sandpiles are developed over a graph G in the following way. Consider a graph G with a special vertex q, called *sink*. A *configuration* c is a vector whose entries are associated with the number of grains of sand at each vertex of G. The sink vertex collects the sand quitting the system. A vertex is *stable* if the number of sand grains on it is lower than its *degree*, that is, the number of edges incident to the vertex. Otherwise, the vertex is *unstable*. A configuration is *stable* if all the non-sink vertices of G are stable. A *toppling* of an unstable configuration consists of selecting an unstable vertex v and moving $\deg(v)$ grains from v to its neighbors, such that each neighbor u receives $m(u,v)$ grains, where $m(u,v)$ denotes the number of edges between u and v. In Figure 1, we show a sequence of topplings.

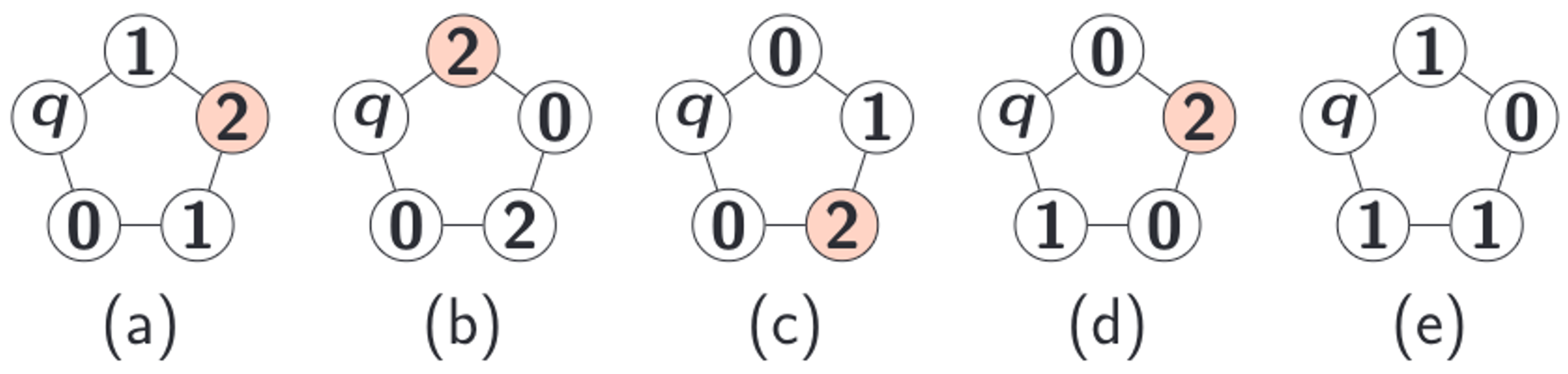

Figure 1: The sequence of topplings starts in (a) with the configuration $(1,2,1,0,q)$ over the cycle with 5 vertices. At each step, the toppled vertex is highlighted in red.

Over connected graphs with a sink, we will always obtain a stable and unique configuration after a finite sequence of topplings. The stable configuration obtained from the configuration c will be denoted by $s(c)$. The sum of two configurations c and d is performed entry-by-entry. Let $c\oplus d := s(c+d)$. A configuration c is recurrent if there exists a non-zero configuration d such that $c = c\oplus d$. Recurrent configurations play a central role in the dynamics of the Abelian sandpile model since recurrent configurations together with the $\oplus$ operation form an Abelian group known as *sandpile group* and denoted $K(G)$. An introduction to the topic can be found in [1].

For example, the recurrent configurations for the cycle with 5 vertices and sink vertex q are $(0,1,1,1,q)$, $(1,0,1,1,q)$, $(1,1,0,1,q)$, $(1,1,1,0,q)$ and $(1,1,1,1,q)$. Could the reader verify that these configurations form an Abelian group with the $\oplus$ operation? Which configuration is the identity?

Smith normal form and graphs

Let $GL_n(\mathbb{Z})$ denote the group of $n\times n$ invertible matrices with entries in the integers whose inverses also have entries in the integers. Two matrices M and N are *equivalent* if there exist two matrices $P, Q \in GL_n(\mathbb{Z})$ such that $M = QNP$. The *Smith normal form* of the matrix M is the unique diagonal matrix $diag(d_1,\dots,d_r,0,\dots,0)$ equivalent to M such that r is the rank of M and $d_i | d_j$ for $i<j$. The integers $d_1,\dots,d_r$ are called *invariant factors*.

Let G be a planar graph with s interior faces $F_1,\dots,F_s$, let $c(F_i)$ denote the number of edges in the cycle bounding F_i. We define the *cycle-intersection matrix*, $C(G) = (c_{ij})$ to be a symmetric $s\times s$ matrix, where $c_{ii} = c(F_i)$, and c_{ij} is the negative of the number of common edges in the cycles bounding F_i and F_j, when $i\neq j$.

Lemma [2]. Let $d_1,\dots,d_r$ be the invariant factors of $C(G)$, where G is a planar graph. Then $K(G)\approx \mathbb{Z}_{d_1}\oplus\cdots\oplus\mathbb{Z}_{d_r}$.

Sandpile groups of outerplanar graphs

We call a graph *outerplanar* if it has a planar embedding with the outer face containing all the vertices. The *weak dual* graph G_* is constructed the same way as the dual graph but without placing the vertex associated with the outer face. A graph G is biconnected outerplanar if and only if its weak dual is a tree. Note that $C(G) + A(G_*) = diag(c(F_1),\dots,c(F_s))$, where $A(G)$ is the adjacency matrix of G.

A *2-matching* M is a set of edges of a graph G such that each vertex of G is incident with at most 2 edges of M. Let denote by G°, the graph G where each vertex has a loop added. Given a *2-matching* M of G°, let $\Omega(M)$ denote the set of loops in M. A 2-matching M of G° is *minimal* if there is no 2-matching M' of G° such that $\Omega(M')$ is not contained in $\Omega(M)$ and $|M'|=|M|$. The set of minimal 2-matchings of a tree with loops T° with k edges will be denoted by $2M_k(T^\circ)$. Let $d(M)$ denote the determinant of the submatrix of $C(G) = diag(c(F_1),\dots,c(F_s)) - A(T)$ created by taking the rows and columns associated with the loops of M of T°.

Theorem [2]. Let G be a planar biconnected graph whose weak dual is the tree T with n vertices. Let $\Delta_k = gcd(\{d(M) : M\in 2M_k(T^\circ)\})$. Then the spanning-tree number $\tau(G)$ coincides with Δ_n and $K(G)\approx \mathbb{Z}_{\Delta_1}\oplus\mathbb{Z}_{\frac{\Delta_2}{\Delta_1}}\oplus\cdots\oplus\mathbb{Z}_{\frac{\Delta_n}{\Delta_{n-1}}}$.

References

[1] C.J. Klivans. The Mathematics of Chip-Firing. CRC Press, Taylor & Francis Group, 2018.
[2] C.A. Alfaro y R.R. Villagrán. The structure of sandpile groups of outerplanar graphs. [arXiv:2005.01314]

Properties of fullerene graphs with icosahedral symmetry

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Abstract

Fullerene graphs are based on a famous carbon allotrope and have become a popular class of graphs (see references in [2]). They are characterized as 3-regular and 3-connected planar graphs, with only pentagonal or hexagonal faces. The fullerene graph with icosahedral symmetry is a particular class of fullerene graphs with precisely 12 pentagonal faces. Moreover, the midpoints of its pentagonal faces form the planning of an icosahedron. They can be described by a vector (i, j), where $j \geq i \geq 0$ and $j > 0$, determining the graph $G_{i,j}$. In 2013, Andova and Skrekovski presented and proved formulas to compute the diameter of the graphs $G_{0,j}$ and $G_{j,j}$. Moreover, they presented a conjecture stating a lower bound for the diameter of all fullerene graphs. Therefore, in this study, we investigate properties of fullerene graphs with icosahedral symmetry. We show that, for $i, j \in \mathbb{N}^*$, $j \geq i$, every graph $G_{i,j}$ contains a reduced $G_{0,j-i}$ and that every graph $G_{i,j}$ is contained in an augmented $G_{j,j}$.

Introduction

An (undirected) graph G is a geometric object composed of a set of vertices and edges. Figure 1 shows a simple graph, i.e., a graph that does not have more than one edge between the same pair of vertices, and has no edges intersecting a vertex to itself. Before investigating the fullerene graphs, we require some graph theory definitions and concepts. A graph G is k-regular if all of its vertices have degree k. A graph G is k-connected if it remains connected whenever fewer than k edges are removed.

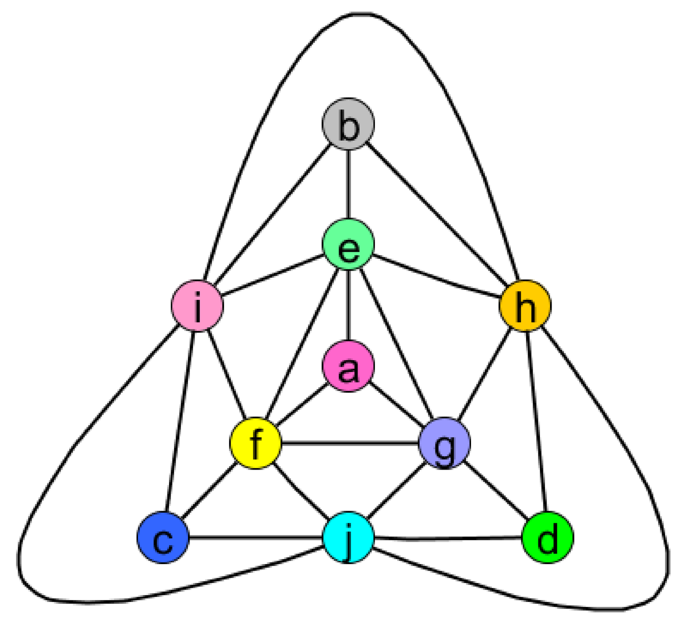

Figure 1: A simple Graph G.

A graph G is planar if it has an immersion in the plane so that its edges intersect only at their endpoints. The diameter of a graph is the maximum distance between any pair of vertices of G.
As an example, Figure 2 displays the Fullerene graph C_{60}: it is planar (no two edges intersect each other); it is 3-regular (all vertices have degree 3); and it is 3-connected (it remains connected if we remove one or two edges). Fullerene graphs are 3-regular and 3-connected planar graphs with only pentagonal and hexagonal faces. Figure 2 shows the Fullerene graph C_{60}.

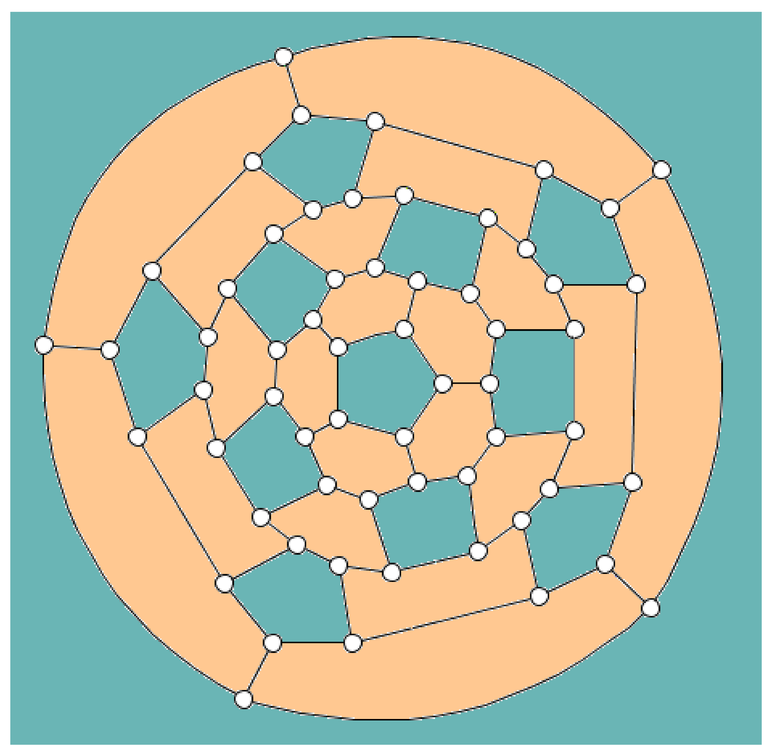

Figure 2: Fullerene graph C_{60}.

Icosahedral Symmetry

Fullerene graphs with icosahedral symmetry have exactly 12 pentagonal faces. All other faces are hexagons. Moreover, their pentagonal faces shape the planning of an icosahedron. They are described by $G_{i,j}$, $i, j \in \mathbb{N}^*$, $j \geq i$, where i and j determine the distance between the vertices, with i as the number of hexagons in direction $\vec{x}$ and j as the number of hexagons in direction $\vec{y}$ (see Figure 3). Figure 3 displays the planning of the graph Fullerene graph with icosahedral symmetry $G_{1,4}$.

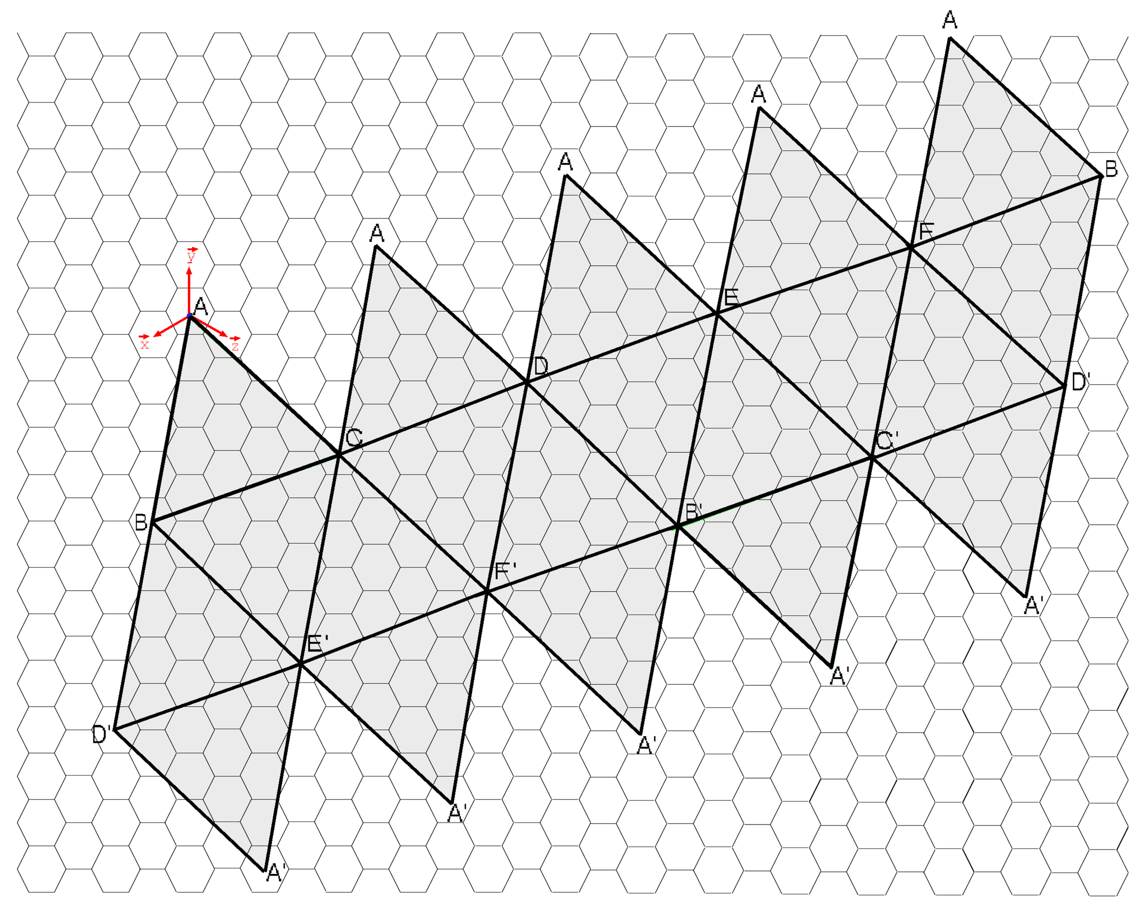

Figure 3: Planning of the graph $G_{1,4}$.

Results

Theorem 1

Every fullerene graph with icosahedral symmetry $G_{i,j}$, $i, j \in \mathbb{N}^*$, $j \geq i$, contains a reduced graph $G_{0,j-i}$.

Theorem 2

Every fullerene graph with icosahedral symmetry $G_{i,j}$, $i, j \in \mathbb{N}^*$, $j \geq i$, is contained in an augmented graph $G_{j,j}$.

The proofs of both theorems are based on vectorial operations of the vector $\vec{x}$ and $\vec{y}$ and the hexagonal lattice's symmetry characteristics. Figure 4 displays the results of Theorems 1 and 2 for the graph $G_{1,4}$. The black triangle corresponds to a section of $G_{1,4}$. As a visual proof of Theorem 1, note that the blue triangle corresponds to the graph $G_{0,3}$, entirely included in $G_{1,4}$. Similarly for Theorem 2, the red triangle corresponds to the $G_{4,4}$, which wholly contains the graph $G_{1,4}$.

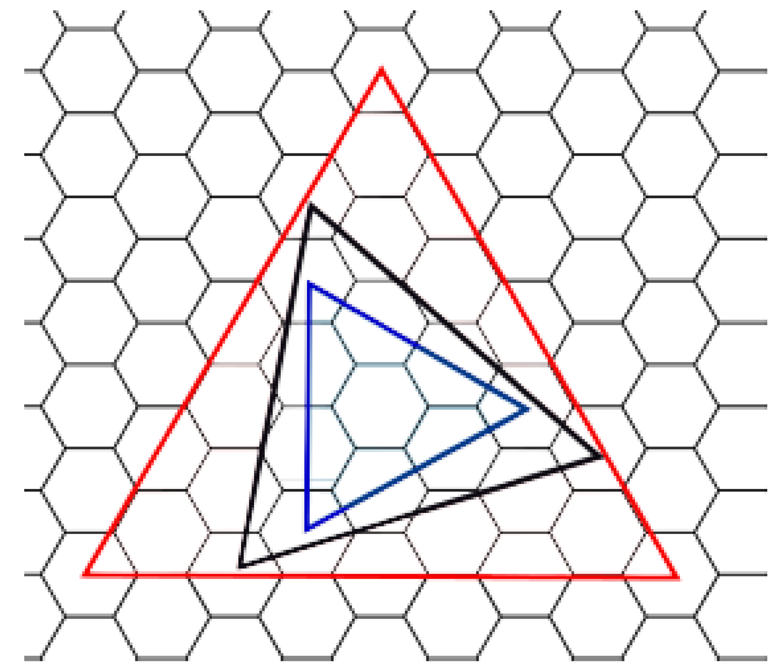

Figure 4: Example of Theorems 1 and 2 for $G_{1,4}$.

References

[1] Andova, V; Skrekovski, R. Diameter of Fullerenes Graphs with Full Icosahedral Symmetry. *MATCH*, 2013.
[2] Nicodemos, D; Stehlik, M. Fullerene graphs of small diameters. *MATCH*, 2017.

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A relationship between D-eigenvalues and diameter.

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Objective

Our goal is to provide examples of connected graphs having diameter d and less than d + 1 D-eigenvalues. This answers a question stated by Atik and Panigrahi in [4, Problem 4.3].

Introduction

It is known, by [1], that if G is a graph of diameter d then the adjacency matrix of G has at least $d+1$ distinct eigenvalues. We can see in [2] that distance-regular graphs actually attains this minimum, that is, they have exactly $d+1$ distinct adjacency eigenvalues.

A simple connected graph G is called *distance-regular* if it is regular, and if for any two vertices $x, y \in V(G)$ at distance i, there are constant number of neighbors c_i and b_i of y at distance $i-1$ and $i+1$ from x, respectively.

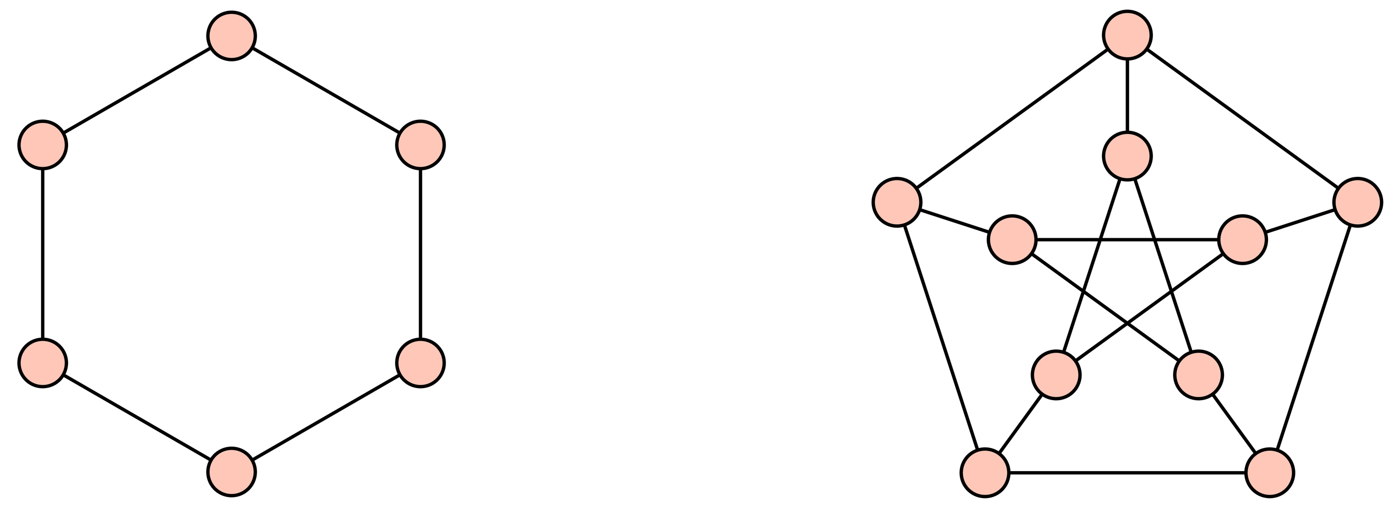

Figure 1: C_6 and Petersen graph are examples of distance regular graphs. More generally, C_n is a distance regular graph.

It seems reasonable to ask whether these results can be extended to the eigenvalues associated with the distance matrix (D-eigenvalues) of a simple connected graph G. Indeed, Lin et al. [5] ask if, for a graph G with diameter d, its distance matrix has at least $d+1$ distinct eigenvalues. Atik and Panigrahi give a negative answer to this problem in [4]. Moreover, they prove that a distance-regular graph with diameter d has at most $d+1$ distinct D-eigenvalues and leave the following question: "*Are there connected graphs other than distance regular graphs with diameter d and having less than $d+1$ distinct D-eigenvalues?*".

In what follows, we answer this question positively by given two examples of connected graphs with diameter d having less than $d+1$ distinct D-eigenvalues.

Examples

In our example we consider two bipartite graphs G_1 and G_2 described in figures 2 and 3. We have that $|V(G_1)| = 20$ and $|V(G_2)| = 70$, and that $diam(G_1) = 5$ and $diam(G_2) = 7$. However, both graphs have exactly four distinct D-eigenvalues. These graphs and their respectively D-spectrum are shown as follows.

In particular, our examples are both *distance-biregular* graphs, for a precise definition see [3].

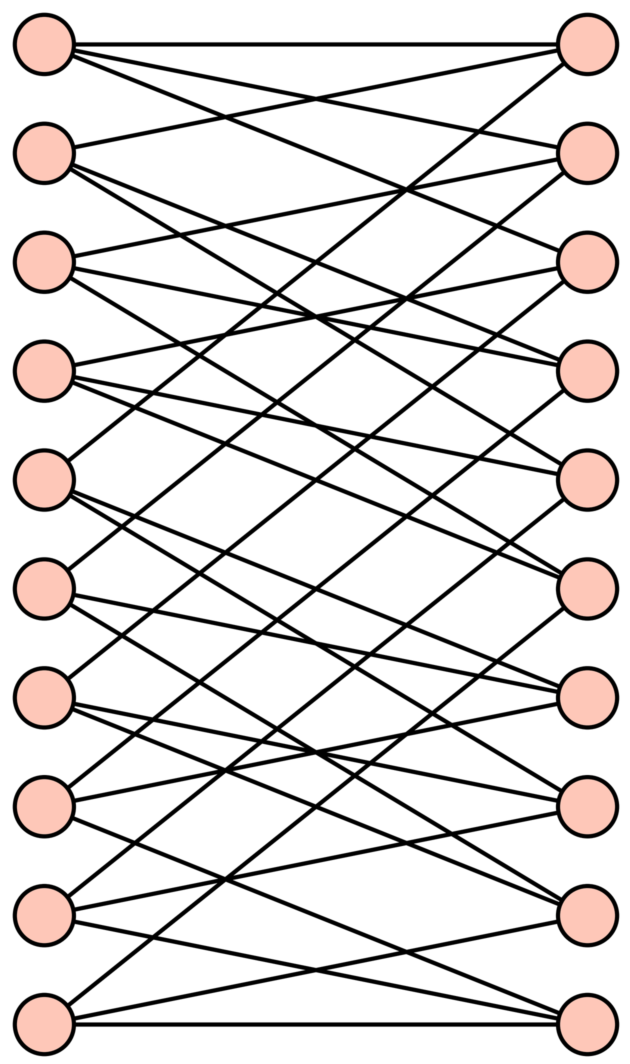

Figure 2: The graph G_1.

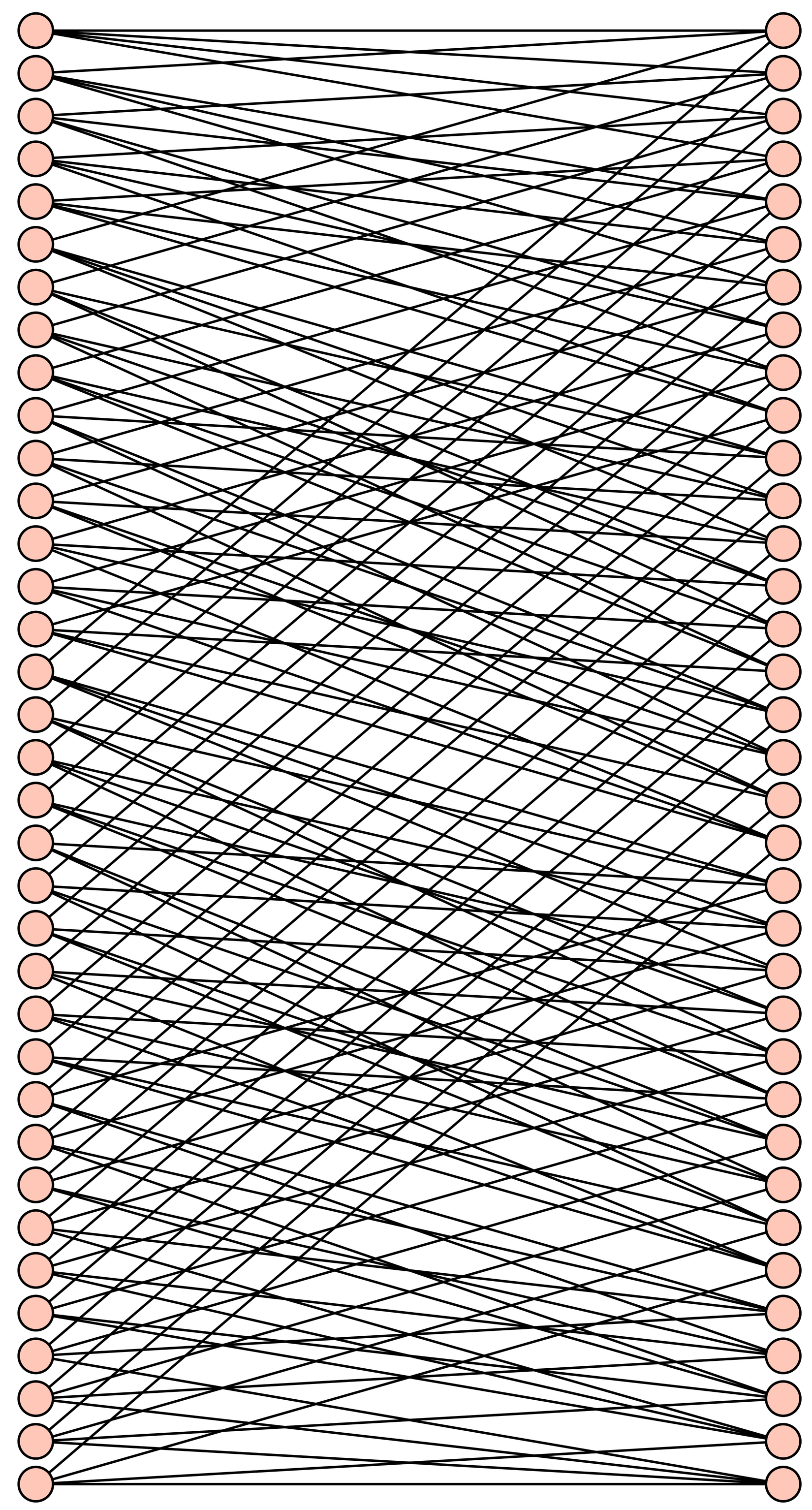

Figure 3: The graph G_2.

$$spect\,(G_1) = \begin{bmatrix} 50 & 0 & -2 & -12 \\ 1 & 14 & 1 & 4 \end{bmatrix} \qquad spect\,(G_2) = \begin{bmatrix} 245 & 0 & -5 & -40 \\ 1 & 62 & 1 & 6 \end{bmatrix}$$

Conclusions

About the problem proposed by Atik and Panigrahi in [4], it can be said that there are other connected graphs with diameter d, in addition to distance regular graphs, having less than $d+1$ distinct D-eigenvalues. More specifically, the graphs presented in this work have exactly 4 distinct D-eigenvalues. For future works, we are interested in characterize a class of distance-biregular graphs with this property.

References

[1] A.E. Brouwer, W.H. Haemers, Spectra of Graphs, Springer, 2011.

[2] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.

[3] C. Delorme, Distance biregular bipartite graphs, European jornal of Combinatorics, v. 15, p. 223–238, 1994.

[4] F. Atik, P. Panigrahi, On the distance spectrum of distance regular graphs, Linear Algebra Appl., v. 478, p. 256–273, 2015.

[5] H. Lin, Y. Hong, J.F. Wang, J.L. Shu, On the distance spectrum of graphs, Linear Algebra Appl., v. 439, p. 1662–1669, 2013.

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Conflict Free Closed Neighborhood Coloring Game

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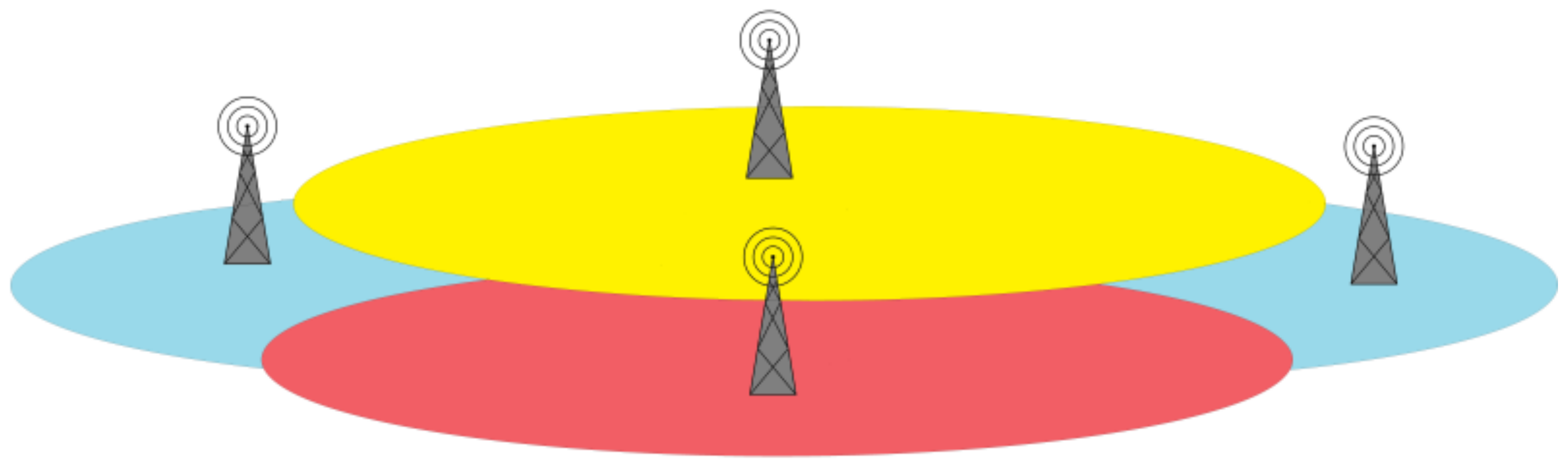
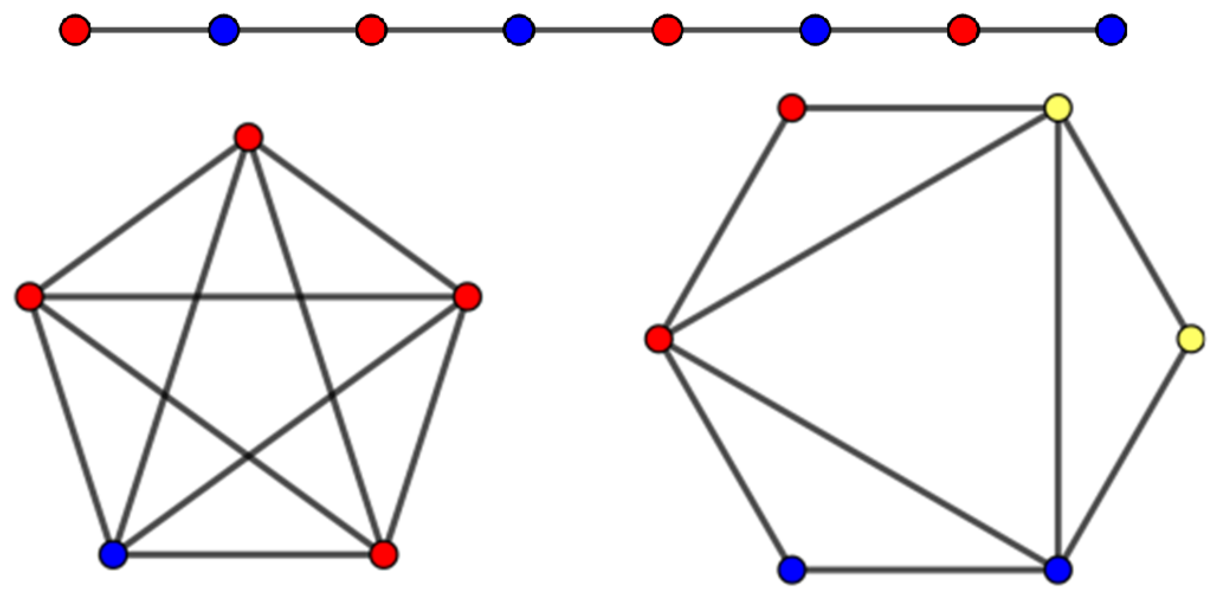
Introduction



In Cellular Networks, communication between bases and mobile devices is established via radio frequencies. Interference occurs if one particular device communicates with two different bases that have the same frequency. So, every device must contact a base with an unique frequency and, since having a lot of different frequencies is expensive, it's important trying to minimize their quantity, in a way that there exists no interference.
With that motivation, in 2002, Even, Lotker, Ron and Smorodinsky [1] introduced the concept of *Conflict Free coloring* in a geometric scenario, which itself led to the study of CFCN coloring in graphs, and, in 2015, Gardano and Rescigno [2] proved that CFCN coloring is NP-complete.
Inspired by this problem, and by the well known coloring game, we introduce a game theoretical approach to CFCN coloring, and determine the minimum number of colors necessary for Alice to have a winning strategy in the case of Complete Graphs.

CFCN Coloring

A *CFCN coloring* of a graph G is an assignment of colors to the vertices of G such that each vertex v in G has an uniquely colored vertex in its *closed neighborhood* $N[v]$ (the set of all vertices adjacent to v including itself). A *CFCN k-coloring* of a graph G is a CFCN coloring with at most k colors. We say that $N[v]$ is *fully colored* if each vertex of $N[v]$ has a color assigned to it. A graph together with a CFCN k-coloring is said to be *CFCN k-colored.*



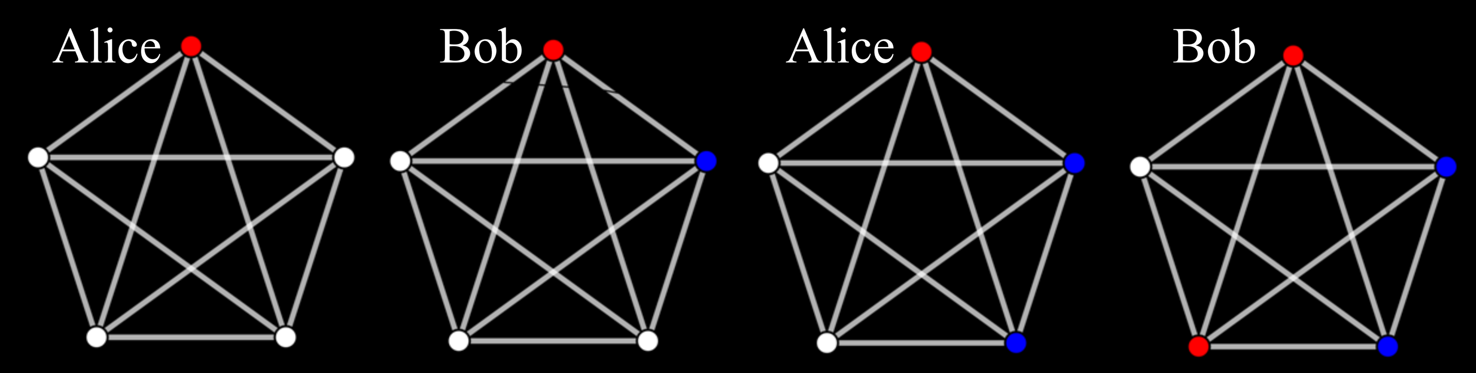
CFCN Coloring of Complete Graphs

Complete graphs have a CFCN 2-coloring, by coloring one vertex with the first color and the others with the second.

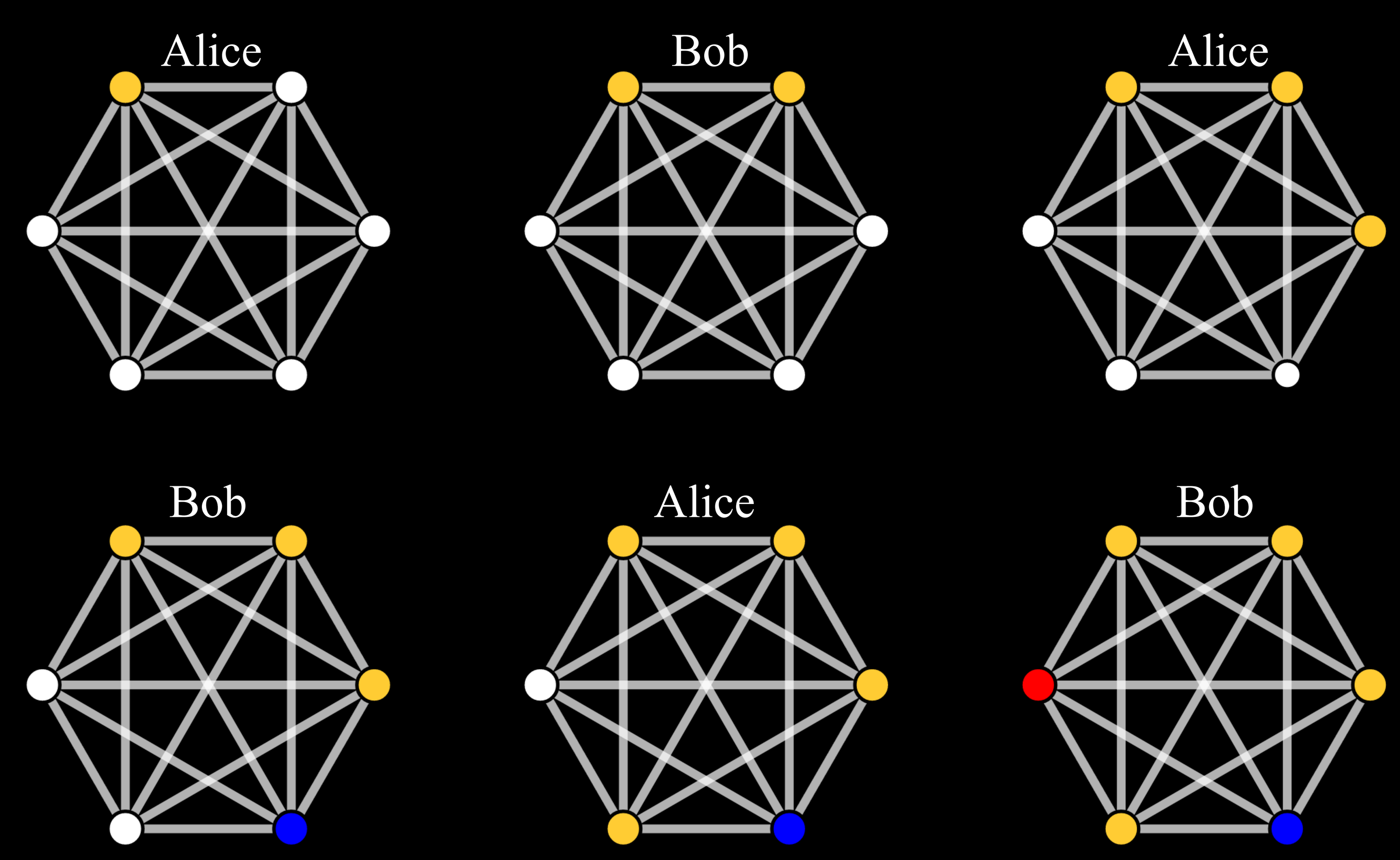
CFCN Coloring Game

Given a graph G and $k>1$ colors, two players, Alice and Bob, take turns coloring vertices of G such that at each turn for every v with a fully colored $N[v]$, the induced subgraph $G[N[v]]$ is CNCF k-colored. The goal of Alice is to obtain a CFCN k-coloring of G while Bob does his best to prevent it. Alice wins if at the end G has a CFCN k-coloring; otherwise Bob wins.

We refer to the next figure for a CFCN 2-coloring game on K_5, where white vertices are uncolored ones. The game ends on the 4th turn because, no matter which color Alice chooses for the 5th turn , it creates a fully colored neighborhood that is not CFCN 2-colored.



The figure below shows a CFCN 3-coloring game on K_6, where white vertices are uncolored ones. The game ends on the 6th turn because the Graph is CFCN 3-colored.



Results

Theorem: *Alice wins CFCN k-coloring game on a complete graph G on n vertices if and only if* $k > \lceil \frac{n}{4} \rceil$.

Sketch of the proof: Let $k>1$ be the number of available colors. Without loss of generality, Alice starts playing in any vertex with color 1.
We claim that Alice always wins if $n \le 4$ and $k=2$ (winning for any k).
Indeed if $n=1$, Alice colors the vertex with 1. If $n=2$, Alice colors a vertex with 1 and then Bob is forced to color the other vertex with 2. If $n=3$, on the 1st turn she colors a vertex with 1 and on the 3rd with 2. If $n=4$, Alice guarantees that by the 3rd turn, without loss of generality, there are two vertices colored with 1 and one vertex colored with 2, thus Bob has to finish the coloring with 1 or another color different from 2.
If n>4, the proof is based on the following strategy.
Assume that $k \le \lceil \frac{n}{4} \rceil$, Bob colors a vertex with 2. If Alice colors the next vertex with 1 (resp. 2), Bob colors a vertex with 2 (resp. 1). On the following turns, independently of the colors chosen by Alice, Bob chooses the other colors twice and the game ends. If Alice colors the next vertex with a color c not in $\{1,2\}$, then Bob colors the next ones with 1, 2, c, and then chooses the remaining colors twice. In any case Bob wins the game. $\square$
Now assume $k > \lceil \frac{n}{4} \rceil$. If the number of vertices is even then Alice always plays 1. If the number of vertices is odd then Alice does the same strategy until her last turn, in which she chooses 1 or one of the remaining colors. In both cases, Alice wins because the graph doesn't have enough vertices for Bob to guarantee that each colors is used twice.

References

[1] G. Even, Z. Lotker, D. Ron, S. Smorodinsky, Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks, SIAM J. Comput. 33 (1) (2004) 94–136.
[2] L. Gardano, A. Rescigno, Complexity of conflict-free colorings of graphs, Theor. Comput. Sci. 566 (2015) 39-49

Acknowledgements

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TEACHING NEWTON's BINOMIAL WITH GENETICS

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Introduction

The teaching of Combinatorial Analysis is still done in a very mechanical way by some teachers who, for the most part, memorize formulas without real content domain. This practice is repeated superficially, thus not stimulating combinatorial reasoning [1]. The vast majority of books and websites present this content only through formulas, without showing their relationship to applicability, making it difficult for students to learn. Thus, we present an application of Newton's Binomial, as a way of intuitively teaching such content. Since the binomial is used in many areas, we choose an interdisciplinary study with Biology, more specifically, in Genetics. In this work, we show how the binomial is presented in Genetics and why it is so important to understand certain characteristics inherited from our ancestors, such as the color of the eyes. We use concepts of Polygenic (or Quantitative) Inheritance [2].

Objective

The aim of this work is to present a new way of teaching Newton's Binomial through an interesting application related to Genetics, without the early use of formulas. In addition, we show the relationship between the binomial and the combinatorial analysis: how is the combination present in terms of the binomial and what do they represent in its expansion?

Methodology

The methodology consisted of studying applications in genetics that involve Newton's Binomial; choosing an application and developing playful material for teaching the content which included simulations and short films.

Application

In genetics, phenotype refers to characteristics of the individual that can be visible or detectable, and polygenes are groups of genes that produce repeated variations. Polygenic inheritance refers to a single inherited *phenotypic trait* that is controlled by two or more different *genes*. The interaction that occurs between genes (polygenes) that convey the inherited characteristics happens in such a way that each one of them is responsible for a portion of the resulting phenotype. The pattern of inheritance distribution, in this case, follows the pattern of Newton's Binomial, $(p + q)^n$, where n is the number of polygenes, p represents the dominant genes (B and G) and q represents the recessive (b). In our study, we develop Newton's binomial for the eye color problem [3].

The eye color results from at least two genes. The first, OCA2 (oculocutaneous albinism II), comes in two forms: B (brown) and b (blue). The second gene, called GEY (green eye color), comes also in two forms: G (green) and b (blue). The first thing to notice is that the gene B is dominant over both G and b. And, as well, G is dominant over b (recessive). In other words, a person heterozygote BbGb, despite having the gene G, she has brown eyes. Thus, we could calculate the probability of their progenies being born with brown, green or blue eyes shown in Table 1 [4]. Other genes produce spots, rays, rings and pigment diffusion patterns.

Table 1: Cross between two heterozygotes

	BB	Bb	bB	bb
GG	BBGG	BbGG	bBGG	bbGG
Gb	BBGb	BbGb	bBGb	bbGb
bG	BBbG	BbbG	bBbG	bbbG
bb	BBbb	Bbbb	bBbb	bbbb

References

[1] ALCANTARA, C. F. B. ; DANTAS, S. ; PARÁ, T. S. . O Jogo da Lanchonete: Sustentabilidade e Ensino dos Princípios Aditivo e Multiplicativo. Educação Matemática em Revista, v. 25, p. 12-25, 2020.

[2] SILVA, S. D. Estudo do Binômio de Newton. 2013. 70 f. Dissertação (Mestrado) - Curso de Matemática, Universidade Federal da Paraíba, João Pessoa - Pb, 2013.

[3] SILVA, P. H. Usos do Binômio de Newton em Diferentes Contextos. 2016. 39 f. TCC (Graduação) - Curso de Matemática, Universidade Estadual Paulista, SP, 2016.

[4] **Eye Color.** Dr. Barry Starr, Stanford University. Retrieved from https://genetics.thetech.org/ask/ask203 ; access in 07/15.

Acknowledgement

The strong pseudoachromatic number of split graphs

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Introduction

Given a graph G and a set of colors C, a vertex coloring $\alpha : V(G) \to C$ is an assignment of colors from C to the vertices of G. If there are no adjacent vertices with the same color, α is proper. Let β be a not necessarily proper vertex coloring of G such that for every two distinct colors, there are adjacent vertices in G assigned these colors. If β is proper, then it is an achromatic (or complete) coloring of G. If β is nonproper, then it is a pseudoachromatic (or nonproper complete) coloring of G. If β is a pseudoachromatic coloring of G and for every color i, there is an edge of G whose both vertices are colored i, then β is a strong pseudoachromatic (or strong nonproper complete) coloring of G. (See Figure 1.) The maximum number of colors of a strong pseudoachromatic coloring is its strong pseudoachromatic number (or strong achromatic number), $\psi^*(G)$.

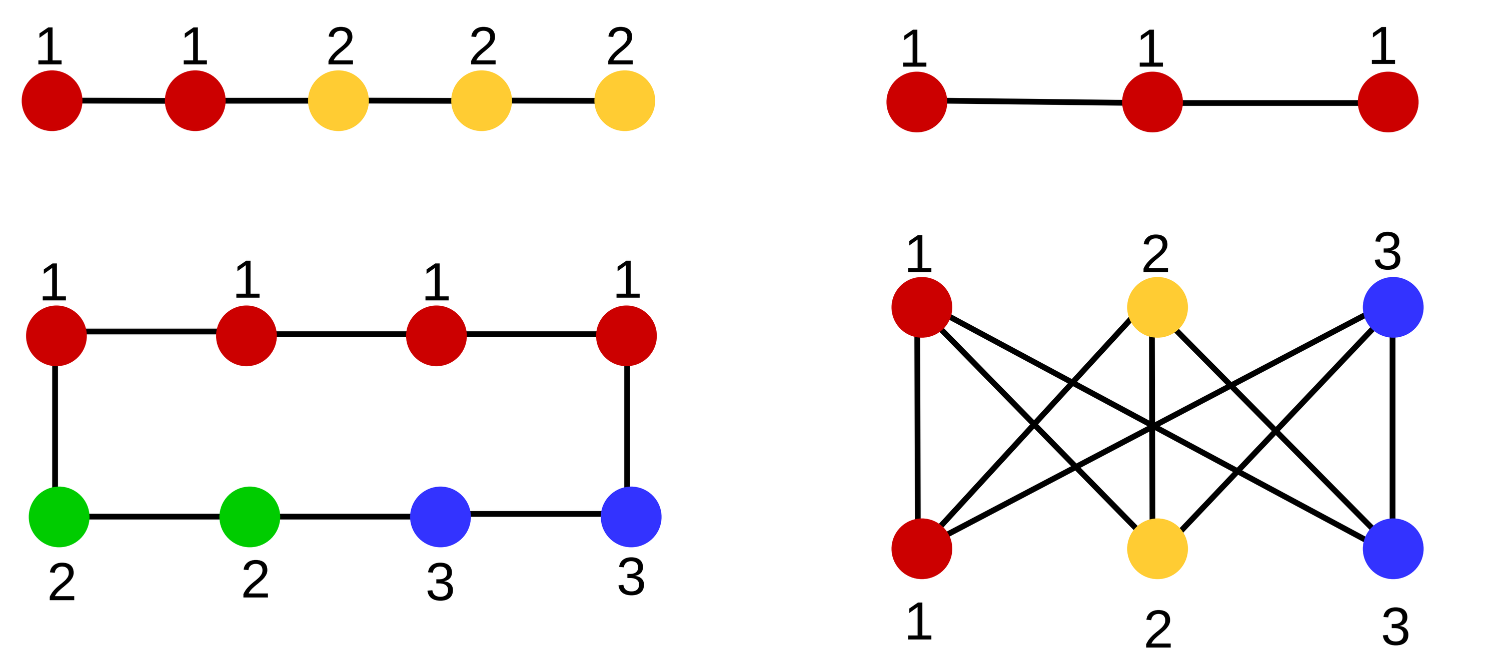

Figure 1: A strong pseudoachromatic coloring for P_5, P_3, C_8 and $K_{3,3}$.

Historical context

Chartrand and Zhang [1, p. 329] presented the strong pseudoachromatic coloring (they use the term "nonproper complete coloring") in the Study Project 6 [1, p. 442]. They ask for bounds to the pseudoachromatic number in terms of the number of edges and suggest investigating the strong pseudoachromatic number of paths and graphs in general.

Previous results

Although there are many studies of the achromatic coloring (see Chartrand and Zhang [1, p. 329]), the only published paper on strong pseudoachromatic coloring is by Liu, Li, and Liu [2]. They present bounds for the strong pseudoachromatic number in the general case and determine the strong pseudoachromatic number of complete graphs, paths, cycles, complete multipartite graphs, complete biequipartite graphs from which a perfect matching is deleted, wheels, fans, and some line graphs.

Motivation

Let G be a graph and $\beta : V(G) \to C$ be a pseudoachromatic coloring of G. By the definition of pseudoachromatic coloring, for each color $i \in C$, there must be an edge whose both vertices are colored i. So, $|C|$ is at most the size of a maximum matching of G, denoted by $\alpha'(G)$. Consequently, $\psi^*(G) \leq \alpha'(G)$. By the previous results [2], this upper bound is tight, since $\psi^*(G) = \alpha'(G)$ when G is a complete graph or a complete multipartite graph. (See Figure 2.)

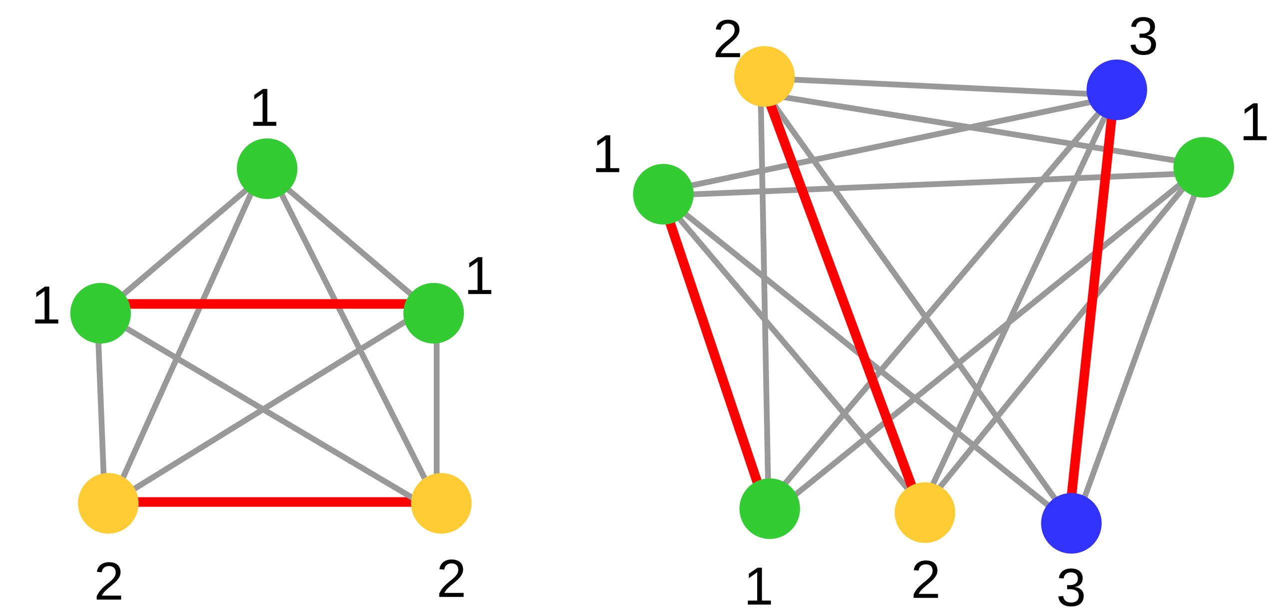

Figure 2: A maximum strong pseudoachromatic coloring and a maximum matching (in red) of K_5 and $K_{2,2,3}$.

A graph G is a split graph iff $V(G)$ can be partitioned into a maximum clique Q and a stable set S. Figure 3 exhibits a split graph. The size of Q is denoted $\omega(G)$. The bipartite subgraph of G obtained by removing the edges between vertices of Q is denoted B_G.

Our contribution

Theorem 2 If G is a split graph, then

$$\alpha'(G) = \alpha'(B_G) + \left\lfloor \frac{\omega(G) - \alpha'(B_G)}{2} \right\rfloor .$$

Theorem 3 If G is a split graph, then $\psi^*(G) = \alpha'(G)$.

Sketch of proof. Since $\psi^*(G) \leq \alpha'(G)$ for any graph G, it is sufficient to exhibit a strong pseudoachromatic coloring with $\alpha'(G)$ colors. Let Q be a maximum clique in G. Consider a maximum matching M_B in B_G and a maximum matching M_Q in $G[Q \setminus V(B_G)]$. For each edge in $M_B \cup M_Q$, assign a new color to its vertices (the same color for both vertices). Assign a color previously used to the remaining vertices. Since, for each color i, there is a vertex in Q colored i and an edge of $M_B \cup M_Q$ whose vertices are colored i, we have a strong pseudoachromatic coloring. $\square$

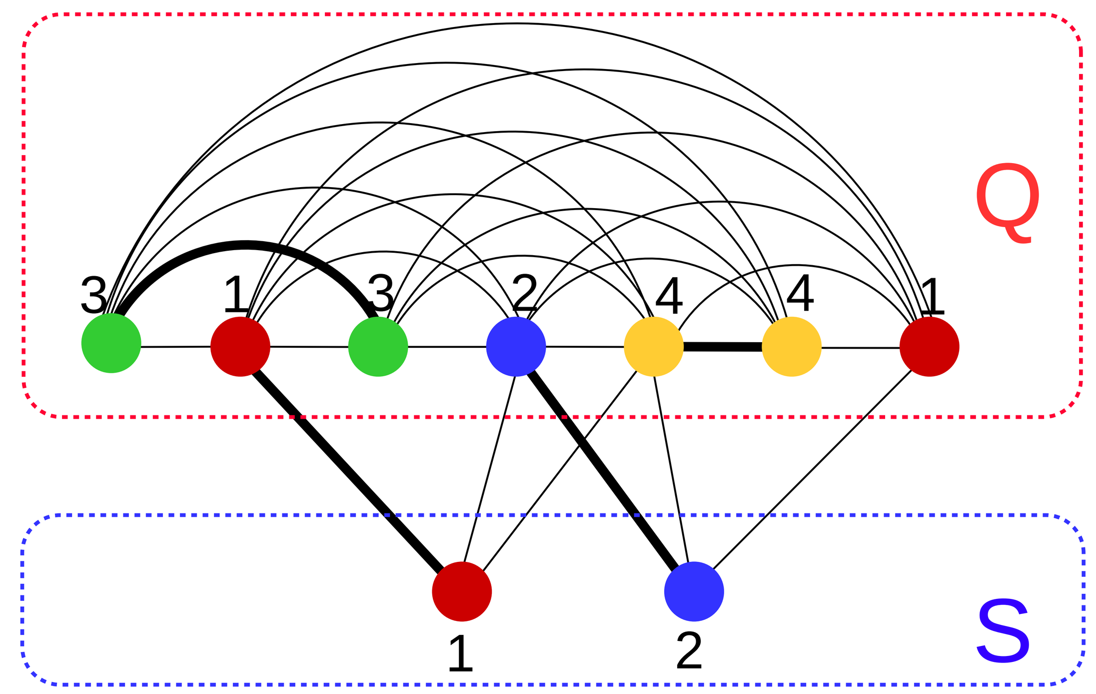

Figure 3: A maximum strong pseudoachromatic coloring of a split graph.

References

[1] Gary Chartrand and Ping Zhang.
Chromatic graph theory.
CRC press, 2009.

[2] Muhuo Liu, Rui Li, and Bolian Liu.
The strong pseudoachromatic number of a graph.
Utilitas Mathematica, 98:3--21, 2015.

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Enumeration of cospectral and coinvariant graphs

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Introduction

Starting from the eigenvalues of a matrix associated to a graph, spectral graph theory seeks to deduce combinatorial properties of the graph. For this, we associate a graph G to a matrix M and analyze the eigenvalues of M. Motivated by the graph isomorphism problem, it is of interest to study, for a graph G, what fraction of all graphs is uniquely determined by the M-spectrum of G. We propose representing a graph using the Smith Normal Form (SNF) of certain distance matrices. We provide numerical evidence that this algebraic representation may do a better job in distinguishing graphs.

Spectrum and invariant factors

The eigenvalues of a matrix $M(G)$ associated with a graph G are called the M-*spectrum* of G, which is the multiset that allows multiple instances for each of its eigenvalues. M-*cospectral* graphs are graphs that share the same M-spectrum.

The *Smith Normal Form* of an integer matrix M, denoted by $SNF(M)$, is the unique diagonal matrix such that $SNF(M) = diag(\mathrm{d}_1, \dots, \mathrm{d}_r, 0, \dots, 0) = PMQ$ for invertible matrices $P, Q \in GL(n, \mathbb{Z})$ such that r is the rank of M and $d_i | d_j$ for $i < j$. The *invariant factors* of M are the integers in the diagonal of $SNF(M)$. We say that graphs G and H are M-*coinvariant,* if the SNFs of integer matrices $M(G)$ and $M(H)$ are the same.

Enumeration

We focus on the following matrices for connected graphs: the adjacency matrix A, the Laplacian matrix L, the distance matrix D, the signless Laplacian matrix Q, the distance Laplacian matrix D^L and the distance signless Laplacian matrix D^Q.

Extensive research has been devoted to understand cospectral graphs, but much less has been dedicated to understand coinvariant graphs and its potential to characterize graphs. The reason for this could be that for matrices A, L, Q and D, there is a large proportion of connected graphs having a coinvariant graph, as Figure 1.1 shows.

Figure 1.2 displays the number of cospectral and coinvariant graphs for matrices D^L and D^Q. We also include the spectral graphs for matrix Q, since according to Figure 1.1, this would be the best invariant for distinguishing graphs using only the spectrum. According to our results, the SNF of D^Q performs better than the spectrum for distinguishing graphs for the considered matrices. Details can be found in [1].

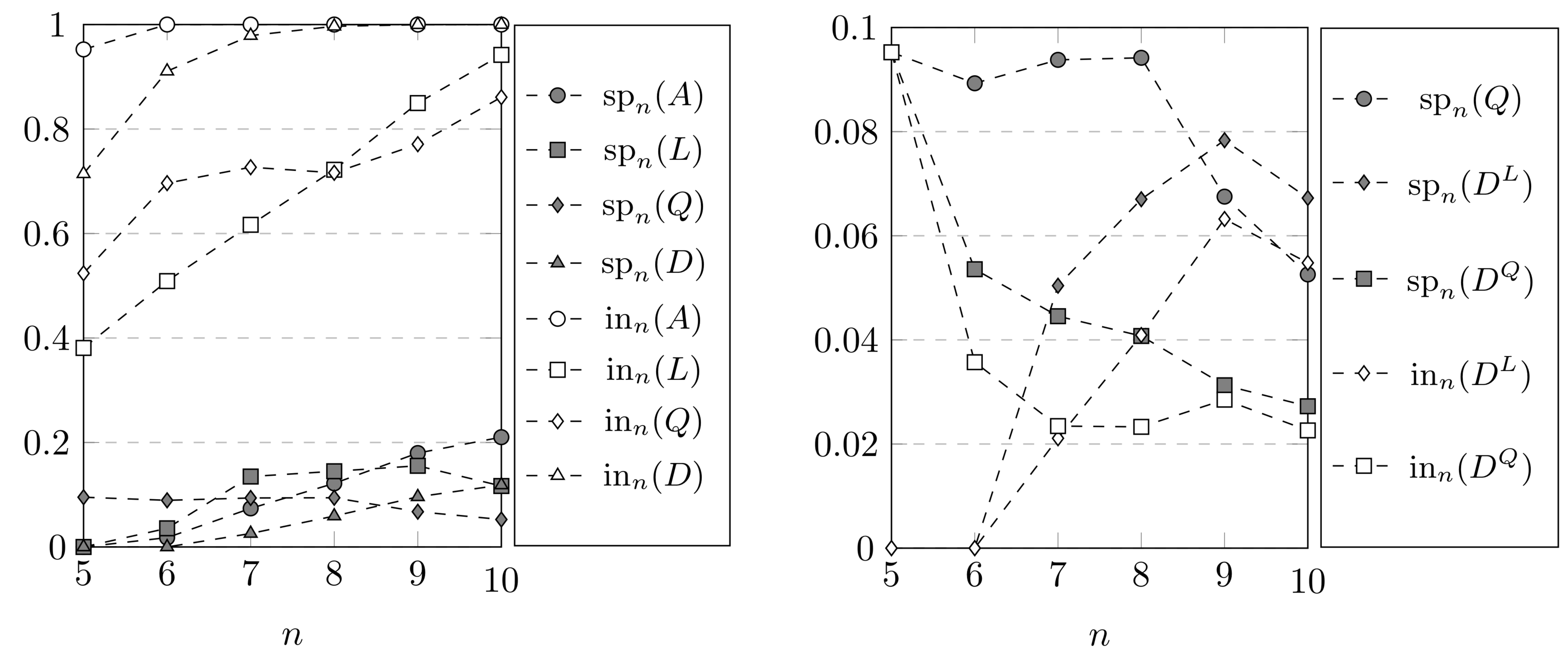

Figure 1: The fraction of connected graphs on n vertices that have at least one cospectral graph with respect to a certain associated matrix is denoted as sp_n. The fraction of connected graphs on n vertices with respect to a certain associated matrix that have at least one coinvariant graph is denoted as in_n.

Coinvariant trees

Aouchiche and Hansen reported in [2] enumeration results on cospectral trees with at most 20 vertices with respect to D, D^L and D^Q matrices. For D, they found that among the 123,867 trees on 18 vertices, there are two pairs of D-cospectral trees. Among the 317,955 trees on 19 vertices, there are six pairs of D-cospectral trees. There are 14 pairs of D-cospectral trees over all the 823,065 trees on 20 vertices. Surprisingly, after the enumeration of all 1,346,023 trees on at most 20 vertices, they found no D^L-cospectral trees and no D^Q-cospectral trees. This fact led Aouchiche and Hansen to conjecture that every tree is determined by its distance Laplacian spectrum, and by its distance signless Laplacian spectrum.

Analogously, for the SNF of D, D^L and D^Q of trees, one can obtain some similar insights. Hou and Woo obtained in [3] that the SNF of the distance matrix for any tree with $n+1$ vertices equals $I_2 \oplus I_{n-2} \oplus (2n)$. From which follows that all trees with n vertices are D-coinvariant graphs. On the other hand, after enumerating coinvariant trees with at most 20 vertices with respect to D^L and D^Q, we found no D^L-coinvariant trees and no D^Q-coinvariant trees among all trees with up to 20 vertices. This fact led us to conjecture that all trees are determined by the SNF of D^L, and, analogously, by the SNF of D^Q.

References

[1] A. Abiad and C.A. Alfaro. Enumeration of cospectral and coinvariant graphs. [arXiv:2008.05786]
[2] M. Aouchiche and P. Hansen. Cospectrality of graphs with respect to distance matrices. Appl. Math. Comput. 325 (2018) 309-321.
[3] Y. Hou and C. Woo. Distance unimodular equivalence of graphs. Linear Multilinear Algebra 56 (2008) 611-626.

Intersection models for 2-thin and proper 2-thin graphs

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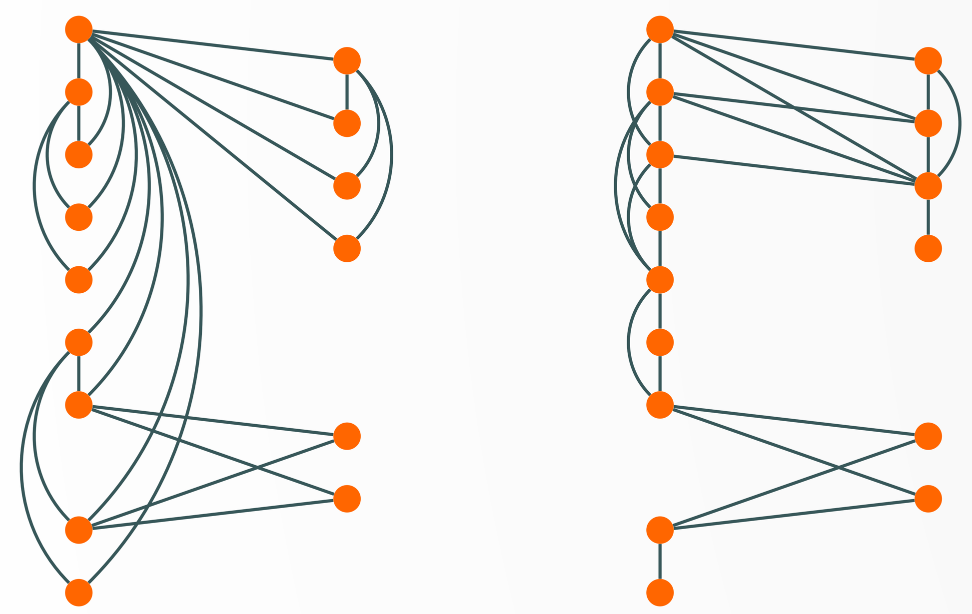
Thinness and proper thinness

A graph $G = (V, E)$ is k-thin if there exist an ordering and a k-partition of V s.t., for $u < v < w$, if u, v belong to the same class and $uw \in E$, then $vw \in E$. The minimum such k is called the *thinness* of G and denoted thin(G) [1].

Interval graphs[1] are exactly the 1-thin graphs, and 2-thin graphs include convex bipartite graphs. Complements of induced matchings have unbounded thinness.



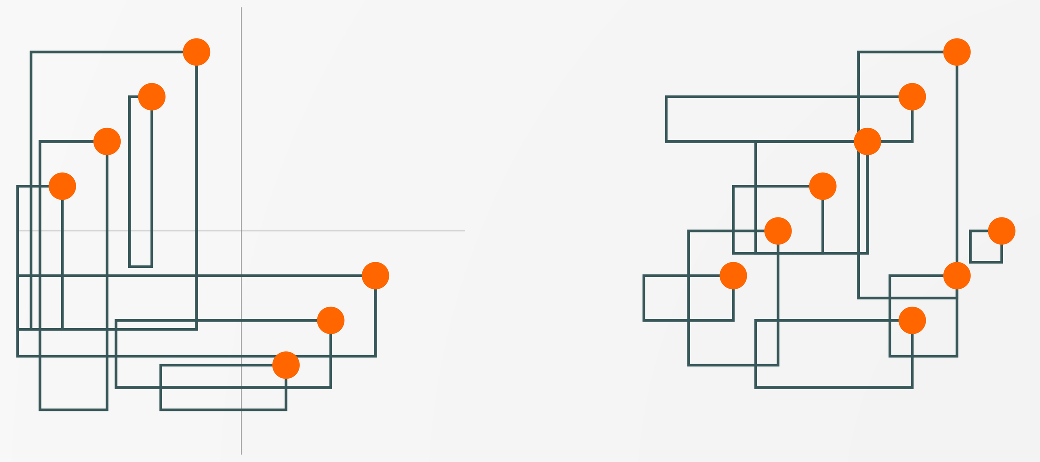
A 2-thin graph and a proper 2-thin graph.

A graph $G = (V, E)$ is proper k-thin if there exist an ordering and a k-partition of V s.t., for $u < v < w$, if u, v belong to the same class and $uw \in E$, then $vw \in E$, and if v, w belong to the same class and $uw \in E$, then $uv \in E$. The minimum such k is called the *proper thinness* of G and denoted pthin(G) [2].

Proper interval graphs are exactly the proper 1-thin graphs, and interval graphs have unbounded proper thinness.

2-diagonal box intersection models

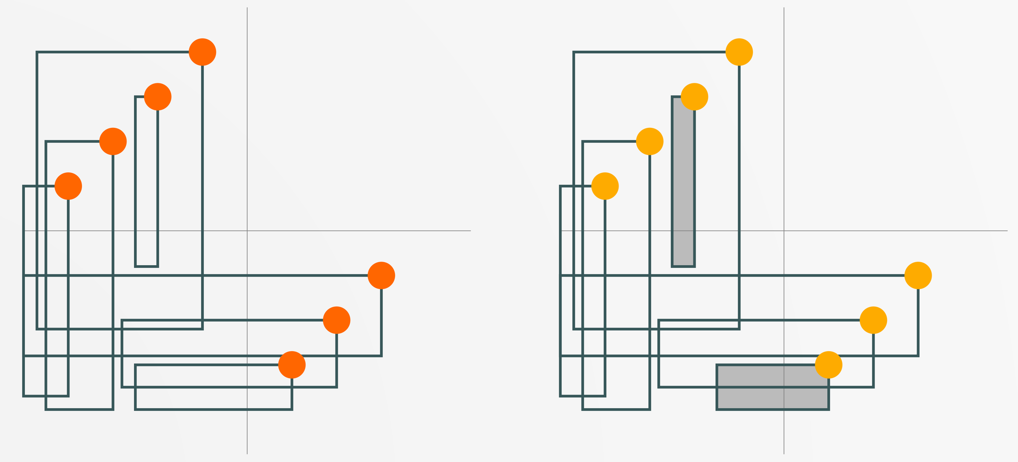
A set of boxes drawn with sides parallel to the Cartesian axes of the plane is 2-diagonal if their upper-right corners are pairwise distinct and lie in two diagonals $y = x + d_1$, $y = x + d_2$, either in the 2nd or in the 4th quadrant, and weakly 2-diagonal if there is no quadrant restriction.



A 2-diagonal and a weakly 2-diagonal model.

Blocking models

A model is blocking if for two non-intersecting boxes b_1, b_2 in the upper and lower diagonal, resp., either the vertical prolongation of b_1 intersects b_2 or the horizontal prolongation of b_2 intersects b_1.



Blocking 2-diagonal model and not.

Characterizations

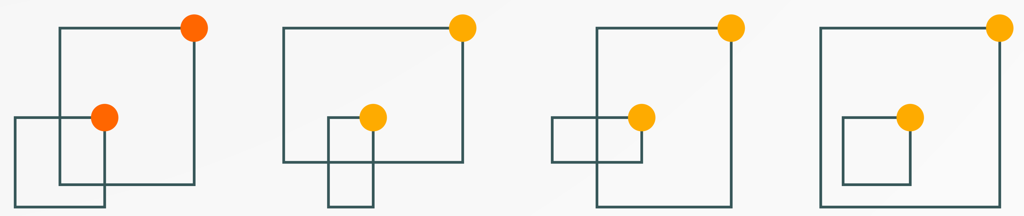
The main results of this work are the following characterizations of 2-thin and proper 2-thin graphs as intersection graphs of boxes drawn with sides parallel to the Cartesian axes of the plane.

Theorem. A graph is 2-thin if and only if it has a blocking 2-diagonal model.

The blocking property is necessary since there are graphs with thinness 3 and a 2-diagonal model.

A model is bi-semi-proper if for two boxes b, b' in the same diagonal, $x_2 < x'_2$ implies $x_1 \leq x'_1$ and $y_1 \leq y'_1$.

Theorem. The following statements are equivalent:
1. G is a proper 2-thin graph.
2. G has a bi-semi-proper blocking 2-diagonal model.
3. G has a bi-semi-proper weakly 2-diagonal model.



Example of bi-semi-proper (first situation) and not bi-semi-proper (last three situations).

The bi-semi-proper property is necessary as interval graphs may have arbitrarily large proper thinness. These models are based on a model by Mannino, Oriolo and Chandran, defined to show that k-thin graphs can be represented as intersection graphs of boxes in the k-dimensional Euclidean space.

2-thin graphs as VPG graphs

A graph is B_k-VPG if it is the vertex intersection graph of paths with at most k bends in a grid. An L-graph is a B_1-VPG graph admitting a representation with all the paths having the same of the four possible shapes L, ⅃, Γ, ꓶ.

- B_0-VPG graphs have unbounded thinness.
- 2-thin graphs are L-graphs (thus B_1-VPG).
- The wheel W_4 is 2-thin and not B_0-VPG.
- 3-thin graphs are B_3-VPG.

Bonus track: new upperbound

The pathwidth (resp. bandwidth) of a graph G can be defined as one less than the maximum clique size of an interval (resp. *proper* interval) supergraph of G, chosen to minimize its maximum clique size [3].

It was proved in [1] that

$$\mathrm{thin}(G) \leq \mathrm{pw}(G) + 1$$

We prove that, if $|E(G)| \geq 1$, then

$$\mathrm{pthin}(G) \leq \mathrm{bw}(G)$$

References

[1] C. Mannino, G. Oriolo, F. Ricci, and S. Chandran. The stable set problem and the thinness of a graph. *Oper. Res. Lett.*, 35:1–9, 2007.

[2] F. Bonomo and D. De Estrada. On the thinness and proper thinness of a graph. *Discrete Appl. Math.*, 261:78–92, 2019.

[3] H. Kaplan and R. Shamir. Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques. *SIAM J. Comput.*, 25(3):540–561, 1996.

[1] Due to lack of space, some standard graph classes, graph parameters, and small graphs are not defined here. The definitions of those concepts can be found in **http://graphclasses.org**

A Near-tight Bound for the Rainbow Connection Number of Snake Graphs★

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Introduction

The *rainbow connection number* of a connected graph G, denoted $rc(G)$, is the least k for which G admits a (not necessarily proper) k-edge-coloring such that between any pair of vertices there is a path whose edge colors are all distinct. This parameter has important applications [3].

Remark ([1]) *If G is a connected and not trivial graph with n vertices, then $diam(G) \le rc(G) \le |E(G)|$.*

We present a near-tight bound for the rainbow connection number of snake graphs, a class commonly studied in labeling problems [2, 5].

Let $\ell \ge 3, k \ge 1, n \ge 2$. An *$\ell$-gon k-multiple snake graph* over n vertices, denoted $S(\ell, k, n)$, is obtained from $P_n\colon v_0 v_1 \ldots v_{n-1}$ by adding k multiple edges between v_i and v_{i+1} for $0 \le i \le n-2$ and making $\ell - 2$ successive subdivisions at each edge added. See Fig. 1.

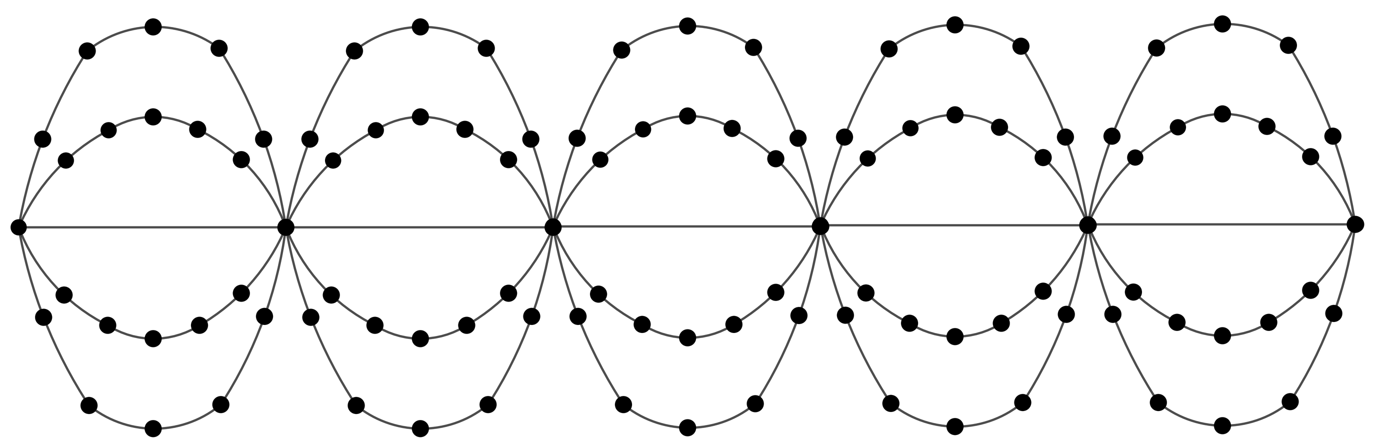

Figure 1. $S(7, 4, 6)$

The rainbow connection number is already known [4] for $G = S(3, k, n)$ with $k \in \{1, 2, 3\}$. In this case,

$$rc(G) = \begin{cases} diam(G) + 1, & \text{if } n = k = 3; \\ diam(G), & \text{otherwise.} \end{cases}$$

★ Partially supported by CNPq (428941/2016-8) and UTFPR.
Snake design vector created by freepik - www.freepik.com

Result

Lemma
Let $G = S(\ell, k, n)$.

$$diam(G) = \begin{cases} \lfloor \ell/2 \rfloor, & \text{if } n = 2 \text{ and } k = 1; \\ \ell - 1, & \text{if } n = 2 \text{ and } k > 1; \\ 2\lfloor \ell/2 \rfloor + n - 3, & \text{if } n > 2. \end{cases}$$

Theorem
Let $G = S(\ell, k, n)$.

$$rc(G) \le \begin{cases} diam(G) + 1, \text{ if } \ell \text{ is even or } n = 2; \\ diam(G) + 2, \text{ if } \ell \text{ is odd.} \end{cases}$$

This bound is near-tight, since we know snake graphs which have $rc(G) = diam(G) + 1$.

Proof (sketch). Fig. 2 shows a rainbow coloring of the *block* $B_{i,i+1}$, for $0 \le i \le n-2$.

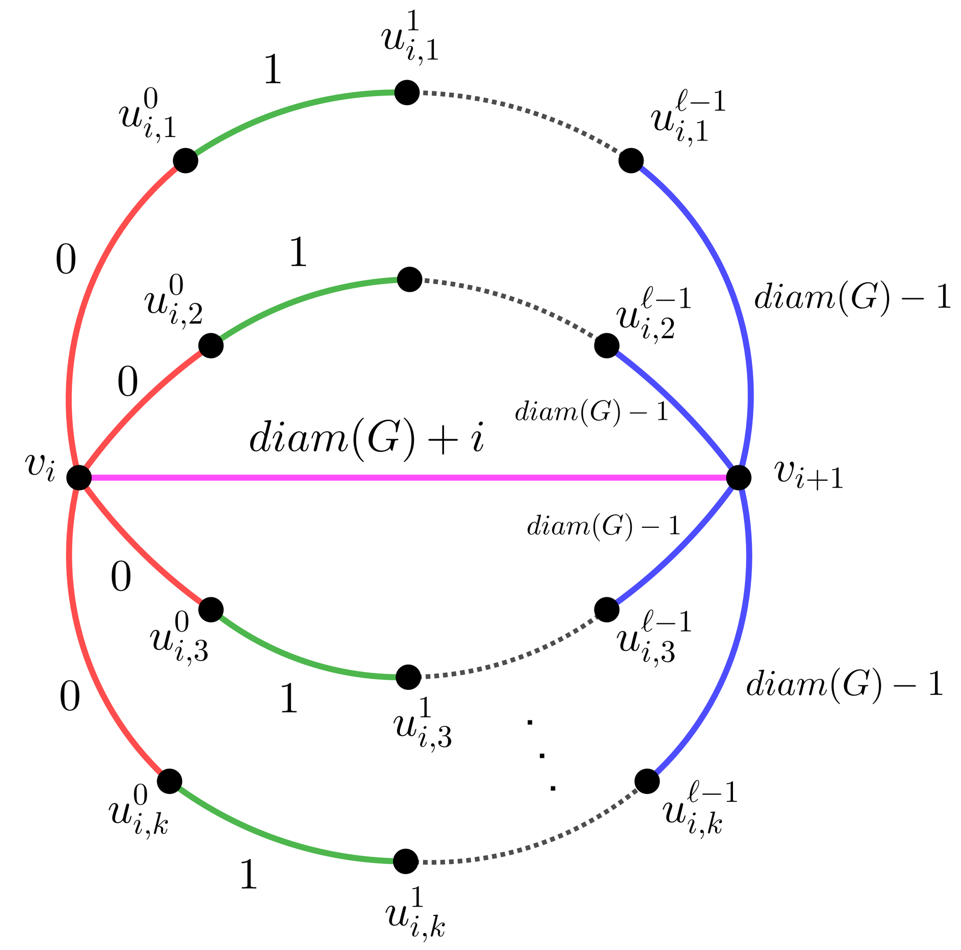

Figure 2. $B_{i,i+1}$

Fig. 3 shows $S(4, 4, 4)$ with the rainbow coloring obtained.

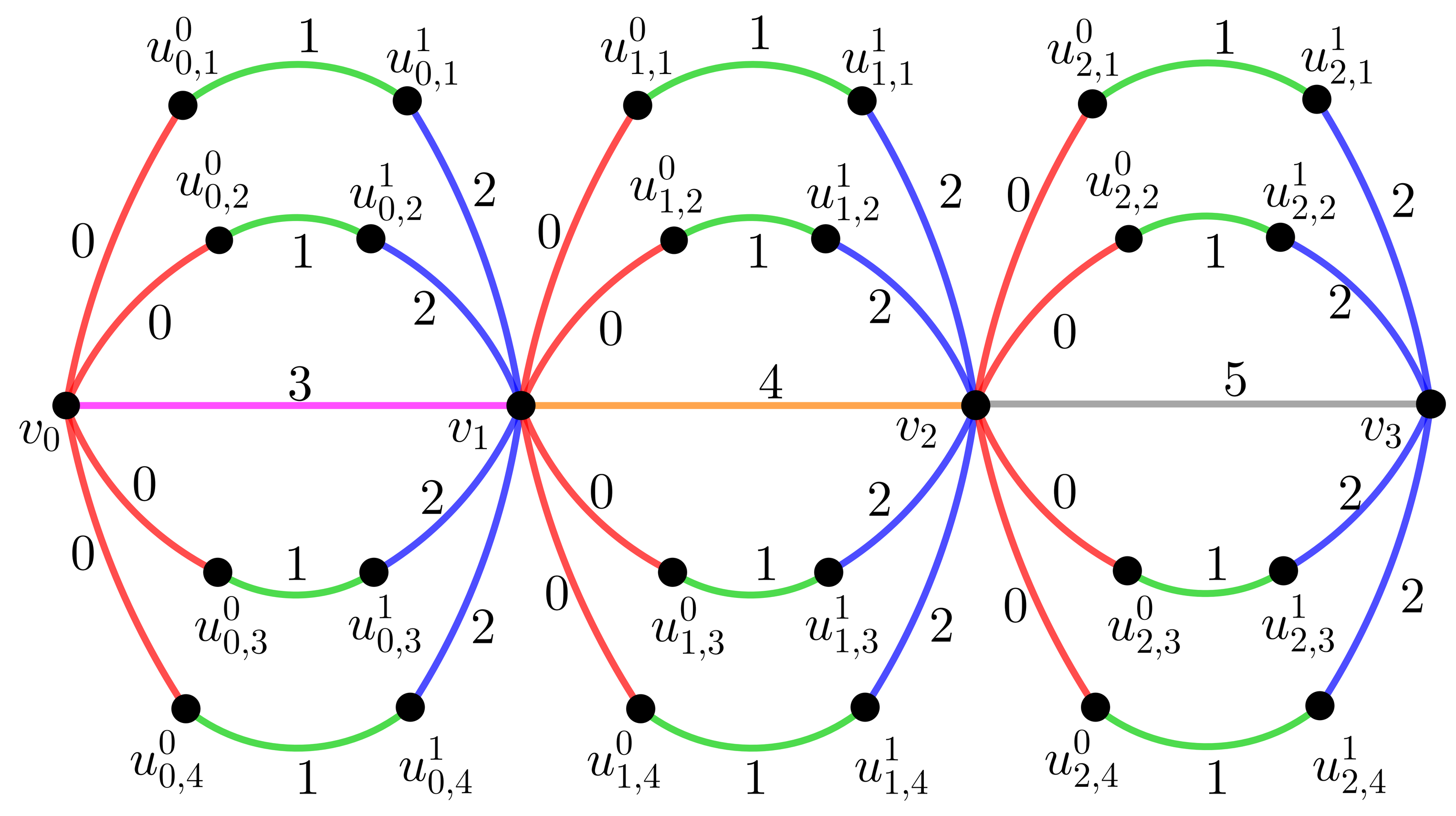

Figure 3. $S(4, 4, 4)$

References

[1] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang. Rainbow connection in graphs. *Mathematica Bohemica*, 133(1):85–98, 2008.

[2] J. A. Gallian. A dynamic survey of graph labeling. *Electron. J. Combin.*, 17:60–62, 2014.

[3] X. Li and Y. Sun. *Rainbow connections of graphs*. Springer Science & Business Media, 2012.

[4] A. Rocha, S. M. Almeida, and L. M. Zatesko. The rainbow connection number of triangular snake graphs. In *Proc. V ETC/CSBC '20*, pages 65–68, Porto Alegre, Brazil, 2020. SBC.

[5] S. S. Sandhya, E. E. R. Merly, and S. D. Deepa. Heronian mean labeling of triple triangular and triple quadrilateral snake graphs. *Int. J. Appl. Math. Sci.*, 9:177–186, 2016.

Reduced indifference graphs are type 1

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Introduction

In the year that Celina Figueiredo, João Meidanis and Célia de Mello celebrate another decade of life, we point out the following result which is an immediate consequence of their papers.

Corollary 1

All reduced indifference graphs are type 1.

Let G be a simple graph. A *total coloring* is an assignment of colors to the vertices and edges of G such that no two adjacent or incident elements receive the same color. See Fig. 1.

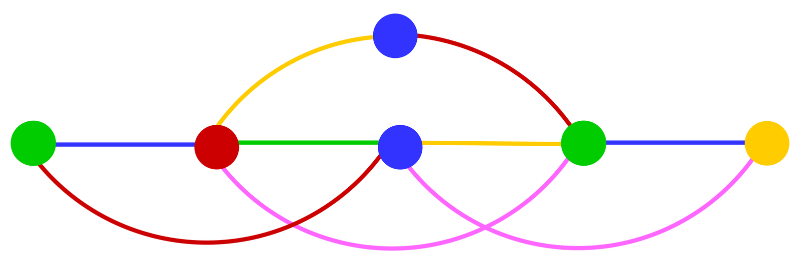

Figure 1: A total coloring of the Hajós graph.

The minimum number of colors for a total coloring of G is the *total chromatic number*, $\chi''(G)$. By definition, $\chi''(G) \geq \Delta(G)+1$. Vizing and Behzad posed the famous Total Coloring Conjecture.

Total Coloring Conjecture (TCC) [1, 2]

$$\chi''(G) \leq \Delta(G) + 2$$

If G has $\chi''(G) = \Delta(G) + 1$, it is *type 1*, otherwise it is *type 2*. By Theorem 1, it is NP-complete to decide if a graph is type 1 for the general case.

Theorem 1 [3]

To decide if a cubic bipartite graph G has $\chi''(G) = \Delta(G) + 1$ is NP-complete.

Total coloring of dually chordal graphs

A graph is *dually chordal* if it is the clique of a chordal graph. Dually chordal graphs generalizes known subclasses of chordal graphs such as interval graphs and indifference graphs. Celina Figueiredo, João Meidanis and Célia de Mello [4] presented the following result.

Theorem 2 [4]

If G is dually chordal, the TCC holds. Moreover, if $\Delta(G)$ is even, G is type 1.

The proof of Theorem 2 gives a polynomial-time algorithm that yields an optimum total coloring of dually chordal graphs with even maximum degree.

Reduced indifference graphs

G is an *indifference graph* if and only if its vertices can be ordered such that those that belong to the same maximal clique are consecutive. This order is known as *indifference order*. Two vertices are *true twins* if they are adjacent and belong to the same maximal cliques. A graph is *reduced* if it does not contain true twins. See Fig. 2.

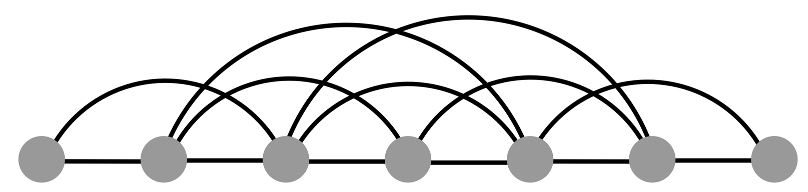

Figure 2: A reduced indifference graph.

Celina Figueiredo, Célia de Mello and Carmen Ortiz [5] presented the following interesting property on indifference graphs.

Theorem 3 [5]

If G is an indifference graph that does not contain maximum degree true twins, then G has a matching M that covers every maximum degree vertex. Moreover, the graph $G - M$, obtained from G by removing the edges of M, is an indifference graph.

Fig. 3 exhibits an indifference graph and a matching that covers its maximum degree vertices.

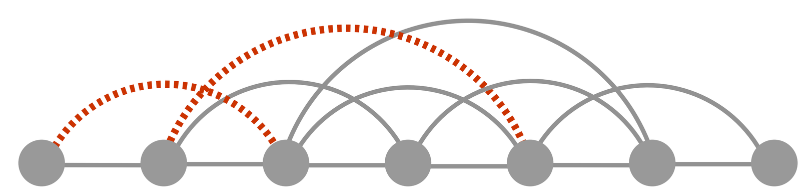

Figure 3: A matching according to Theorem 3.

We use the same technique presented in the proof of Theorem 2 and the property presented in Theorem 3 to prove Theorem 4 and, consequently, Corollary 1. Our proof also gives a polynomial-time algorithm for an optimum total coloring of reduced indifference graphs.

New result

Theorem 4

If G is an indifference graph that does not contain maximum degree true twins, then G is type 1.

Sketch of proof. If $\Delta(G)$ is even, $\chi''(G) = \Delta(G) + 1$, by Theorem 2. Suppose that $\Delta(G)$ is odd. Since G does not contain maximum degree true twins, it has a matching M that covers all maximum degree vertices, by Theorem 3. By Theorem 2, $\chi''(G - M) = \Delta(G)$. Consider an optimum total coloring of $G - M$ as in the proof of Theorem 2. Assign a new color for the edges of M. If the endvertices of an edge in M receive the same color, G has maximum degree true twins, a contradiction. So, $\chi''(G) = \Delta(G) + 1$. □

Fig. 4 presents a total coloring for the graph of Fig 2.

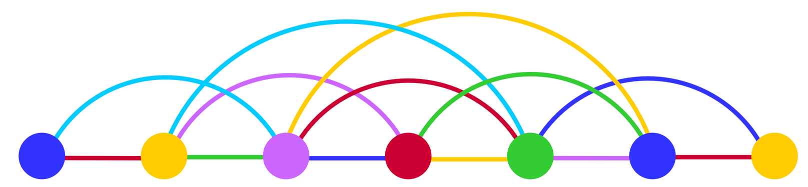

Figure 4: An optimum total coloring according to Theorem 4.

Corollary 1 is an immediate consequence of Theorem 4.

References

[1] M. Behzad, *Graphs and Their Chromatic Numbers*. PhD thesis, Michigan State University, East Lansing, 1965.

[2] V. G. Vizing, "Some unsolved problems in graph theory," *Russian Mathematical Surveys*, vol. 23, no. 6, p. 125, 1968.

[3] A. Sánchez-Arroyo, "Determining the total colouring number is np-hard," *Discrete Mathematics*, vol. 78, no. 3, pp. 315–319, 1989.

[4] C. M. H. de Figueiredo, J. Meidanis, and C. P. de Mello, "Total-chromatic number and chromatic index of dually chordal graphs," *Information processing letters*, vol. 70, no. 3, pp. 147–152, 1999.

[5] C. M. H. de Figueiredo, C. P. de Mello, and C. Ortiz, "Edge colouring reduced indifference graphs," in *Latin American Symposium on Theoretical Informatics*, pp. 145–153, Springer, 2000.

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On Embedding Trees in Grids

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Introduction

The problem of *grid embedding* is that of drawing a graph G onto a rectangular two-dimensional grid (called simply *grid*) such that each vertex $v \in V(G)$ corresponds to a grid point (an intersection of a horizontal and a vertical grid line) and the edges of G correspond to paths of the grid. Grid embedding of graphs has been considered with different perspectives [2, 5, 6]. In [5], linear-time algorithms are described for embedding planar graphs having their edges drawn as non-intersecting paths in the grid, such that the maximum number of bends of any edge is minimized, as well as the total number of bends.

Objective

We are interested in embedding trees T with $\Delta(T) \leq 4$ in a rectangular grid, such that the vertices of T correspond to grid points, while edges of T correspond to non-intersecting straight segments of the grid lines. The aim is to minimize the maximum number of bends of a path of T. We provide a quadratic-time algorithm for this problem. With this algorithm, we obtain an upper bound on the number of bends of EPG models of VPT∩EPT graphs [3, 4].

Embedding trees in a grid

Let T be a tree such that $\Delta(T) \leq 4$. Consider the problem of embedding such a tree in a grid $\mathcal{G}$, so that the vertices must be placed at grid points and the edges drawn as non-intersecting paths of $\mathcal{G}$ with no bends, which we will call a *model* of T. See Figures 1-5 for key notations.

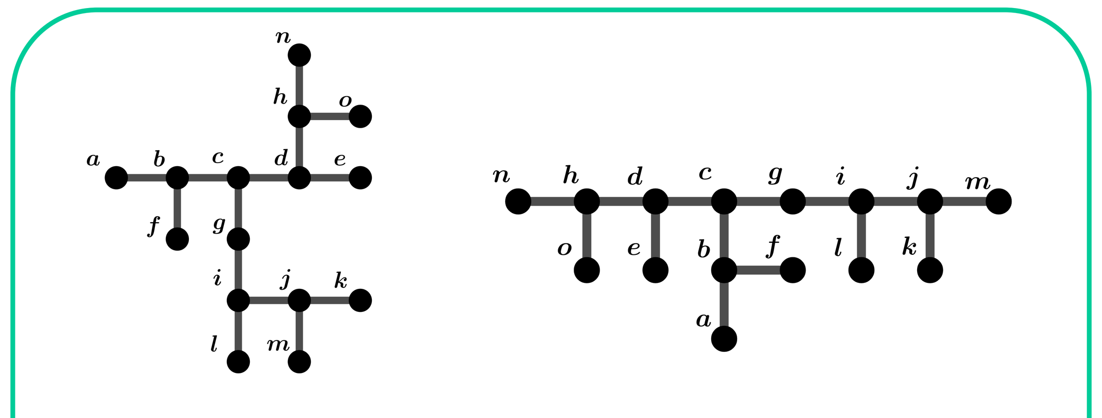

Figure 1: Two possible models M_1 (left) and M_2 (right) of the same tree T.

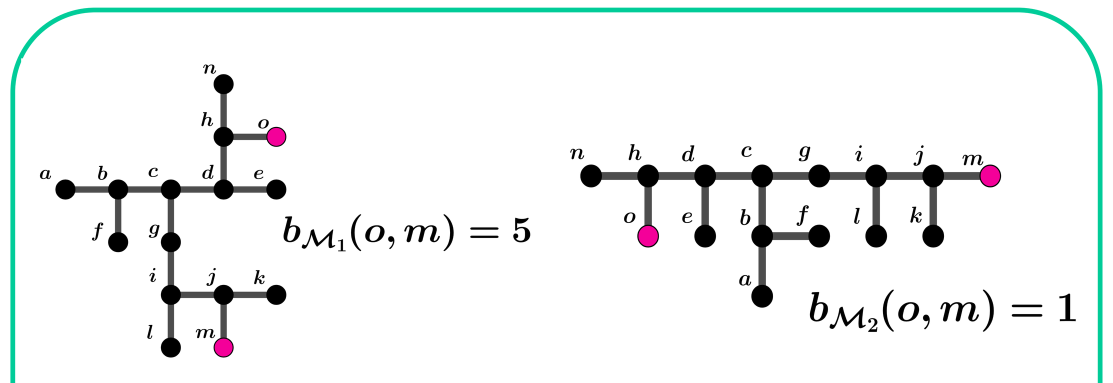

Figure 2: The number of bends of the path connecting u and v in M is denoted by $b_M(u, v)$.

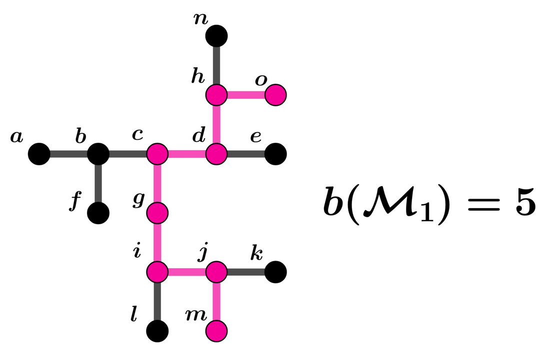

Figure 3: The number of bends of model M is $b(M) = \max\{b_M(u, v) \mid u$ and v are leaves of $T\}$.

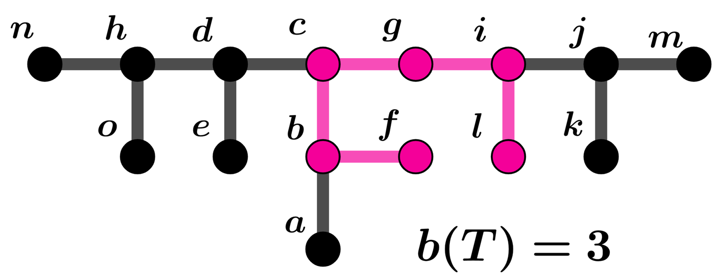

Figure 4: The number of bends of tree T is $b(T) = \min\{b(M) \mid M$ is a model of $T\}$.

$N(i) = \{g, j, l, \emptyset\}$

$u_1(i) = g$ $u_2(i) = j$ $u_3(i) = l$ $u_4(i) = \emptyset$

$b_1(i) = 2$ $b_2(i) = 1$ $b_3(i) = 0$ $b_4(i) = -1$

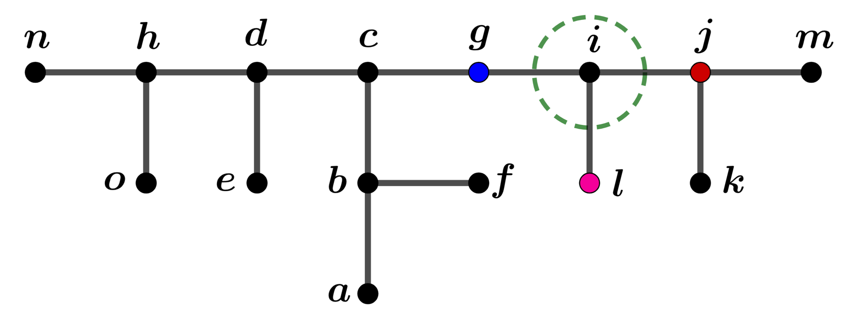

Figure 5: The neighborhood of vertex i ordered according to $b_\ell(i, j)$ for all $j \in N(i)$.

Given a model M, let $b_\ell(p,v)$ be the maximum number of bends of a path in M having as extreme vertices p and a leaf $\ell \in V(T)$, over all paths that contain $v \in V(T)$.

Let M be a model of T and $v \in V(T)$. Let $N(v) = \{u_i(v) \mid 1 \leq i \leq d(v)\}$ be the neighborhood of v and $b_i(v) = b_\ell(v, u_i(v))$.

For $d(v) < i \leq 4$, define "virtual" neighbors $u_i(v) = \emptyset$ for which $b_i(v) = -1$. Assume that the neighbors (both real and virtual) are ordered so that $b_i(v) \geq b_{i+1}(v)$ for all $1 \leq i < 4$. See example in Figure 5.

Let $v \in V(T)$ and M a model of T. We say that v is *balanced* if $u_1(v)$ and $u_2(v)$ are mutually in the same horizontal or vertical grid line in M (and, therefore, so are $u_3(v)$ and $u_4(v)$).

Question

Over all possible models, consider the problem of finding one in which the maximum number of bends of a path of T, over all of them, is minimum.

The algorithm

Algorithm 1: Determining $b(T)$

Input : a tree T such that $\Delta(T) \leq 4$
Output: a model $\mathcal{M}$ of T such that $b(\mathcal{M}) = b(T)$
Let $S = (v_0, \emptyset), (v_1, p_1), \ldots, (v_{n-1}, p_{n-1})$ be such that T is incrementally built by S
Let $\mathcal{M}$ be a model having a single vertex v_0 at some grid point
for $i \leftarrow 1$ **to** $n - 1$ **do**
  Add to $\mathcal{M}$ the vertex v_i attached to the grid point of p_i, in any free horizontal or vertical grid line of p_i
  BALANCE ($\mathcal{M}$, v_i, p_i)

Procedure BALANCE ($\mathcal{M}$, p, v) **:**
  for $u \in N(v) \setminus \{p\}$ **do**
    BALANCE ($\mathcal{M}$, v, u)
  If v is not balanced, then make it balanced by rearranging in $\mathcal{M}$ the drawing of the four subtrees of v rooted at $u^i(v)$ (for $1 \leq i \leq 4$), potentially rotating and rescaling them to fit [***balance step***]

A tree T can be built from a single vertex v_0 by a sequence $v_1, v_2, \ldots, v_{n-1}$ of vertex additions, each new vertex v_i adjacent to exactly one vertex p_i of T for all $1 \leq i < n$. We will call that T is *incrementally built* by $(v_0, \emptyset), (v_1, p_1), \ldots, (v_{n-1}, p_{n-1})$.

Algorithm 1 consists of iteratively adding vertices to T and, for each new vertex v, traversing T in post-order having v as the root. The operation to be carried out in each visited vertex is to balance v if it is not balanced.

Theorem

Given a tree T, let M be the model produced by the execution of the algorithm on input T. Then, $b(M) = b(T)$.

EPG models of VPT∩EPT graphs

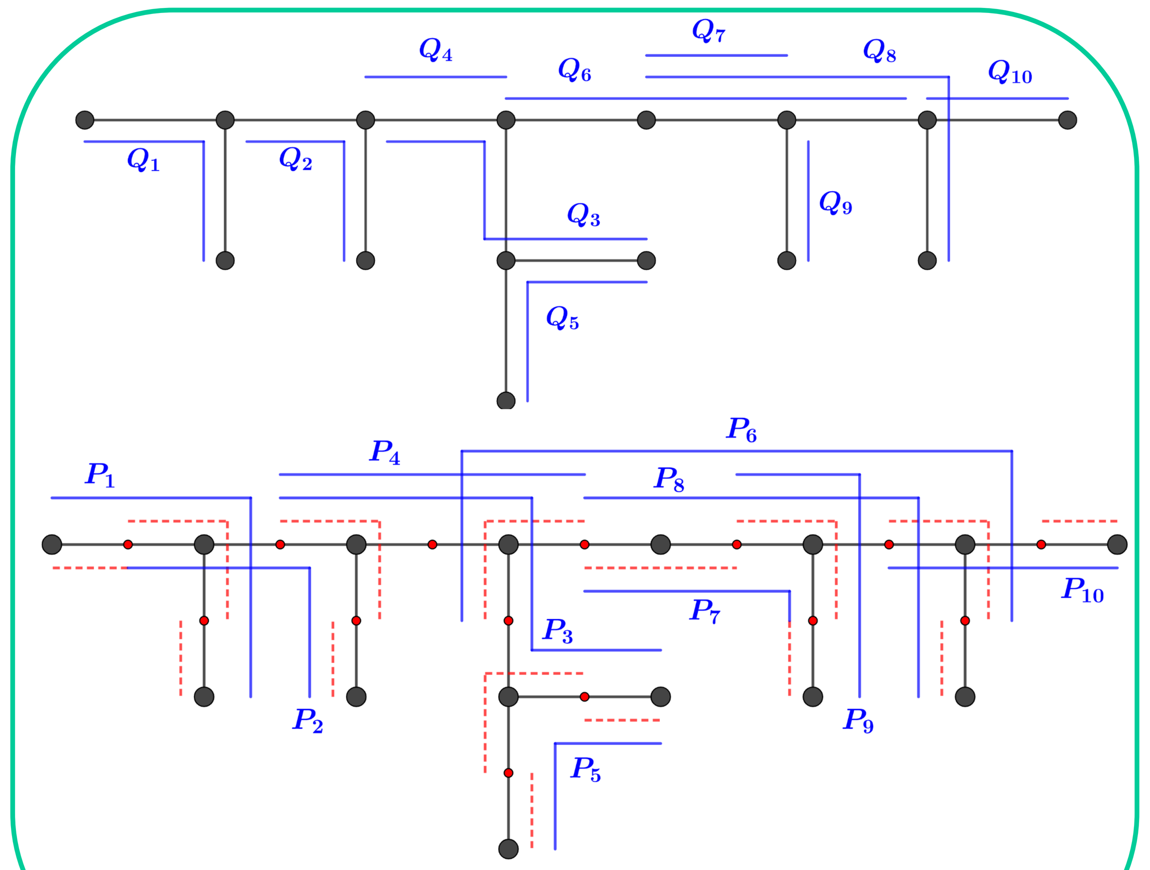

Figure 6: Construction of a B_k-EPG representation with $k \leq b(T)$.

We provide an upper bound on the number of bends of an EPG representation of VPT∩EPT graphs. The VPT∩EPT graphs are those that can be represented in host trees with maximum degree at most 3 [3]. In [1], this class is characterized by a family of minimal forbidden induced subgraphs. An EPG model $R = \{P_i \mid 1 \leq i \leq 10\}$ is shown in Figure 6, obtained from the family $P = \{Q_i \mid 1 \leq i \leq 10\}$.

References

[1] ALCÓN, L., GUTIERREZ, M., AND MAZZOLENI, M. P. (2015). Characterizing paths graphs on bounded degree trees by minimal forbidden induced subgraphs. Discrete Mathematics, 338(1):103–110.
[2] BECK, M. AND STORANDT, S. (2020). Puzzling grid embeddings. In: 2020 Proceedings of the Twenty-Second Workshop on Algorithm Engineering and Experiments (ALENEX), 94–105.
[3] GOLUMBIC, M. C. AND JAMISON, R. E. (1985). Edge and vertex intersection of paths in a tree. Discrete Mathematics, 55(2):151–159.
[4] GOLUMBIC, M. C. ; LIPSHTEYN, M. STERN ; M. . Edge intersection graphs of single bend pathson a grid, Networks: An International Journal, volume 54:3, pages 130–138, 2009.
[5] LIU, Y., MORGANA, A., AND SIMEONE, B. (1998). A linear algorithm for 2-bend embeddings of planar graphs in the two-dimensional grid. Discrete Applied Mathematics, 81(1-3):69–91.
[6] SCHNYDER, W. (1990). Embedding planar graphs on the grid. In Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms, 138–148.

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B_1-EPG representations using block-cutpoint trees

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Introduction

EPG graphs were first introduced by Golumbic et al in [2] motivated from circuit layout problems [1]. In B_1-EPG representations, each path has one of the following shapes x = {⌞,⌜, ⌟,⌝}, besides horizontal or vertical segments. One may consider more restrictive subclasses of B_1-EPG by limiting the types of bends allowed in the representation, that is, only the paths in a subset of x are allowed.
Ex.: The ⌞⌝-EPG graphs are those in which only the "⌞" or the "⌝" shapes are allowed.

Objective

We show that two superclasses of trees are B_1-EPG (one of them being the cactus graphs). On the other hand, we show that the block graphs are ⌞-EPG and provide a linear time algorithm to produce ⌞-EPG representations of generalization of trees. These proofs employed a new technique from previous results based on block-cutpoint trees of the respective graphs.

Preliminaries

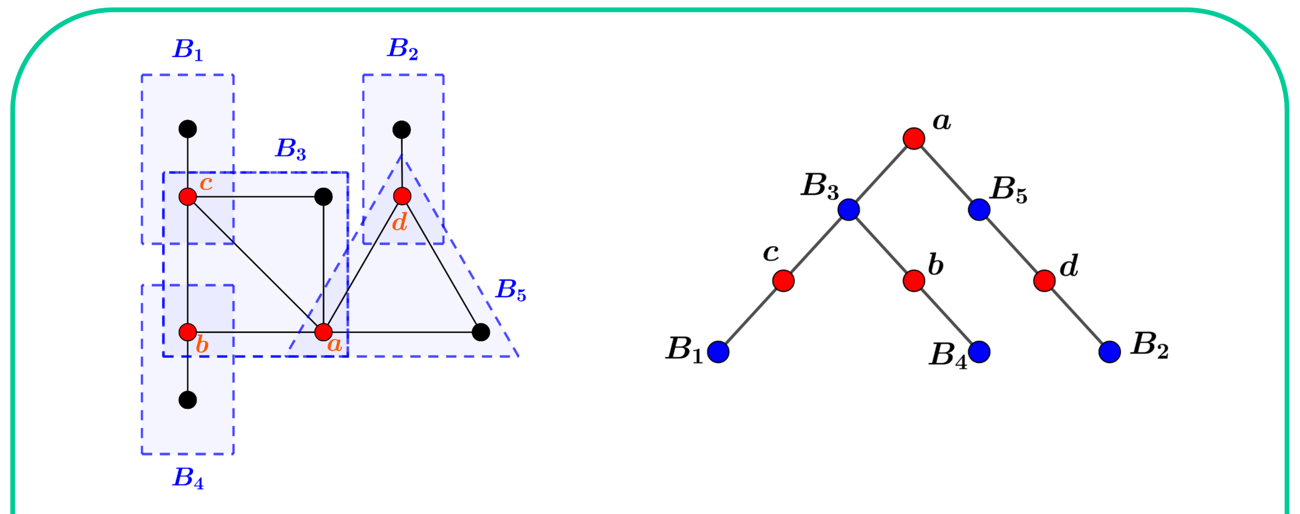

Figure 1: A graph and its respective BC-tree. The cut vertices are marked in red.

Consider a graph G. Let T be a bipartite graph in which the parts X and Y are such that X contains one vertex b for each block B of G, called a *block vertex*, and Y contains one vertex c for each cut vertex c' of G, called as such in T. Vertices b and c form an edge if $c' \in V(B)$. It is easy to see that T is in fact a tree. We define T as the *block-cutpoint tree of G* [3] (BC-tree). See Figure 1.

B1-EPG representations

We describe a B_1-EPG representation of a superclass of trees, inspired on the representation of trees described in [2]. The novelty of our results is the usage of BC-trees to obtain EPG representations, which will be employed to obtain B_1-EPG representations of more general classes of graphs.

Theorem 1

Let G be a graph such that every block of G is B_1-EPG and every cut vertex v of G is a universal vertex in the blocks of G in which v is contained. Then, G is B_1-EPG.

Proof. (Sketch) The theorem is proved by induction. Actually, we prove a stronger claim, stated as follows: given any graph G satisfying the theorem conditions and a BC-tree T of G rooted at some cut vertex r, there exists a B_1-EPG representation $R=\{P_v \mid v \in V(G)\}$ of G in which:

i. P_r is a vertical path with no bends in R;
ii. all paths but P_r are constrained within the horizontal portion of the grid defined by P_r and at the right of it.

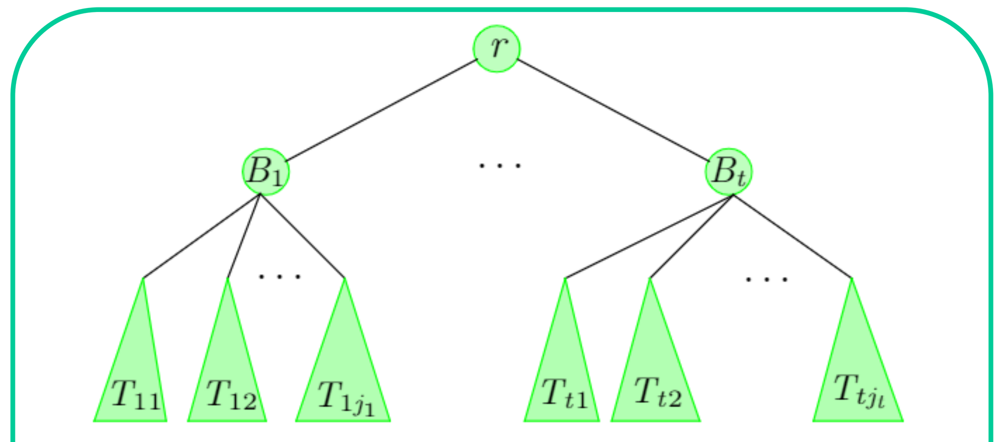

Figure 2: The rooted BC-tree T of a graph.

From T (the BC-tree of G shown in Figure 2), build the representation R of G as follows. First, build an arbitrary vertical path P_r in the grid $\mathcal{G}$, corresponding the root r. Next, divide the vertical portion of $\mathcal{G}$ defined by P_r and at the right of it into t vertical subgrids, $\mathcal{G}_1$, $\mathcal{G}_2$, . . . , $\mathcal{G}_t$, with a row space between them such that the i-th subgrid will contain the paths corresponding to the cut vertices that are descendants of B_i in T. So, each subgrid $\mathcal{G}_i$ is constructed as shown in Figure 3.

We first represent the children of B_i as disjoint ⌞-shaped paths, all sharing the same grid column in which P_r lies. For each B_i, we build the paths in B_i', that correspond to vertices of B_i that are not cut vertices of G (as those in black in Figure 1), and the paths in T_{ij}, belonging to G[T_{ij}], for all $1 \le j \le j_i$. So, it remains to define how the paths belonging to the regions B_i' and T_{ij} will be built.

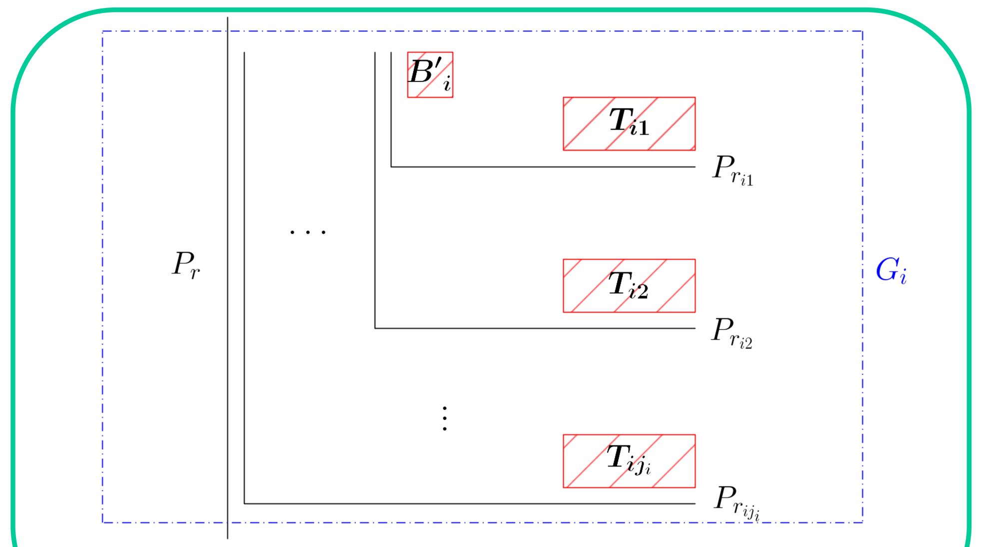

Figure 3: A subgrid $\mathcal{G}_i$.

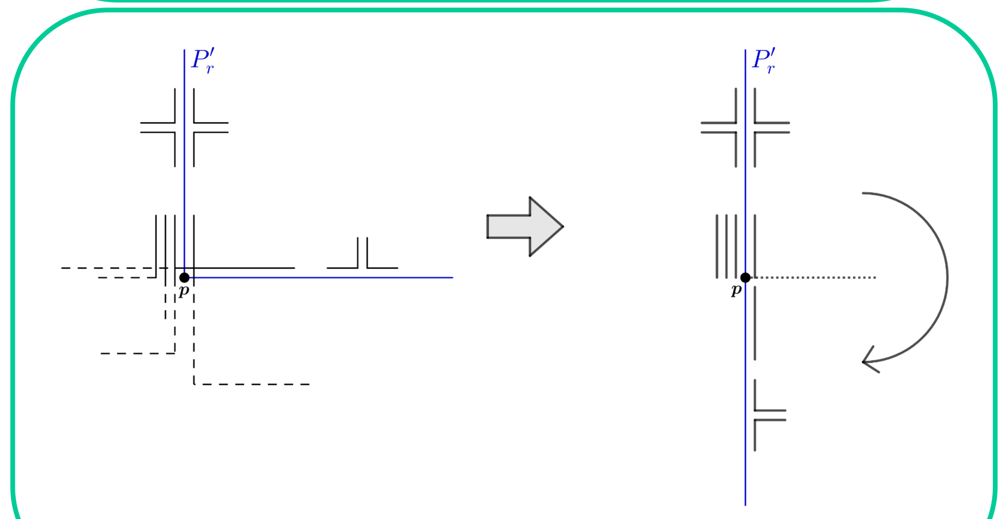

Figure 4: Transforming R'.

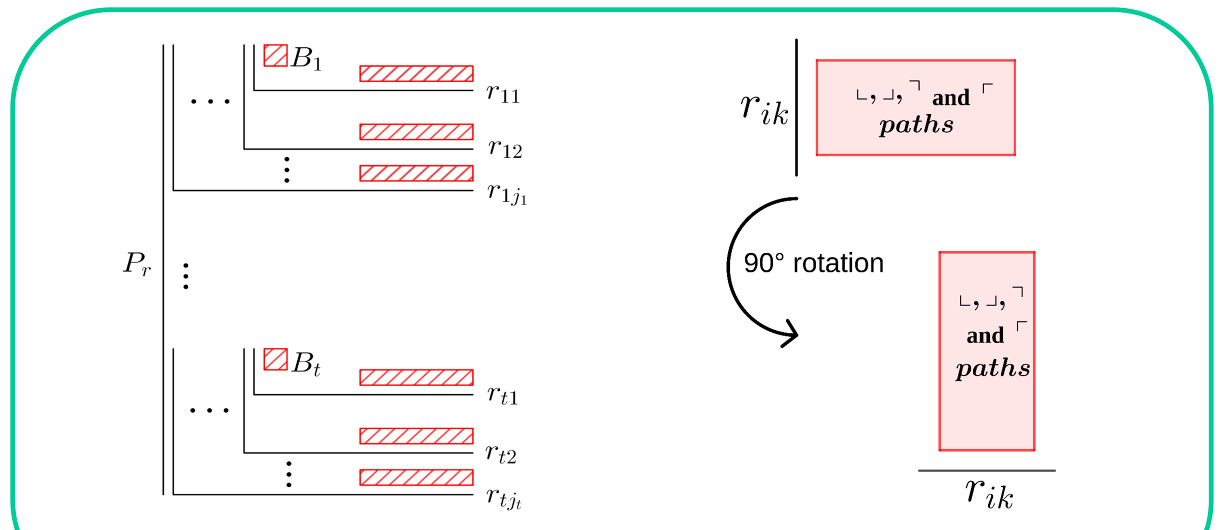

Figure 5: B_1-EPG representation of G after induction step.

Let R' be a B_1-EPG representation of B_i and let P_r' be an ⌞-path corresponding r in R'. Since r is universal to B_i, it is possible to transform R' such that all universal vertices become vertical paths as shown in Figure 4. For the T_{ij} portion of the representation, let r_{ij} be the root of T_{ij}. Applying induction hypothesis, we obtain B_1-EPG representations of each subtree that have vertical paths representing each root and the entire representation is bounded as described previously in (ii).

Thus, we can attach each one of the representations to its respective portion of the model being built, rotated 90 degrees in counter-clockwise (see Figure 5).

Theorem 2

Let G be a graph such that every block of G is ⌞ -EPG and every cut vertex v of G is a universal vertex in the blocks of G in which v is contained. Then, G is B_1-EPG.

Proof. (Sketch) This proof follows the same reasoning lines as those in the proof of Theorem 1. However, the assumption that every block B_i is ⌞-EPG allows their EPG representations to be transformed into interval models. It is possible to show how to build an interval model of each block, given an ⌞-EPG representation of it. Furthermore, the EPG representations of the subtrees T_{ij}, $1 \le j \le j_i$, of B_i, for all i, obtained after the induction step can be transformed into ⌞-EPG models by 90 degree clockwise rotation so that the entire representation is ⌞-EPG.

Theorem 3

Cactus graphs are B_1-EPG

Proof. (Sketch) This proof follows the same reasoning lines as those in the proof of Theorem 1. The difference here is that every block is either an edge or a cycle. It is possible therefore to construct B_1-EPG representations of every block B_i. Furthermore, the B_1-EPG representations of the subtrees T_{ij}, $1 \le j \le j_i$, of B_i, for all i, obtained after the induction step can be shown possible to be attached into vertical or horizontal regions of the cycle/edge so that the entire representation is B_1-EPG.

References

[1] BRADY, M. L. ; SARRAFZADEH, M. . Stretching a knock-knee layout for multilayer wiring, IEEE Transactions on Computers, volume 39, pages 148-151, 1990.

[2] GOLUMBIC, M. C. ; LIPSHTEYN, M. STERN ; M. . Edge intersection graphs of single bend pathson a grid, Networks: An International Journal, volume 54:3, pages 130–138, 2009.

[3] HARARY, F. . Graph Theory, Addison-Wesley, Massachusetts, 1969.

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Contact L-graphs and their relation with planarity and chordality

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remote 9th LAWCG and MDA
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B_0-CPG graphs

- An undirected graph $G = (V, E)$ is called a VPG graph ([1]) if one can associate a path in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point.
- An undirected graph $G = (V, E)$ is then called a B_k-*VPG graph*, for some integer $k \geq 0$, if one can associate a path with at most k bends in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point.
- An undirected graph $G = (V, E)$ is said to be B_0-CPG if one can associate a horizontal or vertical path in a rectangular grid with each vertex, such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point without crossing each other and without sharing an edge of the grid.

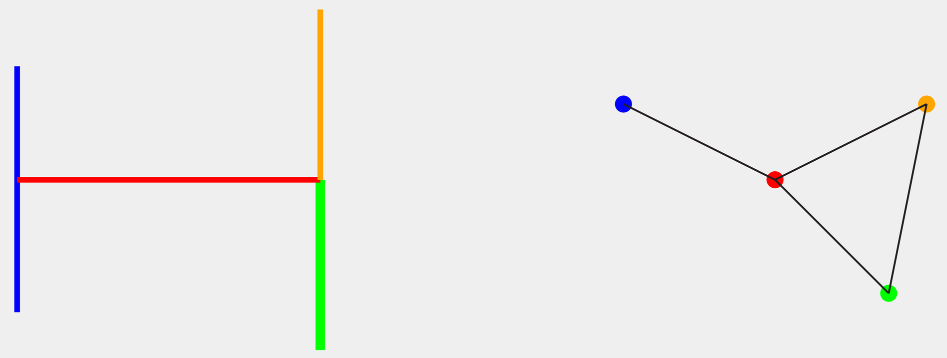

Figure: On the left, a B0-CPG representation of the graph on the right.

L-Contact graphs

- An $\llcorner$-*graph* is a graph with a B_1-VPG representation such that all the paths in the representation have the shapes $\{|, -, \llcorner\}$. We will say that the graph is an *strict* $\llcorner$-*graph* if the paths only have the shape $\llcorner$.
- An *(strict)* $\llcorner$-*contact graph* is an (strict) $\llcorner$-graph such that all the paths in the representation do not cross each other and do not share an edge of the grid.
- A representation of a strict $\llcorner$-contact graph such that no path intersects another in a bend point will be called a *basic* representation.

Figure: Two representations of K_3 as a strict $\llcorner$-contact graph.

Relation with planarity

- B_0-CPG $\subseteq$ $\llcorner$-contact and there are non-planar B_0-CPG graphs.

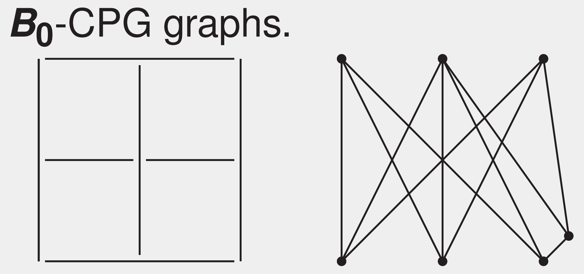

Figure: On the left, a B0-CPG representation of the non-planar graph on the right.

- As a consequence, there are non-planar $\llcorner$-contact graphs.
- $\llcorner$-contact $\subseteq$ B_1-CPG.

Theorem ([3])

For every $k \geq 0$ there is a planar graph G such that G is B_{k+1}-CPG but not B_k-CPG.

Theorem

If G is strict $\llcorner$-contact then G is planar.

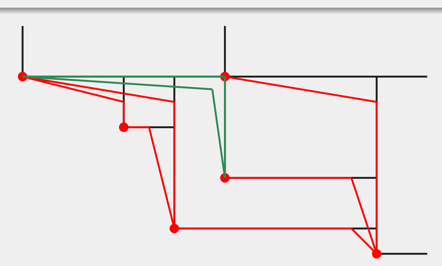

Figure: The planar representation obtained from the strict $\llcorner$-contact representation of a graph.

Laman graph

- A *Laman graph* is a graph on n vertices such that, for all k, every k-vertex induced subgraph has at most $2k - 3$ edges, and such that the whole graph has exactly $2n - 3$ edges.
- An $\llcorner$*-contact representation is a B_1-CPG representation which is strict and basic.
- an $\llcorner$*-contact representation is maximal if every endpoint that is neither bottommost, topmost, leftmost, nor rightmost makes a contact, and there are at most three endpoints that do not make a contact.

Theorem ([4])

If a graph G has a maximal $\llcorner$-contact representation in which each inner face contains the right angle of exactly one $\llcorner$, then G is a planar Laman graph.*

- As a consequence, we have the following result.

Theorem

Every maximal strict $\llcorner$-contact graph is a planar Laman graph.

Relation with chordality

Lemma

A clique in a strict $\llcorner$-contact graph has size at most three.

Theorem

Let G be a chordal graph. G is strict $\llcorner$-contact if and only if G is K_4-free. Moreover, G admits a basic representation.

Let $\mathcal{T}$ be the family of graphs defined as follows. $\mathcal{T}$ contains H_0 as well as all graphs constructed in the following way: start with a tree of maximum degree at most three and containing at least two vertices; this tree is called the *base tree*; add to every leaf v in the tree two copies of K_4 (sharing vertex v), and to every vertex w of degree 2 one copy of K_4 containing vertex w. Notice that all graphs in $\mathcal{T}$ are chordal.

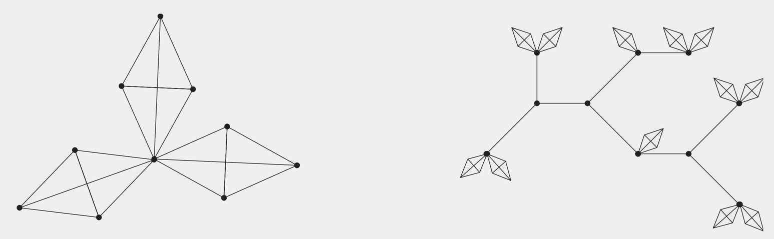

Figure: On the left the graph H_0. On the right a typical graph in $\mathcal{T}$.

Theorem ([2])

Let G be a chordal graph. Let $\mathcal{F} = \mathcal{T} \cup \{K_5, \text{diamond}\}$. Then, G is a B_0-CPG graph if and only if G is $\mathcal{F}$-free.

- It is immediate that $\llcorner$-contact graphs are K_5-free and that B_0-CPG $\subseteq$ $\llcorner$-contact.
- Following the same ideas as in the chordal B_0-CPG characterization, all the graphs in $\mathcal{T}$ are forbidden subgraphs.
- As a consequence, we have the following result concerning block graphs.

Theorem

Let G be a block graph. G is $\llcorner$-contact if and only if G is B_0-CPG.

References

[1] Asinowski, A., E. Cohen, M. Golumbic, V. Limouzy, M. Lipshteyn and M. Stern, *Vertex intersection graphs of paths on a grid*, Journal of Graph Algorithms and Applications **16** (2012), pp. 129–150.

[2] Bonomo, F., M. Mazzoleni, M. Rean, and B. Ries, *On some special classes of contact B_0-VPG graphs*, Discrete Applied Mathematics (2019).

[3] Champseix, N., E. Galby, B. Ries, *Planar CPG graphs*, ArXiv (preprint) (2018).

[4] Kobourov, S. T. Ueckerdt K. Verbeek, *Combinatorial and Geometric Properties of Planar Laman Graphs.* Proceedings of 24th ACM-SIAM Symposium on Discrete Algorithms (SODA 2013).

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Gráficas Cuadrado-Complementarias 4-regulares de cuello grande.

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Introducción

Todas las gráficas con las que trabajamos son gráficas finitas y simples. Este trabajo fue motivado por un problema abierto del artículo de *Graphs whose complement and square are isomorphic*. Dada una gráfica G, el cuadrado de está denotado por G^2, es la gráfica que consta del mismo conjunto de vértices de G, pero $uv \in G^2$ si y solamente si la distancia de u, v en G es 1 o 2. $\overline{G}$ denota al complemento de la gráfica G, en la cual tenemos el mismo conjunto de vértices y $uv \in \overline{G}$ si y sólo si $uv \notin G$. Decimos que una gráfica G es cuadrado complementaria si cumple $G^2 \cong \overline{G}$ o equivalentemente $G \cong \overline{G^2}$. Usaremos el término de *squco* para refirirnos al término cuadrado-complementario, por su abreviatura en inglés; ***squ**are-**co**mplement*. Algunos ejemplos de gráficas squco son la gráfica trivial K_1, el ciclo de 7 vértices C_7 y la gráfica de Franklin:

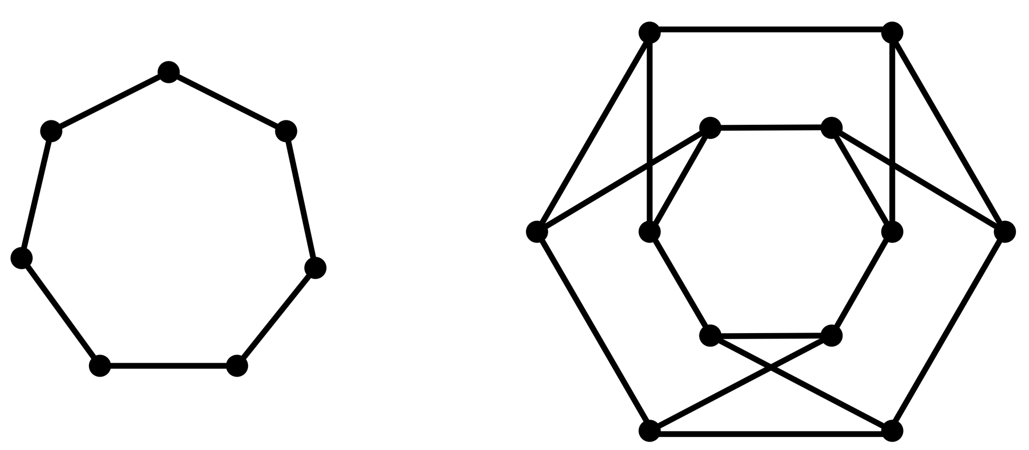

Figura 1: Ejemplos de gráficas cuadrado complementarias.

Objetivo

Dar una respuesta al problema abierto planteado en [1, 4]: **Dado un número par $d \geq 4$, ¿existe una gráfica *squco*, d-regular con $d^2 + d + 1$ vértices?**

Desarrollo del problema

Por definición si tenemos una gráfica d-regular, tenemos que $\forall\ x \in G$ tenemos que $deg_G(x) = d$. La longitud del ciclo más pequeño dentro de una gráfica se denomina cuello (*girth*) y es denotada por $g(G)$. Notemos que si tenemos una gráfica cuadrado-complementaria no trivial G que sea d-regular, entonces G tiene como máximo $d^2 + d + 1$ vértices; debido a que G es regular de grado d, sin pérdida de generalidad escogemos un vértice cualquiera llamémosle u el cual tiene d vecinos, a una distancia dos de u tenemos a lo más $d(d-1)$ vecinos más y por último el número de vecinos a una distancia de al menos 3 tenemos d esto ya que G satisface la condición de ser cuadrado-complementaria, $G \cong \overline{G^2}$ y estos últimos vecinos a distancia 3 serán los vecinos a distancia 1 de u en $\overline{G^2}$ que también tiene que ser d-regular. Por lo tanto $|G| \leq 1 + d + d(d-1) + d = d^2 + d + 1$. Recordemos que la estructura descrita no depende del vértice elegido al inicio pues G es d-regular. Cuando consideramos la longitud del ciclo más pequeño, i.e. el cuello de G, se satisface que $\boldsymbol{g(G) \geq 5}$ **si y sólo si** $\boldsymbol{|G| = d^2 + d + 1}$. Las gráficas que buscamos deben cumplir: $g(G) = 5$ y ser 4-regulares, además de ser cuadrado-complementarias lo que implica que $|G| = 21$. Un ejemplo de la estructura buscada es:

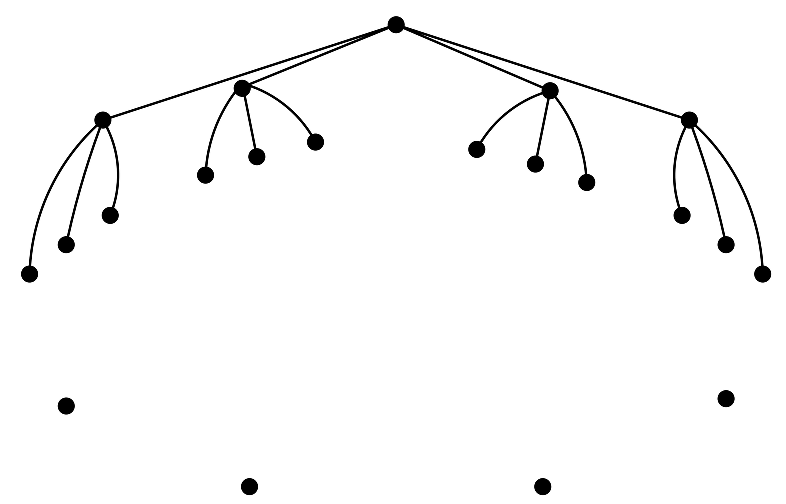

Figura 2: Andamiaje para G cuadrado-complementaria 4-regular con $g(G) = 5$.

El problema se encontraba en encontrar el conjunto de 26 aristas que completaba a G de la figura 2, y la hacían ser cuadrado-complementaria de un conjunto total de 108 posibiles aristas; las cuales provienen de las $\binom{16}{2} = 120$ y a este conjunto le quitamos las aristas que forman un 3-ciclo en cada conjunto de vértices que se encuentran a distancia 2 del vértice superior (12 aristas), dando como resultado un conjunto de búsqueda de:
$\binom{108}{26} = 6'909, 598'959, 706'679, 434'990, 092 \approx 7 \times 10^{24}$. Para abordar el problema se optó por usar la herramienta de programación de GAP [3] y el paquete de YAGS [2], con el cual se desarrolló un algoritmo basado en principio en la técnica de *Backtracking* o *branch and bound*, con la finalidad de podar posibilidades que no condujeran a alguna solución disminuyendo tiempo y opciones de soluciones fallidas a explorar.

Backtracking

Backtracking es una técnica algorítmica de búsqueda en espacios combinatorios con estructura arbórea; con énfasis en el podado de ramas inútiles, como lo es en nuestro caso, para hacer uso de esta técnica debemos hacer énfasis en el podado de ramas inútiles ya que por medio de estas logramos acortar el amplio espacio de búsqueda que tenemos para las soluciones. Analizando las posibles soluciones encontramos que las gráficas tienen una simetría muy buena lo cual nos permitió encontrar un punto clave para desechar posibilidades fallidas ya analizadas y con ello optimizar el tiempo empleado a la resolución de nuestro problema, pues al ir escogiendo aristas podemos desechar algunas configuraciones isomorfas a otras analizadas con anterioridad.

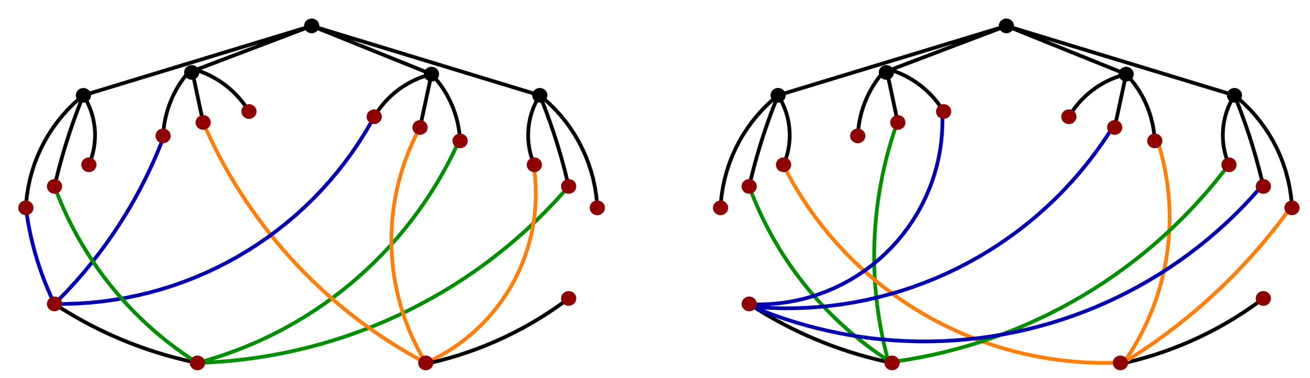

Figura 3: Andamiajes con configuraciones diferentes pero isomorfas.

Resultados

Algoritmo

Después de varias versiones que lograran disminuir el tiempo y el conjunto de posibles soluciones al problema obtuvimos un algoritmo que se compone de varias funciones, las cuales verifican cada una de las características que buscamos verificar que satisfagan las gráficas que buscamos. Entre las cuales se encuentran:

- Que no existan triángulos, ciclos de tamaño 3, en G y en $\overline{G^2}$.
- Que no existan cuadrados, ciclos de tamaño 4, en G y en $\overline{G^2}$.
- Función que analiza las simetrías en la gráfica.
- Verifica que cada vértice no exceda el grado 4.
- Función interna del Backtrack, verifica si la solución que tenemos hasta el momento se puede completar.
- Función interna del Backtrack, indica si hemos encontrado una solución de 26 aristas que completen a G.
- Verificar que se satisfaga $G \cong \overline{G^2}$.

Además de esto se considera analizar el problema en 6 casos; los cuales provienen de ser todas las maneras diferentes hasta isomorfismo de colocar aristas en la parte inferior de nuestro árbol sin llegar a formar triángulos (ciclos de tamaño 3), determinados de la siguiente manera:

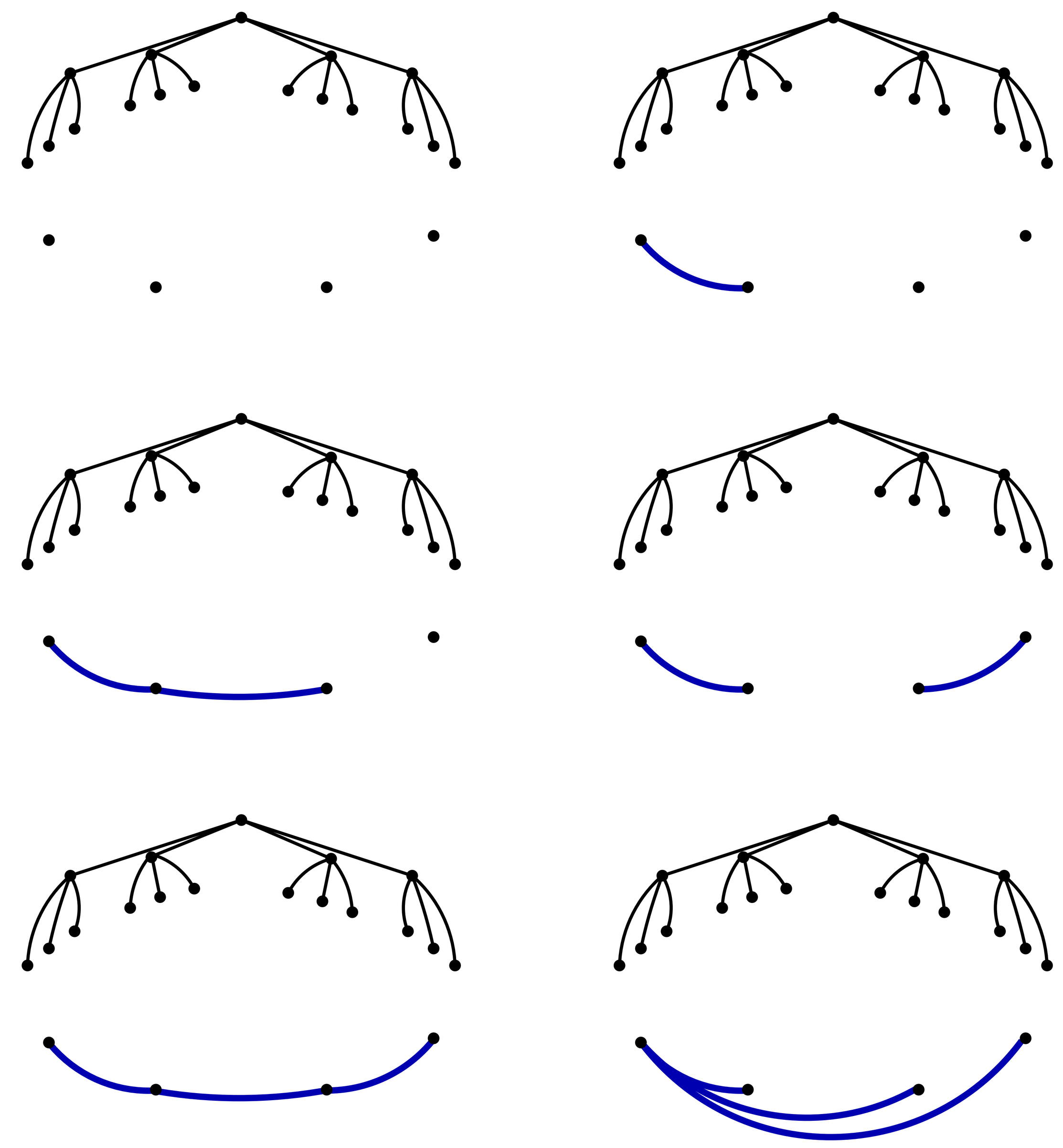

Figura 4: Casos a considerar con las aristas posibles con los vértices en el tercer nivel del andamiaje con 0, 1, 2 y 3 aristas respectivamente.

Con todo ello se redujo el trabajo de analizar casi 7×10^{24} de casos a tan sólo 43 casos.

Conclusiones

Después de las horas empleadas a programar dicho algoritmo que ayude a saber si existen las gráficas cuadrado-complementarias 4-regulares con cuello 5, logramos dar respuesta de que **dichas gráficas no existen**, reduciendo el conjunto de búsqueda considerablemente para dar solución a la interrogante en tan solo 10 minutos. Además de buscar alguna característica que ayude a reducir más el conjunto de posibilidades para dar una prueba con un número pequeño de casos. Claro el trabajo continúa analizando que pasa para el caso general con gráficas cuadrado-complementarias d-regulares con $g(G) \geq 5$, y en particular para $d \geq 6$.

Referencias

[1] V. Baltić, S. Simić and V. Tintor. *Some remarks on graph equation $G^2 = \overline{G}$*. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **5** (1994) 43–48.

[2] C. Cedillo, R. MacKinney-Romero, M.A. Pizaña, I.A. Robles and R. Villarroel-Flores. *YAGS - Yet Another Graph System, Version 0.0.5*, 2019. http://xamanek.izt.uam.mx/yags/.

[3] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10*, 2019. (http://www.gap-system.org).

[4] M. Milanič, A.S. Pedersen, D. Pellicer and G. Verret. *Graphs whose complement and square are isomorphic*. Discrete Mathematics **327** (2014) 62 – 75. https://doi.org/10.1016/j.disc.2014.03.018.

Total Coloring in Some Split-Comparability Graphs

remote 9th LAWCG and MDA November 25th, 2020

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1 Introduction

Let G be a simple graph. For $S \subseteq V(G) \cup E(G)$ and $C\{1, 2, \ldots, k\}$, let $c : S \to C$ be a mapping such that $c(x) \neq c(y)$ for each adjacent or incident elements $x, y \in S$. We say c is a k-total coloring when $S = V(G) \cup E(G)$ and a k-edge coloring when $S = E(G)$. See Fig. 1 for an example. The least j and the least k for which G has a j-total coloring and a k-edge coloring are denoted by $\chi''(G)$ and $\chi'(G)$, respectively.

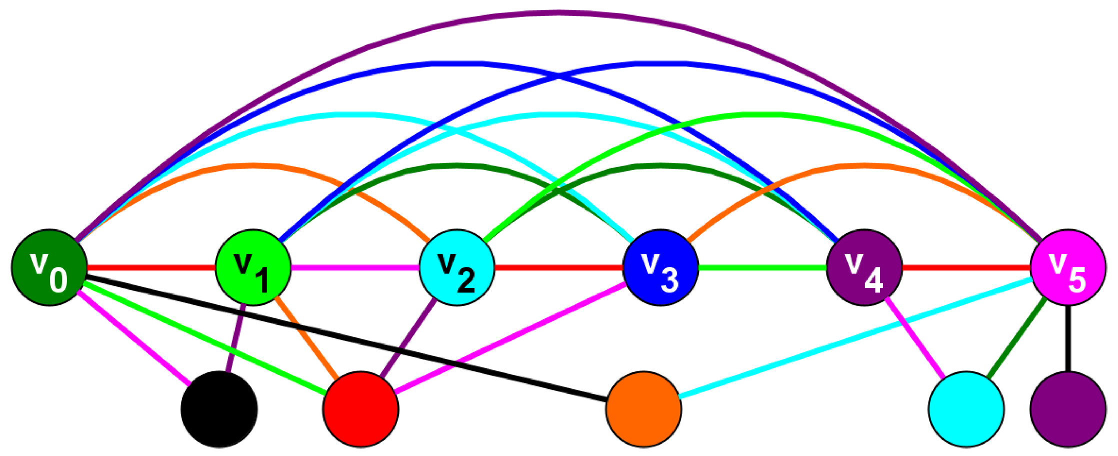

Figure 1: 9-total coloring for G

The Total Coloring Conjecture (TCC) [1, 7] asserts that $\chi''(G) \leq \Delta(G) + 2$ for any G. If $\chi''(G) = \Delta(G) + 1$, G is Type 1; otherwise it is Type 2. To decide if G is Type 1 is NP-Complete [6]. A graph $G[Q, S]$ is split if $V(G)$ can be partitioned into $[Q, S]$ so that Q is a clique and S an independent set.

Theorem 1 [2] Let G be a split graph. Then $\chi''(G) \leq \Delta(G) + 2$. In particular, when $\Delta(G)$ is even G is Type 1.

Ortiz and Villanueva [5] characterized the split-comparability graphs.

Theorem 2 [5] A split graph $G[Q, S]$ is a comparability graph iff Q has a partition $[Q_l, Q_t, Q_r]$ and its vertices can be ordered Q_l, Q_t, Q_r so that for any vertex $s \in S$: $N(s) \cap Q_t = \emptyset$; if $v_k \in (N(s) \cap Q_l)$ then $v_{k-1} \in (N(s) \cap Q_l)$; and if $v_k \in (N(s) \cap Q_r)$ then $v_{k+1} \in (N(s) \cap Q_r)$.

The subset of S whose vertices are not adjacent to Q_r are denoted as S_l, those not adjacent to Q_l denoted as S_r, and $S_t = S \setminus S_l \cup S_r$.

Here we show that certain split-comparability graphs with odd maximum degree are Type 1.

2 Previous Results

When $|E(G)| > \left\lfloor \frac{|V(G)|}{2} \right\rfloor \Delta(G)$ we say G is overfull and if G has a subgraph H with $\Delta(H) = \Delta(G)$ that is overfull, then it is subgraph-overfull. Whenever G is overfull or subgraph-overfull, then $\chi'(G) = \Delta(G) + 1$.

Theorem 3 [3] A split-comparability graph G has $\chi'(G) = \Delta(G)$ iff G is not subgraph-overfull.

Hilton proved the following result for graphs with a universal vertex, i.e. a vertex with degree $|V(G)| - 1$.

Theorem 4 [4] A graph G with a universal vertex is Type 1 iff

$$|E(\overline{G})| + \alpha'(\overline{G}) \geq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor.$$

3 Our Contribution

Theorem 5 A split-comparability graph G, with $|Q_l| \geq |Q_r|$, is Type 1 if

$$|Q| \geq \left(\frac{|S_l|}{|S_l| - 0.5} \right) |Q_l|.$$

Sketch of proof. We assume $|S_r| \neq 0$, $|S_l| \neq 0$ and $\Delta(G)$ is odd, otherwise $\chi''(G)$ is known by Theorems 1 and 4. By Theorem 2, $Q_l \cap Q_r = \emptyset$. Assume $|Q_l| \geq |Q_r|$; so $|Q_r| \leq \frac{|Q|}{2}$. We define a split-comparability supergraph G' of G by adding a vertex v_f twin to the largest degree vertex $v_0 \in Q_l$. Since $|Q_l| \geq |Q_r|$ and $|Q| - |Q_l| \geq \frac{|Q|}{2|S_l|}$, G' is not subgraph-overfull. So, it has a $\Delta(G')$-edge coloring c', by Theorem 3. Fig. 2 shows G' obtained from the graph of Fig. 1.

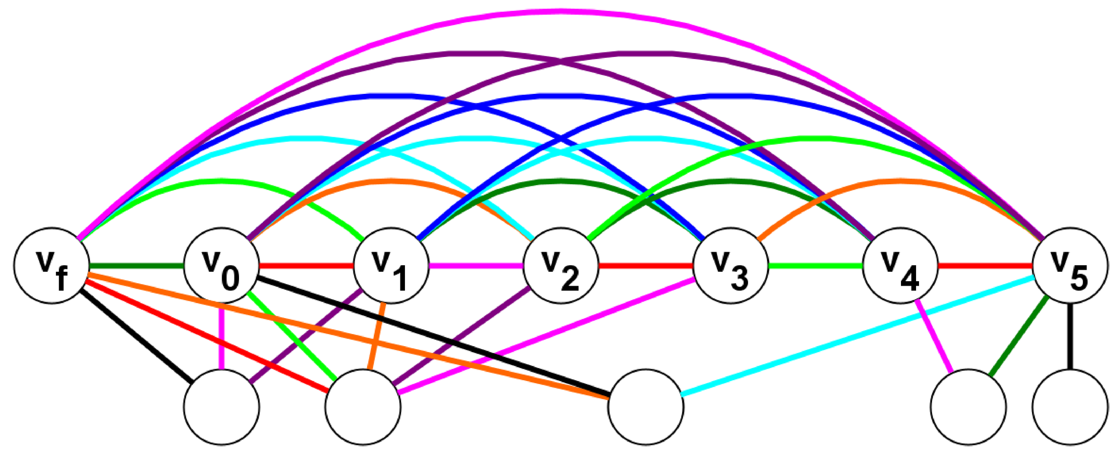

Figure 2: 9-edge coloring for G'

Assign the color $c'(v_f, x)$ to x, for all x in order to obtain a total coloring of $G - S_r$. (Fig. 3 exhibits a partial total-coloring for the graph of Fig. 1.) As $|Q_r| \leq \frac{|Q|}{2}$, at most $|Q|$ colors are used in vertices adjacent or edges incident to vertices of S_r. Since $|Q| < \Delta(G)$ some color is available to be assigned to each vertex $y \in S_r$, and $\chi''(G) = \Delta(G) + 1$.

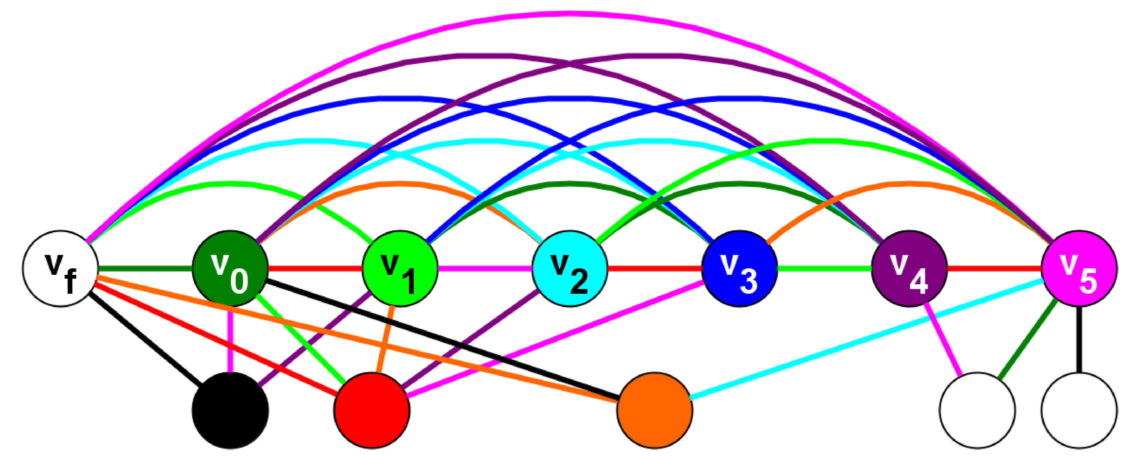

Figure 3: Extending to a total coloring

References

[1] M. Behzad. Graphs and their chromatic numbers. *Michigan State University*, 1965.

[2] B-L. Chen, H-L. Fu, and M. T. Ko. Total chromatic number and chromatic index of split graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 17, 1995.

[3] J. B. S. Cruz, C. N. Silva, and SM S. M. Almeida. The overfull conjecture on split-comparability graphs. *arXiv preprint arXiv:1710.03524*, 2017.

[4] A. J. W. Hilton. A total-chromatic number analogue of plantholt's theorem. *Discrete Mathematics*, 79(2):169 – 175, 1990.

[5] C. Ortiz and M. Villanueva. On split-comparability graphs. In *Proc. II ALIO-EURO Workshop on Practical Combinatorial Optimization, Valparaiso, Chile*, pages 91–105, 1996.

[6] A. Sánchez-Arroyo. Determining the total colouring number is np-hard. *Discrete Mathematics*, 78(3):315–319, 1989.

[7] V. G. Vizing. Some unsolved problems in graph theory. *Russian Mathematical Surveys*, 23(6):125, 1968.

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L(2,1)-labeling of Loupekine Snarks

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Introduction

An $L(2,1)$-*labeling* of a simple graph $G = (V,E)$ is a function $f\colon V \to \{0,\dots,t\}$ such that $|f(u) - f(v)| \geq 2$ if $d(u,v) = 1$ and $f(u) \neq f(v)$ if $d(u,v) = 2$, where $d(u,v)$ denotes the distance between two vertices u and v of G and $t \in \mathbb{N}$. We say that *a conflict occurs* if any of the necessary conditions to have an $L(2,1)$-labeling are not met. The *span* of an $L(2,1)$-labeling f is the largest integer (label) assigned by f to a vertex of G. The λ-*number* of G, denoted by $\lambda(G)$, is the smallest number t such that G has an $L(2,1)$-labeling with span t. Figure 1 exhibits an $L(2,1)$-labeling of the Petersen graph with the smallest span.

The $L(2,1)$-labeling problem was introduced by Griggs and Yeh [3] in 1992, motivated by problems of frequency assignment to transmitters. The main unsolved problem regarding $L(2,1)$-labelings is the *Griggs and Yeh's Conjecture*, which states that every simple graph G with maximum degree $\Delta(G) \geq 2$ has $\lambda(G) \leq \Delta(G)^2$.

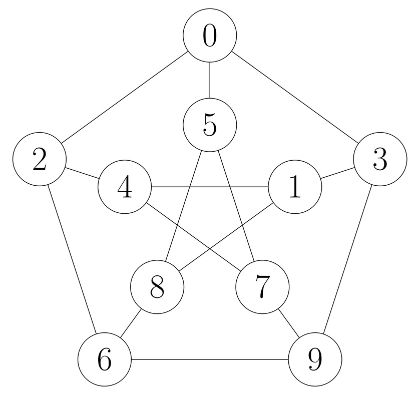

Figure 1: An $L(2,1)$-labeling of Petersen Graph with span 9.

Since Griggs and Yeh's seminal work, $\lambda(G)$ has been determined for various families of graphs [2, 3, 4]. In particular, Georges and Mauro [2] verified Griggs and Yeh's conjecture for some families of 3-regular graphs and, based on their results, posed Conjecture 1.

Conjecture 1. With the exception of the Petersen Graph, every connected 3-regular graph G has $\lambda(G) \leq 7$.

In this work, we verify Conjecture 1 for a family of Loupekine snarks called LP_1-snarks and present a lower bound on $\lambda(G)$ for its members.

Loupekine Snarks

A *snark* is a simple, connected, bridgeless 3-regular graph such that its edges cannot be colored with only three colors such that every two adjacent edges are assigned distinct colors. Snarks are related to fundamental problems in graph theory such as the *4-Color Problem* and the 5-*Flow Conjecture*.

Loupekine snarks were originally defined by Loupekine and first presented by Isaacs [1]. LP_1-snarks are an infinite family of Loupekine snarks and their construction is presented below.

Let k be an odd positive integer. A k-LP_1-snark G is constructed from $k \geq 3$ subgraphs called *blocks*, obtained from the Petersen graph P as follows: given k copies $R_0,\dots,R_{k-1}$ of P, block B_i is obtained from R_i by deleting the vertices of an arbitrary path $P_3 \subset R_i$, for $0 \leq i \leq k-1$. Figure 2 illustrates an arbitrary block B_i with its vertices named. Vertices x_i, u_i, w_i, v_i, y_i are called *border vertices*.

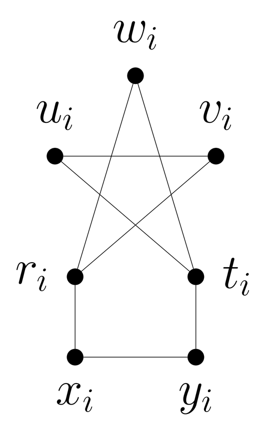

Figure 2: Block B_i.

For all $i \in \{0,\dots,k-1\}$, the border vertices v_i and y_i of block B_i are linked to the border vertices u_{i+1} and x_{i+1} of block B_{i+1} (indices taken modulo k) by edges called *linking edges*. The linking edges can be $\{v_i x_{i+1},\ y_i u_{i+1}\}$ or $\{v_i u_{i+1}, y_i x_{i+1}\}$, but not both.

Any three distinct border vertices w_i, w_j, w_ℓ are linked to a new vertex $u_{i,j,\ell}$, called *star vertex*, by adding $u_{i,j,\ell}$ and three new edges $w_i u_{ij\ell}$, $w_j u_{ij\ell}$ and $w_\ell u_{ij\ell}$ to G. The previous operation can be done an odd number q of times, with $1 \leq q \leq k$. Since k is odd, an even number $k-q$ of border vertices remain. If $k-q > 0$, the remaining border vertices are paired up and each pair w_i and w_j is linked by a new edge $w_i w_j$, thus concluding the construction of a k-LP_1-snark. Figure 3 shows a 3-LP_1-snark with an $L(2,1)$-labeling with span 7.

Results

Theorem 1. Every LP_1-snark G has $\lambda(G) \leq 7$.

Sketch of the proof. Given a k-LP_1-snark G, we construct an $L(2,1)$-labeling f of G with span 7. Initially, choose a block B_i such that its border vertex w_i is adjacent to another border vertex w_j of G. Name this block by B_{k-1} and name the remaining blocks consecutively from this one. If there is no such block, start the enumeration from any block. For every $i \in \{0,\dots,k-1\}$, label the vertices of block B_i as follows: $f(u_i) = f(x_i) = 2 \cdot (2i \bmod 3)$, $f(v_i) = f(y_i) = 2 \cdot (2i+1 \bmod 3)$, $f(r_i) = 6$ and $f(t_i) = 7$. Conflicts occur in this partial labeling when $k \not\equiv 0 \pmod 3$ and, in order to resolve them, some vertex labels in B_{k-1} are changed.

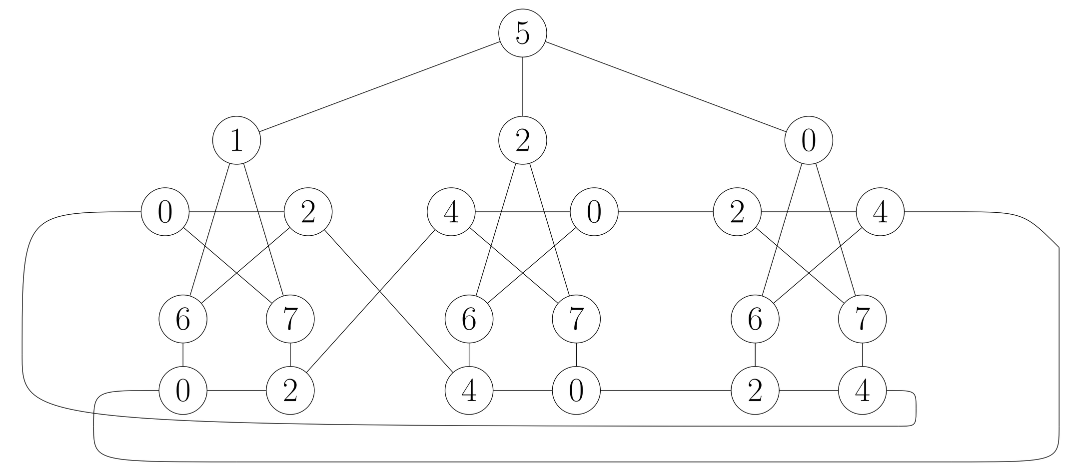

Figure 3: $L(2,1)$-labeling of a 3-LP_1-snark with span 7.

If $k \equiv 1 \pmod 3$, define $f(u_{k-1}) = f(x_{k-1}) = 1$ and $f(v_{k-1}) = f(y_{k-1}) = 3$. If $k \equiv 2 \pmod 3$, define $f(u_{k-1}) = f(x_{k-1}) = 5$, $f(v_{k-1}) = f(y_{k-1}) = 3$ and $f(r_{k-1}) = 1$. Other conflicts can occur depending on the adjacencies of the star vertices. All of them are resolved so that we finally verify that a valid label can always be assigned for every remaining unlabeled vertex without conflict.

Theorem 2. Every LP_1-snark G has $\lambda(G) \geq 6$.

Sketch of the proof. It follows from $L(2,1)$-labeling's definition and G being 3-regular that $\lambda(G) \geq 5$. If $\lambda(B_i) \geq 6$, then $\lambda(G) \geq 6$ since $B_i \subset G$. We suppose that $\lambda(B_i) = 5$. Then, we prove that this assumption restricts to 1 and 4 the labels that a border vertex w_i can have. This restriction leads to a contradiction. Thus, $\lambda(B_i) \geq 6$.

Acknowledgements

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References

[1] R. Isaacs (1976). Loupekine's snarks: a bifamily of non-Tait-colorable graphs. Technical Report 263, Dpt. of Math. Sci., The Johns Hopkins University, Maryland, U.S.A., 1976.

[2] J. P. Georges and D. W. Mauro (2002). On generalized Petersen graphs labeled with a condition at distance two. *Discrete Mathematics*, 259:311–318.

[3] J. R. Griggs and R. K. Yeh (1992). Labelling graphs with a condition at distance 2. *SIAM Journal on Discrete Mathematics*, 5(4):586–595.

[4] D. Ma, H. Zhu and J. He (2014). λ-numbers of several classes of snarks. *Journal of combinatorial Theory*, 28:787–799.

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Two Level Hamming-Huffman Trees

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Introduction

In information theory, there is a common trade-off that arises in data transmission processes, in which two goals are usually tackled independently: data compression and preparation for error detection. While data compression shrinks the message as much as possible, data preparation for error detection adds redundancy to messages so that a receiver can detect, or fix, corrupted ones. Data compression can be achieved using different strategies, often depending on the type of data being compressed. One of the most traditional methods is the method of Huffman [1], that uses ordered trees, known as Huffman trees, to encode the symbols of a given message. In 1980, Hamming proposed the union of both compression and error detection through a data structure called Hamming-Huffman tree [2], which extends the Huffman tree by allowing the detection of any 1-bit transmission error. Determining optimal Hamming-Huffman trees is still an open problem.

Contribution

In this work, we describe an algorithm to determine optimal two level Hamming-Huffman trees when the symbols have uniform frequencies. That is, the algorithm builds optimal Hamming-Huffman trees in which all leaves lay in at most two different levels. Also, considering experimental results, we conjecture that, for uniform frequencies, optimal two levels Hamming-Huffman trees are optimal in general.

Hamming-Huffman Trees

A *Huffman tree* (HT) T is a rooted strict binary tree in which each edge (u, v), v being a left (resp. right) child of u, is labeled by 0 (resp. 1) and there is a one-to-one mapping between the set of leaves of T and the set Σ of symbols of the message M to be sent. Given T, each symbol a of M is encoded into a binary string $c(a)$. Such encoding is obtained by the directed path from the root of T to the leaf corresponding to a. Over all possible trees, the HT for M is a tree in which its cost, defined as the sum of $p(a)|c(a)|$ over all $a \in \Sigma$, is minimized, where $p(a)$ stands for the probability of occurrence of a and $|c(a)|$ is the length of the string $c(a)$.
A *Hamming-Huffman tree* (HHT) T is an extension of the HT in which, for each leaf labeled with $a \in \Sigma$, there exist leaves $e_1, \ldots, e_k$ with $k = |c(a)|$ such that each $c(e_i)$, $1 \leq i \leq k$, differs from $c(a)$ in exactly one position. The leaves $e_1, \ldots, e_k$ are called *error leaves* of a. When $c(e)$ is identified during the decoding process, where e is an error leaf, it means that a transmission error is detected. The cost of HHT's is defined exactly in the same way as the cost of HTs. We define an HHT as *optimal* if its cost is minimum.
Figure 1 depicts an HT with cost 2.4 and an optimal HHT with cost 3.8, both having 5 symbols with uniform frequencies, that is, symbols with a same probability of occurrence.

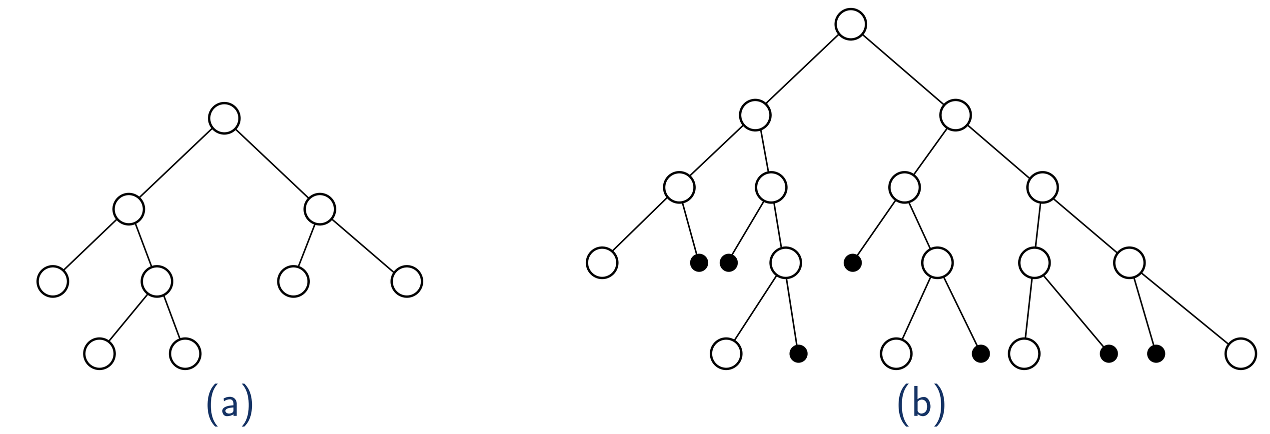

Figure 1: Examples of (a) Huffman and (b) optimal Hamming-Huffman trees, for 5 symbols with uniform frequencies. White (resp. black) leaves represent symbol (resp. error) leaves.

Hamming-Huffman trees with leaves in two levels

Consider the problem of finding an optimal HHT for ℓ uniform-frequency symbols such that these symbols are placed on at most two levels. We will describe an efficient algorithm for this problem. There is a one-to-one mapping between the leaves of a full binary HHT having height n and the vertices of an hypercube Q_n, in which a leaf a corresponds to $c(a) \in V(Q_n)$. The problem of finding the minimum number of error leaves in a full binary HHT T, of height n with ℓ symbol leaves, is equivalent to that of finding one that minimizes $|N(L)|$, over all independent sets L of cardinality ℓ in n-cubes. Define $\varphi(\ell, n)$ as this minimum value.
Concerning an optimal HHT with leaves on two levels $h_1 < h_2$, consider that there are ℓ_1 symbol leaves on level h_1, for some $1 \leq h_1 \leq \lceil \log \ell \rceil + 1$ and $1 \leq \ell_1 \leq \min\{\ell, 2^{h_1 - 1}\}$. Therefore, the minimum number of error leaves is $\varphi(\ell_1, h_1)$ and thus $r(\ell_1, h_1) = 2^{h_1} - (\ell_1 + \varphi(\ell_1, h_1))$ is the number of leaves that neither are symbol nor error leaves. The remaining $\ell_2 = \ell - \ell_1$ symbols are distributed among the subtrees rooted at these $r(\ell_1, h_1)$ leaves. To accomplish this, each subtree is required to have precisely height $h'(\ell_1, h_1) = \lceil \log \frac{\ell_2}{r(\ell_1, h_1)} \rceil + 1$. The strategy is to choose among all the possible trees, one that has minimum cost. Given h_1, ℓ_1 and ℓ, the cost of each tree is given by

$$T(h_1, \ell_1, \ell) = \begin{cases} \ell h_1, \text{ if } \ell = \ell_1 \\ +\infty, \text{ if } r(\ell_1, h_1) = 0 \text{ and } \ell > \ell_1 \\ \ell h_1 + \ell_2 h'(\ell_1, h_1), \text{ otherwise.} \end{cases}$$

The cost of an optimal tree for ℓ symbols can be obtained by

$$\min\{T(h_1, \ell_1, \ell) : 1 \leq h_1 \leq \lceil \log \ell \rceil + 1 \ , \ 1 \leq \ell_1 \leq \min\{\ell, 2^{h_1 - 1}\}\}$$

Concerning the complexity, for each h_1, there are at most $2^{h_1 - 1}$ possible values for ℓ_1. Therefore, there are at most $1 + 2 + 2^2 + \ldots + 2^{\lceil \log \ell \rceil} = \Theta(\ell)$ distinct pairs of values h_1, ℓ_1 to be computed for T. Moreover, for each computation of $T(h_1, \ell_1, \ell)$, the evaluation of $\varphi(\ell_1, h_1)$ is required, which can be done in time $O(h_1^2)$ [3]. So, the complexity of the method is $O(\ell \log^2 \ell)$. Figure 2 depicts this strategy. Nodes with "s" represent symbol leaves, black nodes represent the error leaves, and dashed nodes represent the free leaves.

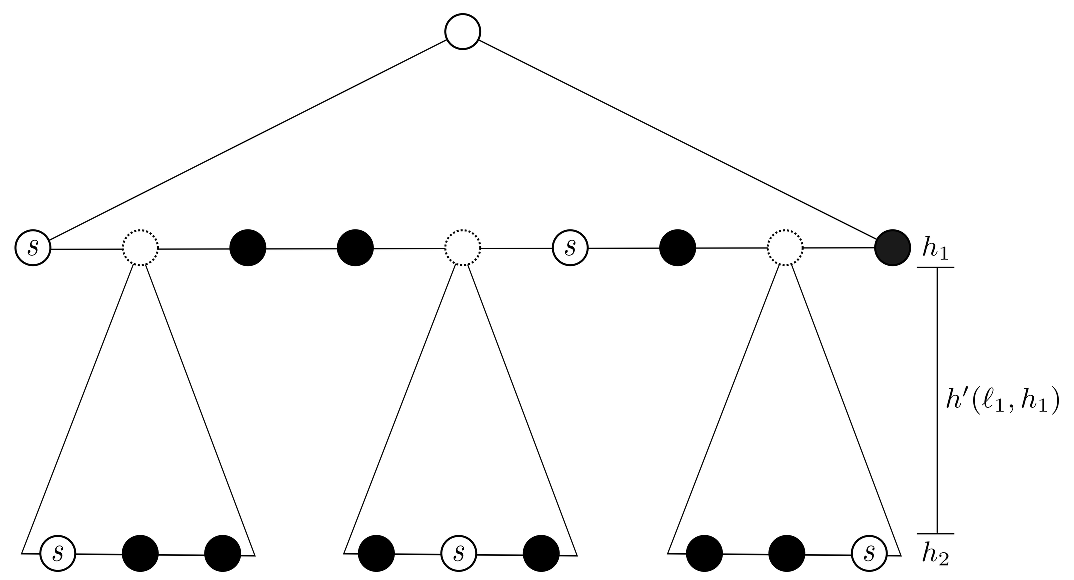

Figure 2: Hamming-Huffman tree with leaves on two levels h_1 and h_2.

Regarding general Hamming-Huffman trees

We have implemented two algorithms. The first one is a backtracking that finds an optimal Hamming-Huffman tree. The second one is a dynamic programming algorithm that evaluates a lower bound for the cost of an optimal Hamming-Huffman tree. Both consider ℓ symbols with uniform frequencies. With respect to the backtracking, we have tested all values of $1 \leq \ell \leq 38$, concluding that there is always an optimal Hamming-Huffman tree with at most two levels. Concerning the dynamic programming algorithm, we have tested all values of $1 \leq \ell \leq 400$. We have verified that, for some cases, the lower bound was equal to the cost of the corresponding optimal two level HHT's.
Considering the results of the experiments, we believe that optimal HHT's for symbols with uniform frequencies indeed have leaves on at most two levels, as formalized in the following conjecture.

Conjecture

Let Σ be a set of symbols having a same frequency. There exists an optimal Hamming-Huffman tree associated with Σ in which all leaves are on at most two levels.

References

[1] David A. Huffman.
A method for the construction of minimum redundancy codes.
Proceedings of the IRE, 40:1098–1101, 1951.

[2] Richard W. Hamming.
Coding and Information Theory.
Prentice-Hall, 1986.

[3] Janos Körner and Victor K. Wei.
Odd and even Hamming spheres also have minimum boundary.
Discrete Mathematics, 51:147–165, 1984.

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A RESULT ON TOTAL COLORING OF FULLERENE NANODISCS

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9th Latin American Workshop on Cliques in Graphs

1. Introduction

A *graph* is a mathematical model used to represent relationships between objects. The general characters that both of these objects and their relationships can assume, allowed the construction of the so-called Graph Theory, which has been applied to model problems in several areas, such as Mathematics, Physics, Computer Science, Engineering, Chemistry, Psychology and industry. Most of them are large scale problems.
Fullerene graphs are mathematical models for carbon-based molecules experimentally discovered in the early 1980s by Kroto, Heath, O'Brien, Curl and Smalley. Many parameters associated with these graphs have been discussed to describe the stability of fullerene molecules.
By definition, fullerene graphs are cubic, planar, 3-connected with pentagonal and hexagonal faces.
The motivation of the present study is to find an efficient method to obtain a 4-total coloring of a particular class of fullerene graphs named fullerene nanodiscs, if it exists.

2. Basic Concepts of Graph Theory

This section is based on the reference Bondy and Murty, 2008.

Definition 1. A **graph** $G = (V(G), E(G))$ is an ordered pair, where $V(G)$ is a nonempty finite set of vertices and $E(G)$ is a set of edges disjoint from $V(G)$, formed by unordered pairs of distinct elements from $V(G)$, that is, for every edge $e \in E(G)$ there is u and $v \in V(G)$ such that $e = \{u, v\}$, or simply $e = uv$.

If $uv \in E$, we say that u and v are adjacent or that u is a neighbor of v, and that the edge e is incident to u and v, and u and v are said to be extremes (or ends) of e. Two edges that have the same end are called adjacent.
The degree of a vertex v in G, represented by $d(v)$, is the number of edges incident to v. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees respectively, of the vertices of the graph G.

A graph G is said **connected** when there is a path between each pair of vertices of G. Otherwise, the graph is called disconnected.

A **cubic** graph is one in which all vertices have three incident edges and in this case, all vertices have degree 3. Cubic graphs play a fundamental role in Graph Theory.

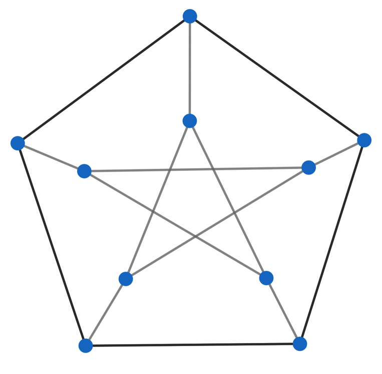

Figure 1: *Cubic Graph.*

A graph G is planar if there is a representation of G in the plane so that the edges meet only at the vertices, that is, the edges do not cross. Such a representation of G is said to be embeddable or planar. A planar representation divides the plane into regions called faces. There is always a single face called external or infinite, which is not limited (has infinite area). The outer boundary or cycle of a connected planar graph face is a closed walk that limits and determines the face.

Two faces are adjacent if they have a common edge between their boundaries. We denote the boundary of f by $\partial(f)$. If f is any face, the degree of f, denoted by $d(f)$, is the number of edges contained in the closed walk that defines it. In a planar connected graph with f faces, n vertices and m edges, we have that $n + f - m = 2$, which is known as Euler's formula.

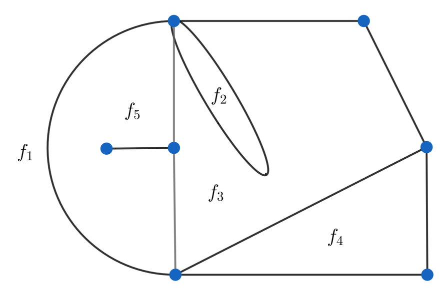

Figure 2: *Planar Graph.*

2.1 Total Coloring

In graph theory, coloring is a color assignment to the graph elements, subject to certain restrictions. The coloring study started with the Four Color Conjecture, which deals with determining the minimum number of colors needed to color a map of real or imaginary countries, so that countries with common borders have different colors. This conjecture was proposed by Francis Guthrie in 1852. After 124 years, the Four Color Conjecture was demonstrated by Kenneth Appel and Wolfgang Haken with the help of a computer. The famous Four Color Theorem is a reference in the area of Graph Theory.

Definition 2. A **total coloring** C^T of a graph G is a color assignment to the set $E \cup V$ in a color set $C = \{c_1, c_2, ..., c_k\}$, $k \in \mathbb{N}$, such that distinct colors are assigned to:

- Every pair of vertices that are adjacent;
- All edges that are adjacent;
- Each vertex and its incident edges.

A k-total coloring of a graph G is a total coloring of G that uses a set of k colors, and a graph is k-total colorable if there is a k-total coloring of G. We define as the **total chromatic number** of a graph G the smallest natural k for which G admits a k-total coloring, and is denoted by $\chi''(G)$.

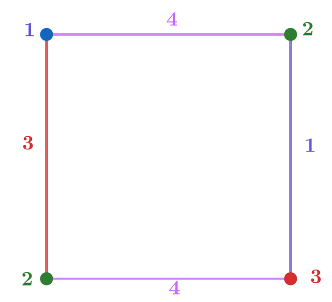

Figure 3: *Graph with 4-total coloring.*

Behzad and Vizing independently conjectured the same upper bound for the total chromatic number.

Conjecture (Total Color Conjecture (TCC))
For every simple graph G,
$$\chi''(G) \leq \Delta(G) + 2.$$

The TCC is an open problem, but has been checked for several classes of graphs. Knowing that $\chi''(G) \geq \Delta(G) + 1$, and from the TCC, we have the following classification: If $\chi''(G) = \Delta(G) + 1$, the graph is **Type** 1; and if $\chi''(G) = \Delta(G) + 2$, the graph is **Type** 2.
For cubic graphs, the TCC has already been demonstrated, which indicates that these graphs have total chromatic number 4 $(\Delta + 1)$ or 5 $(\Delta + 2)$. However, the problem of deciding which are Type 1 or Type 2 is difficult.

3. Fullerene Graphs

3.1 Fullerene: A small history

In 1985 a new carbon allotrope was reported in the scientific community: C_{60}. A group of scientists, led by Englishman Harold Walter Kroto and Americans Richard Errett Smalley and Robert Curl, trying to understand the mechanisms for building long carbon chains observed in interstellar space, discovered a highly symmetrical, stable molecule, composed of 60 carbon atoms different from all the other carbon allotropes.

The C_{60} has a structure similar to a soccer hollow ball (Figure 4), with 32 faces, being 20 hexagonal and 12 pentagonal. They decided to name the C_{60} *buckminster-fullerene*, in honor of American architect Richard Buckminster Fuller, famous for his geodesic dome constructions, which were composed of hexagonal and pentagonal faces.
At the end of the 1980s, other carbon allotrope molecules with similar spatial structure to the C_{60} were reported called fullerene molecules (Kroto et al., 1985).

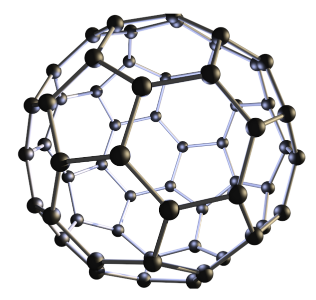

Figure 4: *Molecular structure of C_{60}.*

The buckminsterfullerene was the first new allotropic form discovered in the 20th century, and earned Kroto, Curl and Smalley the Nobel Prize in Chemistry in 1996. Nowadays fullerene molecules are widely studied by different branches of science, from medicine to mathematics. These molecules are supposed to contribute to transport chemotherapy, antibiotics or antioxidant agents and released in contact with deficient cells.

3.2 Fullerene Graphs

Each fullerene molecule can be described by a graph where the atoms and the bonds are represented by the vertices and edges of the graph, respectively. In addition, fullerene graphs preserve the geometric properties of fullerene molecules, i.e., fullerene graphs are planar and connected. Moreover, all vertices have exactly 3 incident edges and all faces are pentagonal or hexagonal (Nicodemos, 2017).

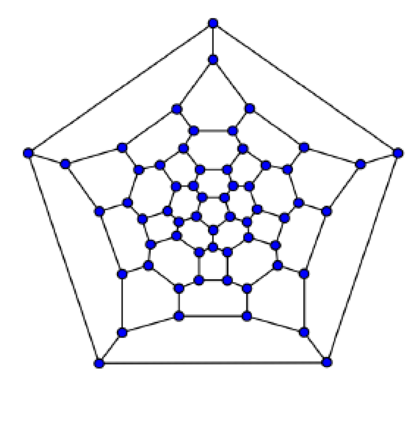

Figure 5: *Fullerene Graph.*

3.3 Fullerene Nanodiscs

The fullerene nanodiscs, or nanodiscs of radius $r \geq 2$ are structures composed of two identical flat covers connected by a strip along their borders. While in the nanodisc lids there are only hexagonal faces, in the connecting strip, 12 pentagonal faces are arranged side by side.
A nanodisc of radius $r \geq 2$, represented by $D_{r,t}$, can be obtained through its flattening. The idea is to arrange the faces in layers around the nearest previous layer starting from a hexagonal face (Nicodemos, 2017).

The sequence

$$\{1, 6, 12, 18, ..., 6(r-1), 6r, 6(r-1), ..., 18, 12, 6, 1\}$$

provides the amount of faces on each layer of nanodisc planning D_r, while $r \geq 2$. In addition, this sequence states that a D_r nanodisc has $(6r^2 + 2)$ faces, $12r^2$ vertices and $(2r + 1)$ layers. The 12 pentagonal faces will always be distributed in the same layer with other $(6r - 12)$ hexagonal faces. This is the key property of fullerene nanodiscs.

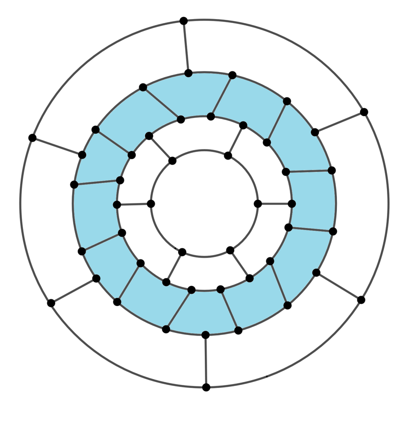

Figure 6: *Nanodisc D_2.*

4. Goals

To prove that a cubic graph is Type 1, it suffices to show a total coloring with 4 colors. However, to demonstrate that a cubic graph is Type 2, we need to show that it has no total coloring with only 4 colors. Thus, finding Type 2 cubic graphs is more complicated.
We define the girth of a graph G as the length of its shortest cycle. Until now, every Type 2 cubic graph we know has squares or triangles. So, we could think that there are no Type 2 cubic graphs with girth at least 5. Thus, we investigate the following question.

Question
(Sasaki, 2013) Does there exist a Type 2 cubic graph with girth at least 5?

Motivated by this question, we analyze the family of fullerene nanodiscs, in search of evidences that can positively or negatively contribute to this question. In this context, we look for an efficient algorithm to find a 4-total coloring of the fullerene nanodisc, if this coloring exists.

5. Results

After a few attempts using the brute force method, we were able to obtain a 4-total coloring of the D_2 nanodisc, with $r = 2$. Therefore, D_2 is Type 1, which contributes to the evidences that the previously proposed question has a negative answer.

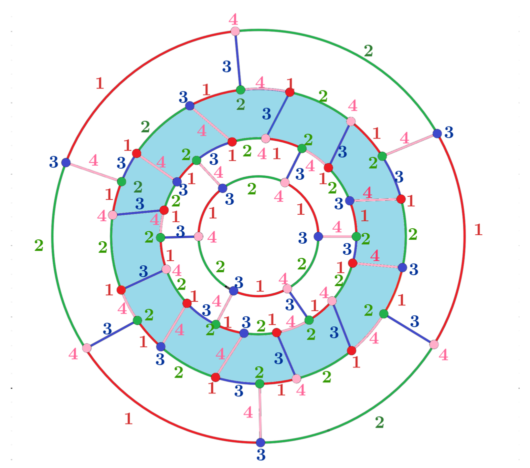

Figure 7: *A 4-total coloring of D_2.*

6. Conclusion

We will continue the study of total coloring of nanodiscs, looking for an algorithm that gives a total coloring of the graphs of the infinite family of fullerene nanodiscs, also seeking to answer the question previously proposed.

References

[1] BONDY, J. A and Murty, U. S. R.; *Graduate Texts in Mathematics 244 - Graph Theory.* Springer, 2008.

[2] KROTO, H. et al; C_{60}: *Buckminsterfullerene* Nature, 1985.

[3] NICODEMOS, D. de S. *Diâmetro de Grafos fulerenos e Transversalidade de Ciclos Ímpares de Fuleróides-(3, 4, 5, 6).* Rio de Janeiro: UFRJ/COPPE, 2017.

[4] SASAKI, D. *Sobre Coloração Total de Grafos Cúbicos* Rio de Janeiro: UFRJ/COPPE, 2013.

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Alliance and Domination on Uniformly Clique-expanded Graphs

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Introduction

This work aims at presenting the uniformly clique-expanded graphs and its results on global defensive alliance and total dominating set problems. Those graphs are related to Sierpiński graphs [5] and subdivided-line graphs [1]. We show the minimum cardinality of the global defensive alliance for some particular situations of uniformly clique-expanded graphs, and we also relate that cardinality to the total dominating set number for graphs having a path or cycle as the root.

Basic Definitions

Consider $G = (V, E)$ a finite, simple, and undirected graph. We write P_n, C_n, and K_n for a *path*, *cycle*, and *clique* of the order n, resp. For the *closed* (resp. *open*) *neighborhood of a vertex* $v \in V$, we denote it by $N[v]$ (resp. $N(v)$). Analogously, we use $N[S]$ (resp. $N(S)$) for the *closed* (resp. *open*) *neighborhood of a vertex subset* $S \subseteq V$. A vertex subset $S \subseteq V$ is said a *dominating set* if $N[S] = V$. Moreover, we call the subset S by *total dominating* set only for $N(S) = V$. Now, S is a *defensive alliance* if it satisfies $|N[v] \cap S| \geq |N(v) \cap (V/S)|$ for every $v \in S$. When S is both a defensive alliance and a dominating set, we say S is a *global defensive alliance*. We denote $\gamma_t(G)$ (and $\gamma_a(G)$) as the minimum cardinality of a total dominating set (and global defensive alliance) of G.

The Main Definition & an Example

We say that a graph H is a *uniformly clique-expanded graph* if there exist a graph G and a clique K_n with $n \geq \Delta(G)$ (maximum degree of G) satisfying: (1) $V(H)$ consists of vertices from K_n^v, which is a copy of the clique K_n, for each vertex v of G, and (2) $E(H)$ contains edges of all clique copies, and every edge $(u)(v)$ linking a vertex $(u) \in K_n^u$ to some $(v) \in K_n^v$ since $uv \in E(G)$ and no edges coincide end-vertices in H besides the ones inside of cliques. G is the so-called *root* of H. See an example in Figure 1.

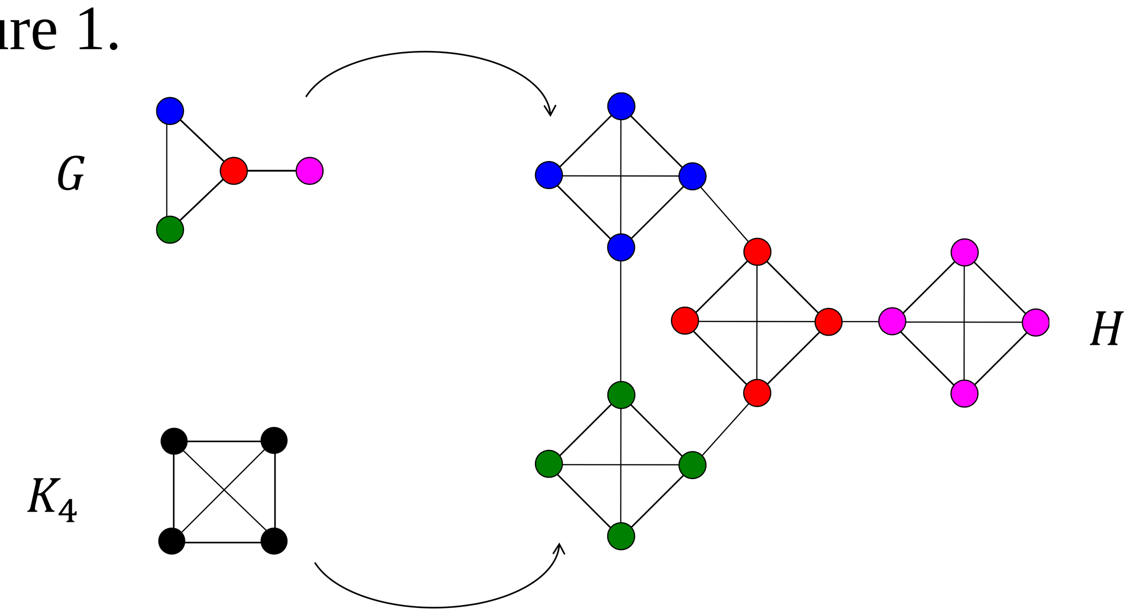

Figure 1: The graph H can be obtained from the root G and the clique K_4, and so it is a uniformly clique-expanded graph.

Results

Theorem 1: Let H be a uniformly clique-expanded graph from a root G and a clique K_n. If n is even and $\Delta(G) \leq \frac{n}{2}$, then $\gamma_a(H) = \frac{n}{2}|V(G)|$.

Theorem 2: Let H be a uniformly clique-expanded graph from a root G and a clique K_n. If n is odd and $\Delta(G) \leq \frac{n-1}{2}$, then:$\gamma_a(H) = \sum_{d(v)<\frac{n-1}{2}} \frac{n+1}{2} + \sum_{d(v)=\frac{n-1}{2}} \frac{n-1}{2}$, for all $u \in V(G)$, where $d(u)$ is the degree of v in G.

Now, the next theorem arises from properties in [2,3,4].

Theorem 3: Let H be a uniformly clique-expanded graph from a root $G \in \{P_q, C_q\}$, $q \geq 2$, and a clique K_n. We have $\gamma_t(H) = q + q \bmod 2$, and if:

i. G is a cycle and:

- *a.* $2 \leq n \leq 3$, then $\gamma_a(H) = \gamma_t(H)$;
- *b.* $4 \leq n \leq 5$, then $\gamma_a(H) = \lfloor \frac{n}{2} \rfloor q$;
- *c.* $n \geq 6$, then $\gamma_a(H) = \lceil \frac{n}{2} \rceil q$.

ii. G is a path and:

- *a.* $n = 2$, then $\gamma_a(H) = \gamma_t(H) - 1$ whether $p \equiv 1 (\mathrm{mod}\ 2)$ or $\gamma_a(H) = \gamma_t(H)$ otherwise;
- *b.* $n = 3$, then, $\gamma_a(H) = \gamma_t(H)$;
- *c.* $n = 4$, then $\gamma_a(H) = \frac{n}{2}q$.
- *d.* $n = 5$, then $\gamma_a(H) = \frac{n-1}{2}q$.
- *e.* $n \geq 6$, then $\gamma_a(H) = \lceil \frac{n}{2} \rceil q$.

Conclusions & Remarks

The uniformly clique-expanded graphs are particular cases of line graphs of bipartite graphs since we can verify that they are (claw,diamond,odd-hole)-free. Thus, we presented preliminary results that somehow are important to the well-known superclass.

References

[1] HASUNUMA T. Structural Properties of Subdivided-Line Graphs. In: Lecroq T., Mouchard L. (eds) Combinatorial Algorithms. IWOCA 2013. Lecture Notes in Computer Science, vol 8288. Springer, Berlin, Heidelberg, 2013. https://doi.org/10.1007/978-3-642-45278-9_19

[2] HAYNES, T. W.; HEDETNIEMI, S.T.; HENNING, M.A. Global defensive alliances in graphs. The Electronic Journal of Combinatorics, 2003.

[3] HENNING, M.A.: Graphs with large total domination number. J. Graph Theory 563 35(1), 21–45 (Sep 2000)

[4] KAHINA OUAZINE, HACHEM SLIMANI, ABDELKAMEL TARI. Alliances in graphs: Parameters, properties and applications – A survey. AKCE International Journal of Graphs and Combinatorics. DOI: 10.1016/j.akcej.2017.05.002, 2018.

[5] KLAVŽAR, S., MILUTINOVIĆ, U., PETR, C. 1-perfect codes in sierpiński graphs. Bulletin of the Australian Mathematical Society66(3), 369–384. (2002)

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Sharp Bounds for the Annihilation Number of the Nordhaus-Gaddum type

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INTRODUCTION

The **annihilation number** is a graph invariant used as a sharp upper bound for the independence number. In this poster, we present bounds and Nordhaus-Gaddum type inequalities for the annihilation number.

We also investigate the extremal behavior of the invariant and showed that both parameters satisfy the **interval property**. In addition, we characterize some extremal graphs, ensuring that the bounds obtained are the best possible.

ANNIHILATION NUMBER

The **independence number** of a graph is the cardinality of a largest set of mutually non-adjacent vertices. It is not always possible to determine the number of independence of a graph, since this is a well-known widely-studied NP-hard problem, and for this reason the approximation of the independence number through inequalities represents a relevant research topic.

The annihilation number is a polynomial time computable upper bound for the independence number introduced by R. Pepper and S. Fajtlowicz [1,2].

Definition

The annihilation number of G, denoted by $a(G)$, can be defined as the largest integer k such that the sum of the smallest k degrees of graph G was at most its number of edges $e(G)$, that is

$$a(G) = \max\left\{ k \in \mathbb{N} : \sum_{i=1}^{k} d_i \leq e(G) \right\},$$

where d_i is the i-th smallest degree of G.

The annihilation number and the independence number are used to investigate the relationship between the reactivity of an organic molecule, represented by a graph, and its independence number. More precisely, the research states that, for a fixed number of vertices, molecules with a lower independence number are, in general, less reactive than molecules with a greater independence number. This study is known in organic chemistry as the independence-stability hypothesis [2].

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NORDHAUS-GADDUM PROBLEM

The Nordhaus-Gaddum problem is related to find lower and upper bounds on the sum and the product of the invariant of a graph and its complement, denoted by G^c [3].

The Nordhaus-Gaddum problem was studied for several domination parameters associated with the annihilation number, such as the independence number, the domination number, the Roman domination number, the total domination number, among others. This establishes a valuable connection between the annihilation number and the Nordhaus-Gaddum problem.

INTERVAL PROPERTY

Let $\mathcal{G}$ be a collection of graphs and $\xi : \mathcal{G} \to \mathbb{R}$ be a graph parameter defined on $\mathcal{G}$. We say that ξ has the interval property on $\mathcal{G}$ if $\xi(\mathcal{G}) = I \cap \mathbb{Z}$, for some interval $I \subset \mathbb{R}$ [4].

In other words, a graph parameter satisfies the interval property if each integer value in an interval is realized by at least one graph. The interval property generalizes the behavior of a parameter in an interval making it a relevant research topic.

BOUNDS FOR ANNIHILATION NUMBER

We present bounds for the annihilation number of a graph and prove that those bounds are the best possible. To state the result, we denote by K_n the **complete graph** on n vertices.

Theorem

Let G be a graph of order n. Then

$$\left\lfloor \frac{n}{2} \right\rfloor \leq a(G) \leq n.$$

Equality holds in the upper bound if and only if G is isomorphic to nK_1.
If G is a non-empty k-regular graph then the equality holds in the lower bound.

As a consequence, we show that the annihilation number satisfies the interval property.

Interval Property for $a(G)$

Let n and k be integers such that $\lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n-1$.
If G is isomorphic to

$$(n-k)K_2 \cup (2k-n)K_1,$$

then $a(G) = k$.

NORDHAUS-GADDUM FOR $a(G)$

We present Nordhaus-Gaddum inequalities associated with the annihilation number and ensure that they are the best possible. To state the result, we denote by S_n the **star graph** on n vertices.

Theorem

Let G be a graph of order n. Then

$$2\left\lfloor \frac{n}{2} \right\rfloor \leq a(G) + a(G^c) \leq n + \left\lfloor \frac{n}{2} \right\rfloor.$$

For n even, the equality holds in the upper bound if and only if G or G^c is isomorphic to nK_1.
For n odd, the equality holds in the upper bound if and only if G or G^c is isomorphic to nK_1 or $S_{d_n+1} \cup (n-d_n-1)K_1$, for $\lfloor \frac{n}{2} \rfloor \leq d_n \leq n-1$.
If G and G^c are non-empty graphs and G is a k-regular graph then the equality holds in the lower bound.

We then show that $a(G) + a(G^c)$ satisfies the interval property.

Interval Property for $a(G) + a(G^c)$

Let n and k be integers such that $2\lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n + \lfloor \frac{n}{2} \rfloor - 1$.
If G is isomorphic to

$$\left(n + \left\lfloor \frac{n}{2} \right\rfloor - k\right) K_2 \cup \left(2k - 2\left\lfloor \frac{n}{2} \right\rfloor - n\right) K_1,$$

then $a(G) + a(G^c) = k$.

CONCLUSION

We obtained important structural information about the graphs that satisfy the equality in the upper bounds. In particular, we can observe that, in general, such graphs have few edges.

The lower bounds are satisfied by a large number of graphs and, consequently, their characterization is important for understanding the extremal behavior of the annihilation number.

REFERENCES

[1] Pepper, R.: Binding independence. University of Houston (2004)
[2] Fajtlowicz, S., Larson, C.: Graph-theoretic independence as a predictor of fullerene stability. Chemical Physics Letters 377(5), 485–490 (2003).
[3] Nordhaus, E.A., Gaddum, J.W.: On complementary graphs. The American Mathematical Monthly 63(3), 175–177 (1956).
[4] Jaume, D.A., Pastine, A., Schvollner, V.N.: 2-switch: transition and stability on graphs and forests (2020)

Un algoritmo lineal para el problema de la k-upla dominación en grafos web

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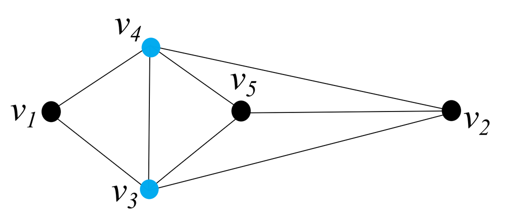
Dados un grafo G con conjunto de vértices $V(G)$ y $k \in \mathbb{N}$, $k \leq min\,\{d(v):\ v \in V(G)\}+1$, $D \subseteq V(G)$ es un **conjunto k-upla dominante** en G si

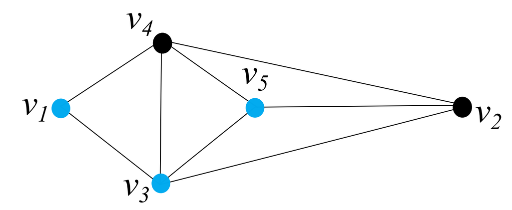
$$|N[v] \cap D| \geq k \quad \forall v \in V(G).$$

$N[v]$: vecindad cerrada del vértice v



$D = \{v_3, v_4\}$

$k = 2$



$D = \{v_1, v_3, v_5\}$

Problema de la k-upla dominación (PkUD), $k \in \mathbb{N}$ fijo

Dado G, el problema consiste en hallar

$$\gamma_{\times k}(G) = min\,\{|D| : D \text{ es conjunto } k\text{-upla dominante en } G\}$$

[Harary & Haynes, 2000].

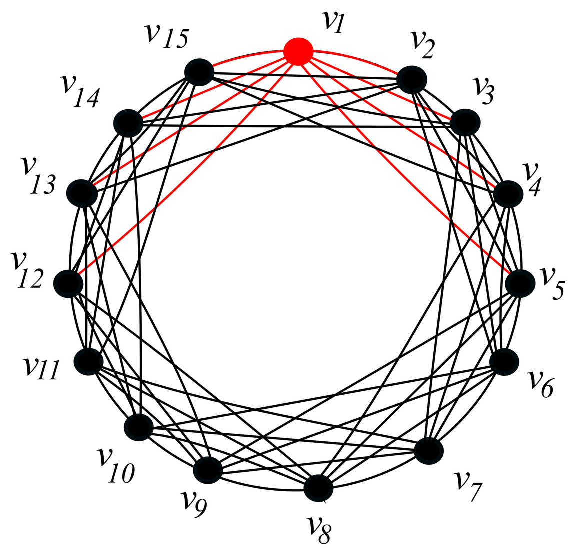
- NP-Completo, aún en grafos cordales [Liao & Chang, 2003].
- P1UD, lineal en grafos arco-circulares [Hsu & Tsai, 1991].
- Complejidad no conocida de PkUD en grafos arco-circulares para $k \geq 2$.

Una subclase de grafos arco-circulares:

Grafo web: $W_n^m \quad n, m \in \mathbb{N} \quad m \geq 1, \quad n \geq 2m+1.$

$V(W_n^m) = \{v_1, v_2, \cdots, v_n\}$.

$E(W_n^m) = \{v_i v_j \ :\ j \equiv i \pm l \ (mod\ n)\,,\ l \in \{1, \cdots, m\}\ \}$.



W_{15}^4

PkUD en grafos web W_n^m: Antecedentes

Teorema [Argiroffo, Escalante & Ugarte, 2010]
$n, m \in \mathbb{N}:\ n = c(2m+1)+r, \quad c \in \mathbb{N}, \quad 0 \leq r \leq 2m.$

- $\gamma_{\times 2}(W_n^m) = \begin{cases} 2c, & r = 0 \\ 2c+1, & 0 < r \leq m \\ 2c+2, & m+1 \leq r \leq 2m. \end{cases}$
- $k\left\lfloor \dfrac{n}{2m+1} \right\rfloor \leq \gamma_{\times k}(W_n^m) \leq k\left\lceil \dfrac{n}{2m+1} \right\rceil, \quad \forall\ k \leq 2m.$

Teorema [Dobson, Leoni & Lopez Pujato, 2019]
$n, m \in \mathbb{N}:\ n = c(2m+1)+r, \quad c \in \mathbb{N}, \quad 0 \leq r \leq 2m.$

$$\gamma_{\times k}(W_n^m) = \left\lceil \frac{kn}{2m+1} \right\rceil, \quad \forall\ k \leq 2m+1.$$

Objetivo: Dado W_n^m, presentar un algoritmo que devuelve un conjunto k-upla dominante en W_n^m de tamaño $\gamma_{\times k}(W_n^m)$.

Notación:

$n, m \in \mathbb{N}:\ n = c(2m+1)+r, \quad c \in \mathbb{N}, \quad 0 \leq r \leq 2m,$
$M := mcd\,(2m+1, r)\,, \quad [1, x]_{\mathbb{N}} := \{z \in \mathbb{N}:\ 1 \leq z \leq x\}\,.$

- Para cada $i \in [1, M]_{\mathbb{N}}$:

$[i]_M \rightarrow$ clase de equivalencia de i módulo M,

$S_i := [i]_M \cap [1, n]_{\mathbb{N}}$.

Lema

- $\{S_i\}_{i=1}^M$ es una partición de $[1, n]_{\mathbb{N}}$.
- $|S_i| = \dfrac{n}{M} \qquad \forall\ i \in [1, M]_{\mathbb{N}}$.

Identificamos:

$j \in S_i \leftrightarrow v_{m+j} \in V(W_n^m)$ (suma mod n en los subíndices, en $[1, n]$).

Corolario

$\{S_i\}_{i=1}^M$ es una partición de $V(W_n^m)$ en conjuntos de tamaño $\dfrac{n}{M}$.

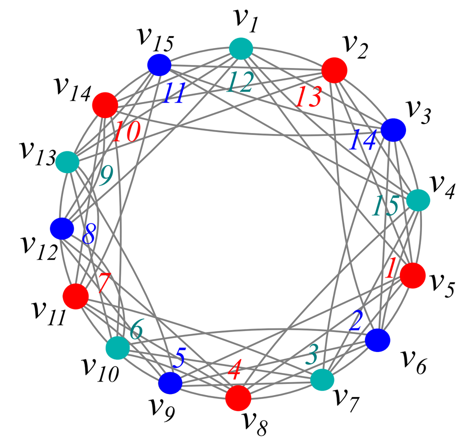
Ejemplo sobre $V(W_{15}^4)$:

$2m+1 = 9,\ r = 6\ ,\ \ mcd(9, 6) = 3 = M.$

$S_1 = [1]_3 \cap [1, 15]_{\mathbb{N}} = \{1, 4, 7, 10, 13\}$

$S_2 = [2]_3 \cap [1, 15]_{\mathbb{N}} = \{2, 5, 8, 11, 14\}$

$S_3 = [3]_3 \cap [1, 15]_{\mathbb{N}} = \{3, 6, 9, 12, 15\}$



Propiedades de los conjuntos de la partición de $V(W_n^m)$:

Proposición 1:

Dado W_n^m, para cada $v \in V(W_n^m)$ y cada $i \in [1, M]_{\mathbb{N}}$ se tiene $|N[v] \cap S_i| = l$, i.e. **S_i es conjunto l-upla dominante de W_n^m**, donde $\ 2m+1 = lM, \ \ l \in \mathbb{N}$.

Proposición 2:

Dados W_n^m y $l \in \mathbb{N}$ tal que $2m+1 = lM$, se tiene

$$\gamma_{\times l}(W_n^m) = \frac{n}{M}.$$

Proposición 3:

Para cada $i \in [1, M]_{\mathbb{N}}$ se tiene

$$S_i = \bigcup_{t \in [0,\, n/M-1]_{\mathbb{N}}} \{w \in [1, n] : w \equiv i + t(2m+1)\ (mod\ n)\}\,.$$

Definición: para $i, j \in V(W_n^m)$, $\ j$ es **1-contiguo** a i si

$$j \equiv i + 2m+1 \ \ (mod\ n).$$

- La 1-contiguidad induce en cada S_i un ordenamiento tal que, empezando por i, cada vértice se obtiene del anterior, como un «movimiento circular» de $2m+1$ posiciones.
- Procedimiento **PROC(n,m,i)** $\rightarrow$ devuelve $\langle S_i \rangle$ (S_i con el ordenamiento).

Procedimiento **DOM** $(n,\ m,\ \langle S_j \rangle,\ \alpha) \rightarrow$ devuelve un conjunto α-upla dominante en W_n^m donde $\alpha \in \mathbb{N},\ \ \alpha \leq l\ $ y $\ 2m+1 = lM$.
$t = 0$ *....(indica cuál fue el último elemento incorporado a D según $\langle S_j \rangle$).*
$h = 1$
DIV $(n,\ 2m+1)$ *....(obtiene el resto r de la división entera).*
$M = mcd(2m+1, r)$
$D = \emptyset$
mientras $h \leq \alpha \ \ \wedge \ \ i \leq \dfrac{n}{M}$ *....(h indica que D será un conjunto h-upla dominante en W_n^m).*
  $i = t+1$ *....(indica cuál será el próximo elemento a incorporar a D según $\langle S_j \rangle$).*
  mientras $s_i^j + 2m < n \ \ \wedge \ \ i \leq \dfrac{n}{M}$
   $D = D \cup \{s_i^j\}$
   $i = i+1$. Fin
  $D = D \cup \{s_i^j\}$
  $h = h+1$
  $t = i$. Fin
Fin Procedimiento

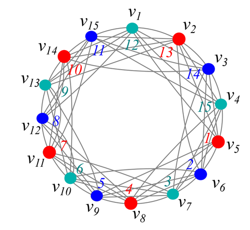
ALGORITMO: Conjunto k-upla dominante mínimo en W_n^m (k-fijo)

Entrada: $n \in \mathbb{N},\ m \in \mathbb{N}$ con $n \geq 2m+1$.
Salida: Un conjunto k-upla dominante mínimo D en W_n^m.

1: **DIV**$(n, 2m+1)$ y obtener resto r.
2: $M := mcd(2m+1, r)$.
3: **DIV**$(2m+1, M)$ y obtener cociente l.
4: **PROC**$(n, m, 1)$ y obtener $\langle S_1 \rangle$.
5: Si $k \leq l$ luego $D =$**DOM**$(n,\ m,\ \langle S_1 \rangle,\ k)$.
sino ($k > l$) hacer **DIV**(k, l) y obtener cociente $\tilde{c}$ y resto $\tilde{r}$.
$D=$ **DOM**$(n, m, \langle S_1 \rangle, \tilde{r}) \cup$ **PROC**$(n, m, 2) \cup \cdots \cup$ **PROC**$(n, m, \tilde{c}+1)$.

$$D = \textbf{DOM}(n, m, \langle S_1 \rangle\,, \tilde{r}) \cup \bigcup_{i=2}^{\tilde{c}+1} S_i.$$

ALGORITMO LINEAL.



Aplicando el algoritmo en W_{15}^4

$\gamma_{\times k}(W_{15}^4)$	D
$\gamma_{\times 1}(W_{15}^4) = 2$	$\{v_5, v_{14}\}$
$\gamma_{\times 2}(W_{15}^4) = 4$	$\{v_5, v_{14}, v_8, v_2\}$
$\gamma_{\times 3}(W_{15}^4) = 5$	$\{v_5, v_{14}, v_8, v_2, v_{11}\}$
$\gamma_{\times 4}(W_{15}^4) = 7$	$\{v_5, v_{14}, v_8, v_2, v_{11}, v_6, v_{15}\}$
$\gamma_{\times 5}(W_{15}^4) = 9$	$\{v_5, v_{14}, v_8, v_2, v_{11}, v_6, v_{15}, v_9, v_3\}$
$\gamma_{\times 6}(W_{15}^4) = 10$	$\{v_5, v_{14}, v_8, v_2, v_{11}, v_6, v_{15}, v_9, v_3, v_{12}\}$
$\gamma_{\times 7}(W_{15}^4) = 12$	$\{v_5, v_{14}, v_8, v_2, v_{11}, v_6, v_{15}, v_9, v_3, v_{12}, v_7, v_1\}$
$\gamma_{\times 8}(W_{15}^4) = 14$	$\{v_5, v_{14}, v_8, v_2, v_{11}, v_6, v_{15}, v_9, v_3, v_{12}, v_7, v_1, v_{10}, v_4\}$
$\gamma_{\times 9}(W_{15}^4) = 15$	$V(W_{15}^4)$

REFERENCIAS:

* Argiroffo, G., Escalante, M., Ugarte, M.E., *On the k-dominating set polytope of web graphs*, Electronic Notes in Discrete Mathematics **36** (2010), 1161–1168.
* Dobson, M.P., Leoni, V., Lopez Pujato, M.I., *Tuple domination on graphs with the consecutive-zeros property*, Electronic Notes in Theoretical Computer Science **346** (2019) 401-411.
* Dobson, M.P., Leoni, V., Lopez Pujato, M.I., *Efficient algorithms for tuple domination on co-biconvex graphs and web graphs*, http://arxiv.org/abs/2008.05345.
* Harary, F., Haynes, T. W., *Double domination in graphs*, Ars Combin. **55** (2000), 201–213.
* Hsu, W.L., Tsai, K.H., *Linear time algorithms on circular-arc graphs*, Inform. Process. Lett. **40**, 3 (1991), 123–129.
* Liao, C.S., Chang, G. J., *k-tuple domination in graphs*, Inform. Process. Lett. **87**, 1 (2003), 45–50.
* Safe, M. D., *Characterization and linear-time detection of minimal obstructions to concave-round graphs and the circular-ones property*, Journal of Graph Theory **93** 2 (2020) 268–298.
* Tucker, A., *Matrix characterizations of circular-arc graphs*, Pacific J. Math. **39.2** (1971), 535–545.

Rotulação $\mathscr{L}(h,k)$ dos Sunlets

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1 Introdução

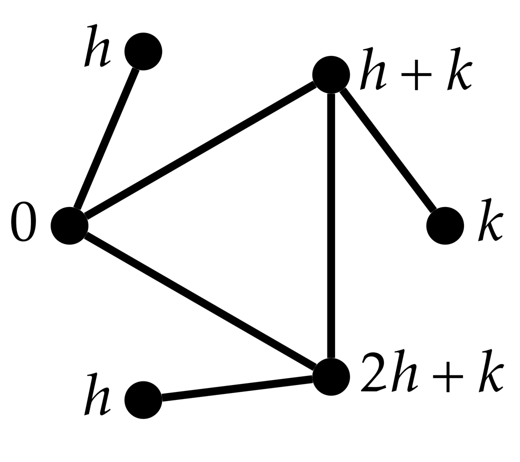
As *rotulações* $\mathscr{L}(h,k)$ foram introduzidas como uma generalização natural das *rotulações* $\mathscr{L}(2,1)$ [1], estas conhecidas por sua importância para o problema de atribuir canais em redes [2].

Rotulação $\mathscr{L}(h,k)$

Sejam $h,k \in \mathbb{Z}_{\geqslant 0}$ e G um grafo simples. Uma *rotulação* $\mathscr{L}(h,k)$ de G é uma função $\sigma\colon V(G) \to \mathbb{Z}_{\geqslant 0}$ tal que:

(i) $|\sigma(u)-\sigma(v)| \geqslant h,\ \forall uv \in E(G)$;

(ii) $|\sigma(u)-\sigma(v)| \geqslant k,\ \forall uw, wv \in E(G),\ u \neq v$.



$\lambda_{h,k}(\acute{C}_3) = 2h+k$

Span $\lambda_{h,k}$

Sendo σ uma rotulação $\mathscr{L}(h,k)$ de G:

$$\lambda_{h,k}(\sigma) = \max_{u,v \in V(G)} \{\sigma(u)-\sigma(v)\};$$

e $\lambda_{h,k}(G) = \min_{\sigma}\{\lambda_{h,k}(\sigma)\}$.

O span foi estudado apenas em classes de grafos básicas, como ciclos e caminhos [3], ou classes em contextos muito restritos [1, 4]. Neste trabalho, determinamos o span dos *Sunlets* $\acute{C}_n$, obtidos a partir do C_n adicionando-se um pingente a cada vértice do ciclo.

Outras classes relacionadas que estão sob investigação são os *Caterpillars* e os *Multisunlets*, os últimos obtidos adicionado-se possivelmente mais de um pingente a cada vértice do ciclo.

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2 O span dos Sunlets

Teorema

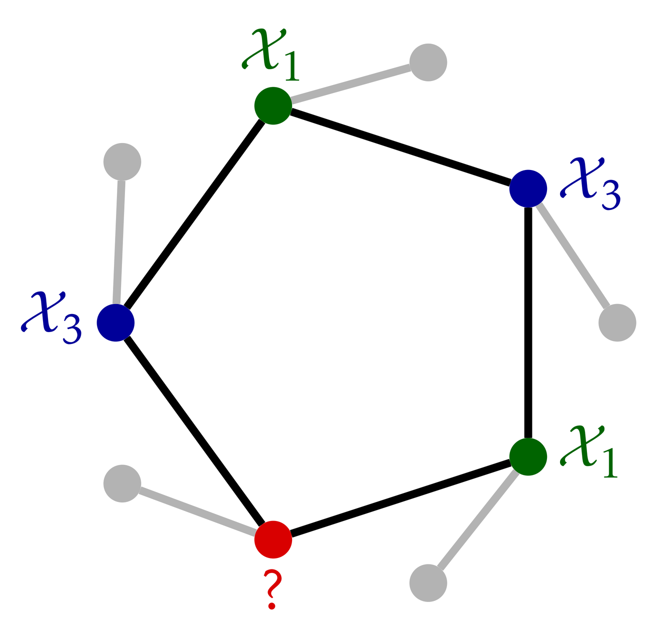
Sejam $h,k,n \in \mathbb{Z}_{\geqslant 0}$ tais que $h \geqslant k$ e $n \geqslant 3$. Então:

$$\lambda_{h,k}(\acute{C}_n) = \begin{cases} h+3k \text{ se } n=5 \text{ e } h<2k; \\ h+3k \text{ se } n \equiv 0 \pmod 4 \text{ e } h \geqslant 2k; \\ h+4k \text{ se } n \equiv 2 \pmod 4 \text{ e } h > 3k; \\ 2h+k \text{ nos demais casos.} \end{cases}$$

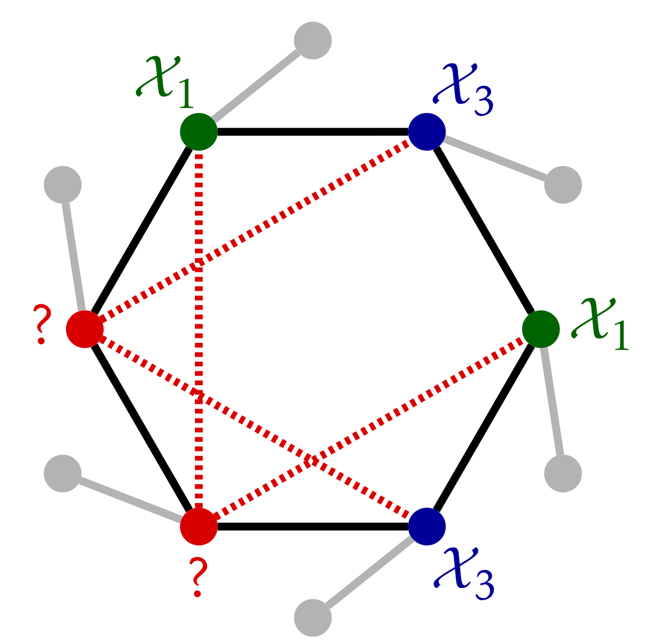
Esboço de demonstração.
($\geqslant$) Por contradição, suponha que exista σ com span menor do que o enunciado pelo teorema. Os rótulos são particionados em três conjuntos. Por exemplo, nos casos em que $\lambda_{h,k}(\acute{C}_n) = 2h+k$,

$$\mathcal{X}_1 = \{0,1,\dots,h-k-1\},$$
$$\mathcal{X}_2 = \{h-k, h-k+1, \dots, h+2k-1\}, \text{ e}$$
$$\mathcal{X}_3 = \{h+2k, h+2k+1, \dots, 2h+k-1\}.$$

Por um lado, mostramos que os rótulos dos vértices do ciclo não podem pertencer a $\mathcal{X}_2$ e, por outro, que não é possível utilizar apenas rótulos de $\mathcal{X}_1$ e $\mathcal{X}_3$ para o ciclo.

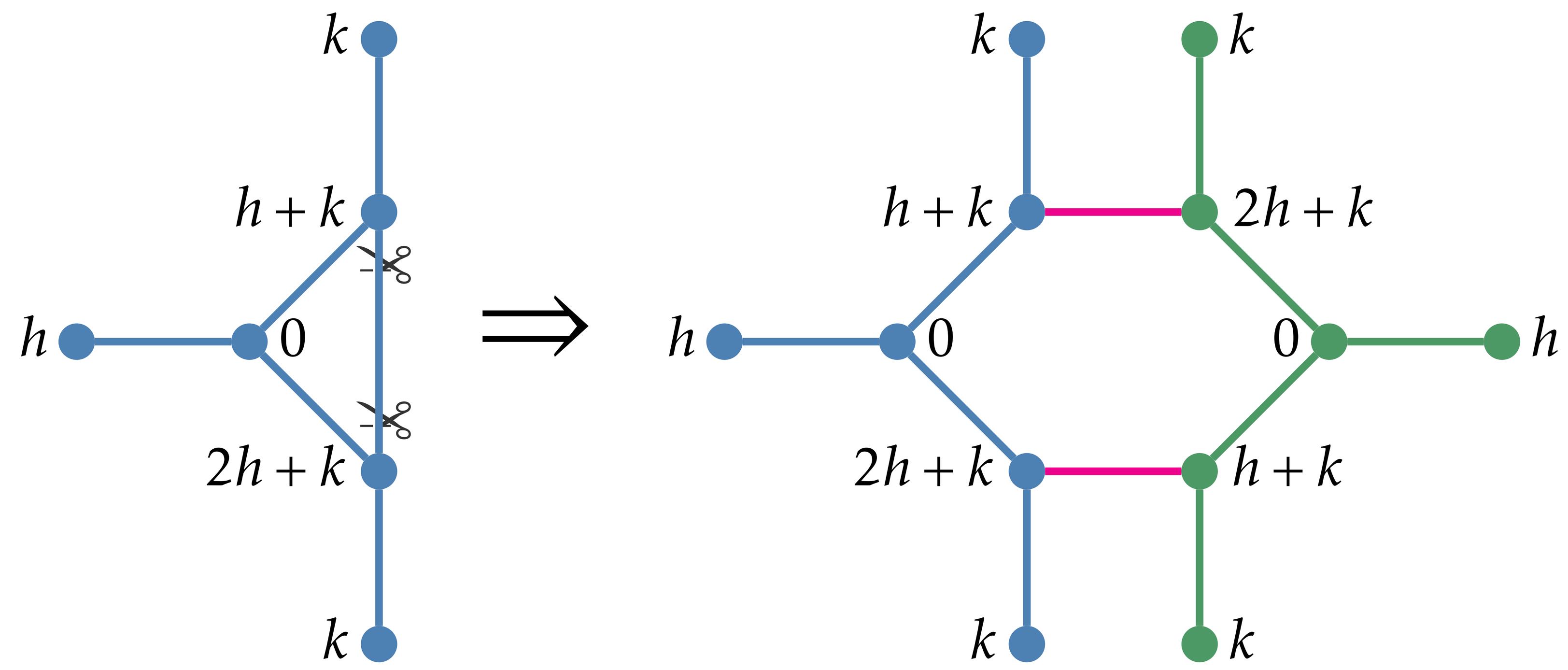


Caso $n \equiv 1 \pmod 2$

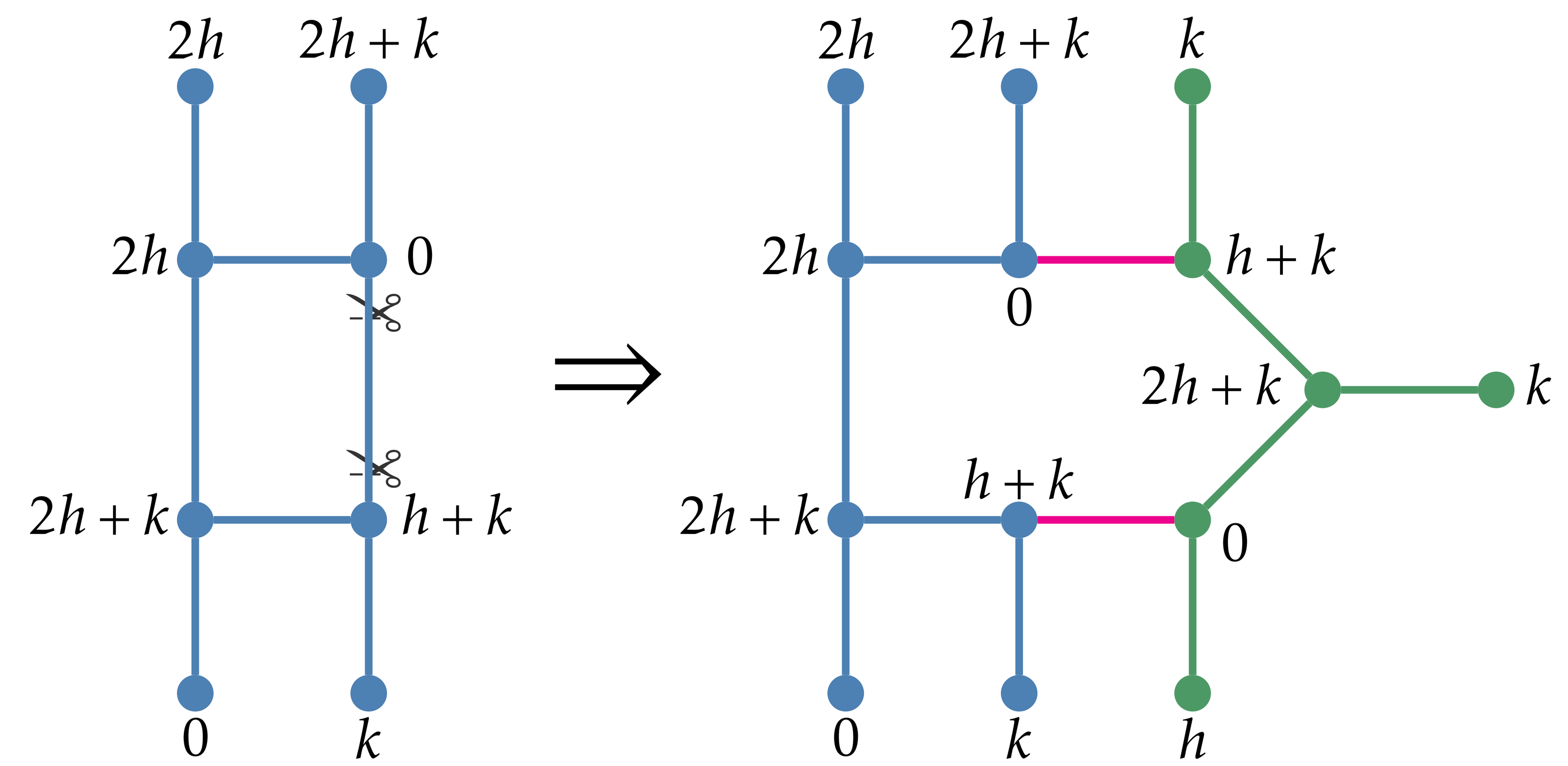


Caso $n \equiv 2 \pmod 4$ e $n \leqslant 3k$

($\leqslant$) Construa a rotulação por blocos pré-definidos a partir de casos-base.



Caso $n \equiv 0 \pmod 3$



Caso $n \equiv 1 \pmod 3$ □

Referências

[1] Calamoneri, T. (2011). The $\mathscr{L}(h,k)$-labelling problem: an updated survey and annotated bibliography. *Comput. J.*, 54:1344–1371.

[2] Griggs, J. R. e Yeh, R. K. (1992). Labelling graphs with a condition at distance 2. *SIAM J. Discrete Math.*, 5:586–595.

[3] Georges, J. e Mauro, D. (1995). Generalized vertex labelings with a condition at distance two. *Congr. Numer.*, 109:141–159.

[4] Georges, J., Mauro, D. e Wang, Y. (2009). Labeling the r-path with a condition at distance two. *Discrete Appl. Math.*, 157:3203–3215.

Emparelhamentos perfeitos no produto cartesiano de árvores

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Resumo

Neste trabalho, investiga-se a existência de emparelhamento perfeito no produto cartesiano de duas árvores sem emparelhamento perfeito, focando-se no caso de árvores do tipo *caterpillar*. Especificamente, é descrita uma família infinita de *caterpillars* com um número par de vértices e sem emparelhamento perfeito tais que o produto cartesiano de duas quaisquer destas árvores possui emparelhamento perfeito.

Palavras-chave: Produto cartesiano de grafos; Emparelhamento perfeito. *Caterpillar*.

Introdução

Sejam G_1, G_2 grafos com conjuntos de vértices $V_1 = \{u_1, \ldots, u_r\}$ e $V_2 = \{v_1, \ldots, v_s\}$, respectivamente. O **produto cartesiano de** G_1 **por** G_2, denotado $G_1 \square G_2$, é o grafo com conjunto de vértices $V = V_1 \times V_2$, no qual (u_i, v_j) e (u_l, v_t) são adjacentes quando u_i é adjacente a u_l em G_1 e $j = t$ ou $i = l$ e v_j é adjacente a v_t em G_2, $1 \leq i, l \leq r$, $1 \leq j, t \leq s$.

Um **emparelhamento** em um grafo $G = (V, E)$ é um subconjunto M do conjunto de arestas E tal que nenhum par de elementos de M possui vértice em comum. Dizemos que o emparelhamento M **satura um vértice** v de G quando alguma aresta de M que incide em v. Dizemos que M é um **emparelhamento perfeito** quando M satura todos os vértices de G. Se o grafo G com n vértices admite emparelhamento perfeito M, então n é par e M tem cardinalidade $n/2$. Um **grafo** que admite um emparelhamento perfeito é chamado **perfeitamente emparelhável**.

É conhecido [1] que se G_1 ou G_2 é perfeitamente emparelhável então $G_1 \square G_2$ também é. Em 2015, A. R.Almeida ([2]), exibiu um grafo G sem emparelhamento perfeito tal que $G \square G$ possui emparelhamento perfeito e levanta a questão: como caracterizar grafos G sem emparelhamento perfeito tais que $G \square G$ possua emparelhamento perfeito?

Dizemos que uma árvore T é do **tipo *caterpillar*** (ou, brevemente, uma ***caterpillar***) se ao retirarmos todos os vértices pendentes, resta um caminho, chamado **corpo** da *caterpillar*.

Neste trabalho, investigamos a questão acima proposta na família das *caterpillars*.

Uma última definição a ser usada em nosso resultado é dada a seguir:

Definição.[3] *Dado $G = (V, E)$, uma partição $P = \{V_1, V_2, \cdots, V_k\}$ de V é dita uma* ***partição por estrelas induzidas de*** *G quando para cada $i, 1 \leq i \leq k$, o subgrafo induzido $G[V_i]$ de G for isomorfo a uma estrela.*

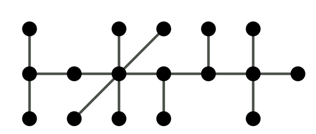

Figure 1:Exemplo de *caterpillar*.

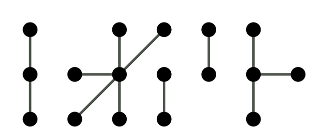

Figure 2:Uma partição por estrelas induzidas formada por $K_{1,2}, K_{1,5}, K_{1,1}, K_{1,1}$ e $K_{1,3}$.

Resultados

Teorema 1 Seja C uma *caterpillar* que admite uma partição por estrelas induzidas que, da esquerda para a direita, é descrita como: uma quantidade ímpar de $K_{1,2}$'s cujos centros coincidem com o corpo, seguida por um número par de $K_{1,1}$'s e, por fim, outra quantidade ímpar de $K_{1,2}$'s com os centros coincidindo com o corpo. Então o produto cartesiano de C por $K_{1,2}$ possui emparelhamento perfeito.

Ideia da prova, com um exemplo:

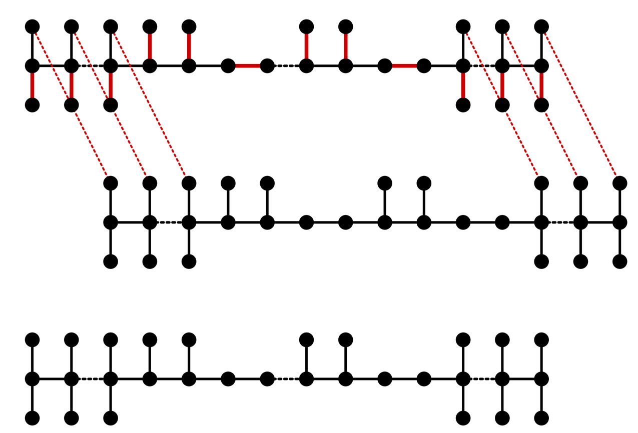

Figure 3:Etapa 1.

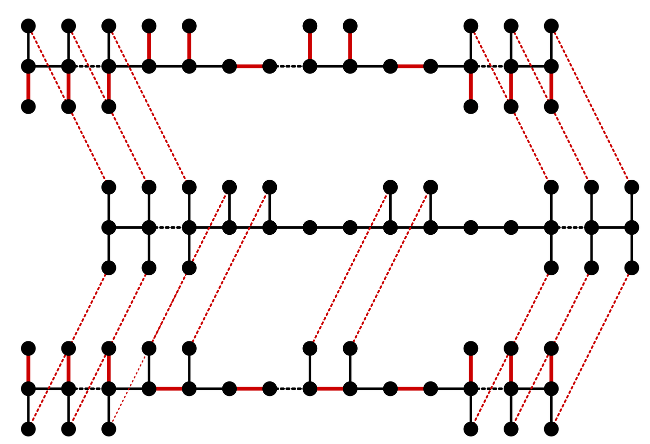

Figure 4:Etapa 2.

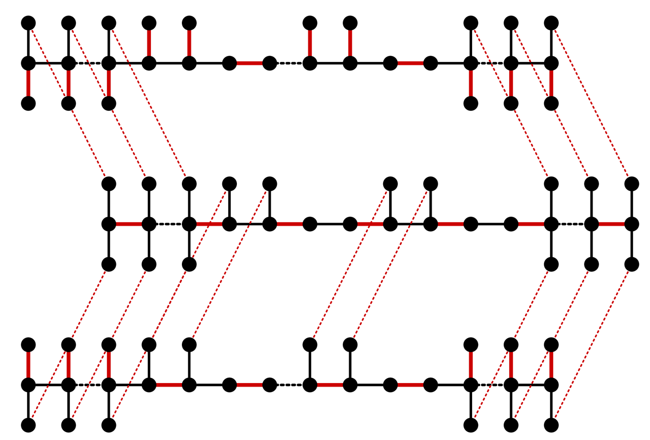

Figure 5:Etapa 3.

Corolário Sejam C_1 e C_2 *caterpillars* tais como a descrita no Teorema 1. Então $C_1 \square C_2$ é perfeitamente emparelhável.

Conclusões

Descrevemos uma família infinita de *caterpillars* sem emparelhamento perfeito tais que o produto cartesiano de qualquer par delas possui emparelhamento perfeito.

Referências

[1] S. KLAVŽAR R. HAMMACK, W. IMRICH. *Product of graphs: Structure and Recognition.* Wiley-Intersciense Series in Discrete Mathematics and Optimization, vol. 56, 2nd edition, 2011.

[2] A.R. ALMEIDA. *Propriedades do produto cartesiano de grafos.* PhD thesis, Instituto de Ciência da Computação, Universidade Federal Fluminense, Programa de Pós-graduação em Ciência da Computação, Niterói, 2015. Tese (Doutorado).

[3] M. A. SHALU, S. VIJAYAKUMAR, and T. P. HANDHYA. The induced star partition of graphs. In *5th International Conference (CALDAM 2019) Proceedings*, pages 14–16, Kharagpur, India, 2019.

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Connected Matchings

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Introduction

Graph matching problems are well known and studied, in which we want to find sets of pairwise non-adjacent edges[1]. This work focus on the study of matchings that induce subgraphs with special properties [2][3]. For this work, we consider the property of being connected, also studying it for weighted or unweighted graphs. For unweighted graphs, we want to obtain a matching with the maximum cardinality, while, for the weighted graphs, we look for a matching whose sum of the edge weights is maximum.

Objective

The problem of maximum connected matching is polynomial[1]. We show ideas that lead to two linear algorithms. One of them, having a maximum matching as input, determines a maximum unweighted connected matching. The complexity of the maximum weighted connected matching problem is unknown for general graphs. However, we present a linear time algorithm that solves it for trees.

Unweighted Connected Matchings

For a graph G and a matching M, we denote $G[M]$ as the subgraph induced by the vertices of M and $N(v)$ as the set of neighbors of v in G. Note that, in the same graph, the cardinalities of a maximum unweighted connected matching and of a maximum weighted connected matching are not always the same. We exemplify in Figure 1. Therefore, we expect that these problems have different computational treatments.

Theorem 1

If G is connected and M is an unweighted maximum matching in G, then the unweighted maximum connected matching has cardinality $|M|$ [2]

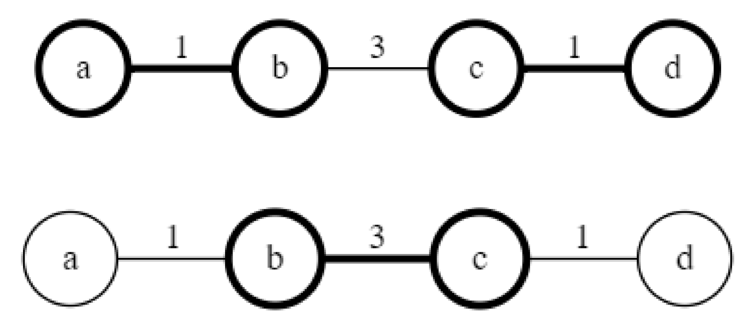

Figure 1: Two maximum connected matchings of a graph.

The proof of Theorem 1[2] is based on the fact that, in a graph G, if M is a maximum matching and $G[M]$ is disconnected, in which C is connected component of $G[M]$, then it is possible to redefine the edges of M in order to increment vertices of C in M, successively, until $G[M]$ has a single component.

We present an idea to do all this process and leave $G[M]$ connected in linear time. Let M be a maximum matching such that $G[M]$ is disconnected and r a M-saturated vertex. Consider C_r to be the component of $G[M]$ which contains r. We use two sets, Q_s and Q_n, to store M-saturated and M-unsaturated vertices, respectively. Additionally, we employ a set C, to which vertices of C_r or new vertices are added. A main loop can be executed until $G[M]$ equals C. Each iteration is divided into two other auxiliary loops and includes at least one vertex at C. The first auxiliary loop, for each vertex v of Q_s, analyzes $N(v)$, and properly adds to this set each vertex of that neighborhood that has not yet entered the set. The second auxiliary loop, for each vertex v of Q_n, if $w \in N(v) \setminus C$ exists, then w is saturated by some edge, (w,u), and we perform the *edge exchange* operation in M. Such operation removes (w,u) and adds the edge (v,w) to M. In the end of this process, $G[M]$ will be connected.

Weighted Connected Matchings

Though it is still unknown the complexity of finding maximum weighted connected matchings, we present an idea that leads to a linear solution for trees. Let T be a tree and r, $v \in V(T)$. We denote T^r as a tree T rooted in r and T_v^r as the subtree of T^r rooted in v. Also, $S(r,v)$ is the set of all sons of v in T_v^r and $weight(v,w)$ is the weight of the edge (v,w).

Theorem 2

Let G be a connected graph and M a maximum connected macthing of G. Then M saturates all articulations in G

Without loss of generality, by Theorem 2, we know that, for a tree T, each articulation v must be saturated. We look for the neighbors of v, which maximize the weighted sum of the edges to build a maximum connected matching in T_v^r. For such a construction, we consider r as any vertex of T, and apply a dynamic programming algorithm described below. We define the sum of the edge weights of a maximum weighted matching in T_v^r as B_v if v is matched with one of its children, and $\overline{B_v}$ if v is matched with its father. We can determine this variables as follows. If v is a leaf, then $B_v = \overline{B_v} = 0$. Else, consider the following equations.

$$\overline{B_v} = \sum_{u\in S(r,v)} B_u$$

$$B_v = \max_{a\in S(r,v)} \left(\overline{B_a} + weight(a,v) + \sum_{u\in S(r,v)\setminus\{a\}} B_u \right)$$

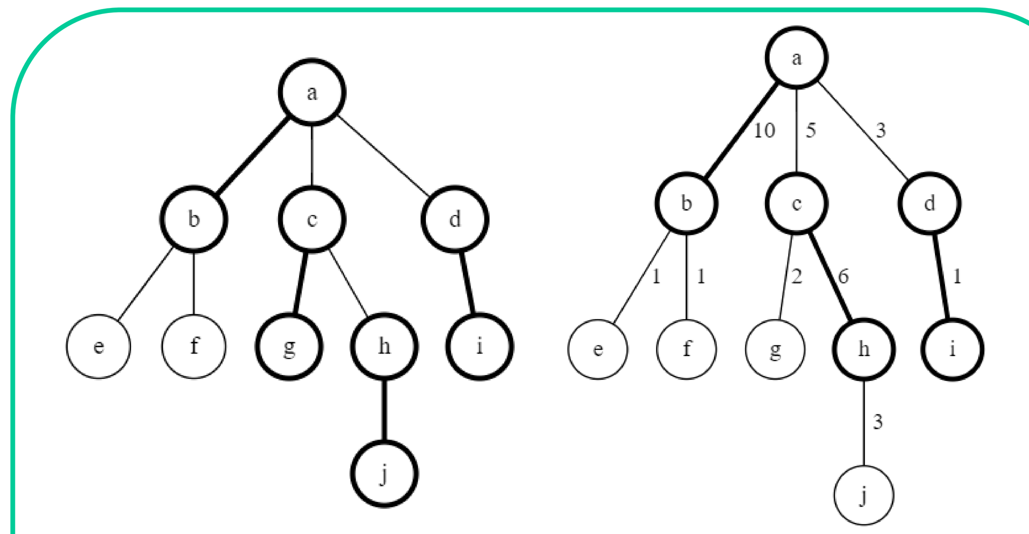

Figure 2: Two maximum connected matchings. The left matching refers to a unweighted while the right, to weighted.

An algorithm can dynamically build a maximum connected matching M as follows. From an arbitrary articulation r elected as root, two searches are made. The first computes the vertices from the leaves to the root r. It obtains, for each vertex u, a child vertex s_u of u that maximizes B_u. In addition, $\overline{B_u}$ is calculated from the sum of B_w for all its children w. The second search is responsible for building M, computing the vertices from r to the leaves, so that, when a vertex u is processed, if u is not part of M yet, we add (s_u, u) to M. In the end, M will be a maximum weighted connected matching.

References

[1] MICALI, S. and VAZIRANI, V. V.. An $O(\sqrt{|V|}|E|)$ algorithm for finding maximum matching in general graphs. In 21st Ann. Symp. on Foundations of Comp. Sc., pages 17–27, 1980.

[2] GODDARD, W., HEDETNIEMI, S. M., HEDETNIEMI, S. T., and LASKAR, R.. Generalized subgraph-restricted matchings in graphs. Discrete Math., 293 (2005), 129-138.

[3] MASQUIO, B. P.. Emparelhamentos Desconexos. Master's Thesis, Universidade do Estado do Rio de Janeiro. Available in https://github.com/BMasquio/papers/raw/master/MastersThesisBrunoMasquio.pdf, 2019.

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Number of spanning trees of a subclass of matrogenic graphs

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Introduction

Laplacian Matrix of a Graph

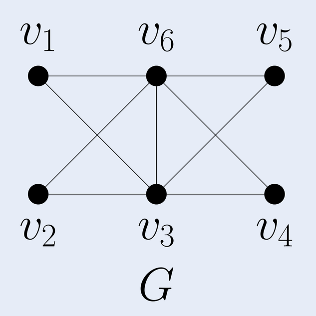
Definition ([1]) Let $G = G(V, E)$ be a simple graph with n vertices. The *adjacency matrix of* G is the matrix $A(G) = (a_{ij})$ with order n, whose entries are given by

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E \text{ for } v_i, v_j \in V; \\ 0, & \text{otherwise.} \end{cases}$$

Let $D(G)$ be the diagonal matrix given by the degree of vertices of G. The *Laplacian matrix* of G is the matrix $L(G)$ defined by

$$L(G) = D(G) - A(G).$$

For example,



$$L(G) = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{bmatrix}.$$

Preliminaries

Matrix-Tree Theorem

Theorem 1.1 ([3]) *The number of spanning trees of a graph G with order n is equal to any co-factor of $L(G)$. In symbols*

$$adj(L(G)) = \tau(G) J_{n \times n},$$

where $adj(L(G))$ is the classical adjoint of $L(G)$, $\tau(G)$ is the number of spanning trees of G and $J_{n\times n}$ is the matrix with order $n \times n$ whose entries are all equal to one.

We emphasize that this counting does not disregard isomorphic trees, that is, the number of non-isomorphic spanning trees is less than or equal to the number of spanning trees.

Corollary 1.2 ([1]) *Let G be a connected graph which n vertices. If $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are all the non-zero eigenvalues of $L(G)$, then*

$$\tau(G) = \frac{\mu_1 \mu_2 \cdots \mu_{n-1}}{n}.$$

This is the spectral version of the Matrix-Tree Theorem, which is very useful, since we've reduced the problem of finding the number of spanning trees in a graph to a problem of characterization of Laplacian eigenvalues.

For more reference see [4] and [5].

Matrogenic Graphs

Literature Results

The *symmetric difference* between two sets A, B is given by $A \oplus B = (A \cup B) \backslash (A \cap B)$.

Definition. If u, v are vertices of a graph G, we say that u *dominates* v if $N_G(v) \backslash \{u\} \subseteq N_G(u) \backslash \{v\}$. When neither u dominates v, nor v dominates u, we say that u and v are *incomparable.*

Definition. A *split graph* is a graph in which the set of vertices can be partitioned into a clique and an independent set. A graph is a *complete split graph* if it is a split graph such every vertex in the independent set is adjacent to every vertex in the clique.

Definition. A graph G is *matrogenic* if and only if for any incomparable vertices, u and v in G, we have that the cardinality of symmetric difference between the sets $N_G(v) \backslash \{u\}$ and $N_G(u) \backslash \{v\}$ is 2.

Proposition 2.1 The split complete graph is matrogenic.

Theorem 2.2 ([2]) *A graph $G = G(V, E)$ is matrogenic if and only if its vertex set V can be partitioned into three disjoint sets K, S and C such that*

(i) $K \cup S$ induces a matrogenic split graph in which K is a clique and S is an independent set.

(ii) C induces a crown, where a crown is either a perfect matching or a hyperoctahedron or a C_5.

(iii) Every vertex in C is adjacent to every vertex in K and to no vertex in S.

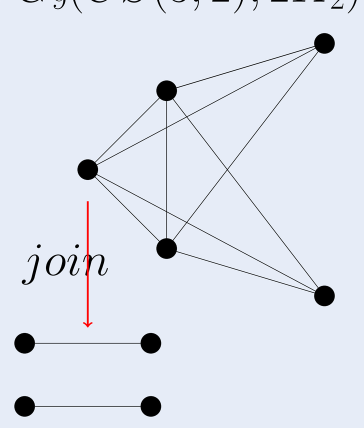
A Subclass of Matrogenic Graphs

From Theorem 2.2 every matrogenic graph of order n can be denoted by $G_n(K \cup S, C)$, where K, S and C are defined in the same theorem.

Given the non-negative integers r, s and t, we consider the class of graphs, $\mathcal{G}$, constituted by the matrogenic graphs of the form $G_n(K \cup S, C)$, where $K \cup S$ induces the complete split graph, $CS(r, s)$, and the subset of vertices C induces t copies of the complete graph K_2, that is,

$$\mathcal{G} = \{G_n(CS(r, s), tK_2) \,|\, r, s, t \in \mathbb{N} \wedge n = r + s + 2t\}.$$

The figure below shows the graph $G_9(CS(3, 2), 2K_2)$.



Application

Main Result

Theorem 3.1.
Let $H = G_n(CS(r, s), tK_2)$, then $\tau(H) = (r + s + 2t)^{r-1}(r + 2)^t r^{s+t-1}$.

Sketch of proof. We have

$$L(H) = \begin{bmatrix} D(tK_2) - A(tK_2) & -J_{2t\times r} & 0_{2t\times s} \\ -J_{r\times 2t} & D(K) - J_{r\times r} + I_{r\times r} & -J_{r\times s} \\ 0_{s\times 2t} & -J_{s\times r} & D(S) \end{bmatrix},$$

where $D(tK_2)$ is the diagonal matrix of the induced subgraph by tK_2, $D(K)$ is the diagonal matrix of induced subgraph by K, $D(S)$ is the diagonal matrix of induced subgraph by S, $I_{r\times r}$ is the identity matrix with order $r \times r$ and $0_{a\times b}$ is the matrix with order $a \times b$ with all entries is equal to 0.

Through eigenvalue calculation techniques we obtain $r \in Spec(L(H))$ with $m(r) \geq s - 1$, when $m(r)$ is the algebraic multiplicity of r as eigenvalue. On other hand, $r + s + 2t \in Spec(L(H))$ with $m(r + s + 2t) \geq r - 1$. In addition, we obtain $r + 2 \in Spec(L(H_1))$ with $m(r + 2) \geq t$.

By a result about reduced matrices, we obtain that $\{r+s+2t, r, 0\} \subset Spec(L(H))$. So, $Spec(L(H)) = \{(r + s + 2t)^{[r]}, (r + 2)^{[t]}, r^{[s+t-1]}, 0\}$.
By the Corollary 1.2, the number of spanning trees of H is

$$\tau(H) = (r + s + 2t)^{r-1}(r + 2)^t r^{s+t-1}.$$

Corollary 3.2. *The number of spanning trees of H depends of the cardinality of each cell of the partition of vertices in H.*

For example, if $H = G_9(CS(3, 2), 2K_2)$, then $\tau(H) = (3 + 2 + 4)^2(3 + 2)^2(3)^3 = 54675$.

References

[1] ABREU, N.; DEL-VECCHIO, R.; TREVISAN, V.; VINAGRE, C. Teoria Espectral de Grafos - Uma Introdução, *III Colóquio de Matemática da Região Sul*, Florianópolis, SC, 2014.

[2] FOLDES, S.; HAMMER, P. On a class of matroid-producing graphs., *Colloq. Math.Soc. J. Bolyai (Combinatorics)*,v. 18, p. 331-352, 1978.

[3] KIRCHHOFF, G. Über der Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteinlung galvanischer Ströme geführt wird, *Ann. Phys. Chem.*, 72:497-508, 1847.

[4] LIU, J-B.; ZHAO, J.; ZHU, Z. On the number of spanning trees and normalized Laplacian of linear octagonalquadrilateral networks. *Int J Quantum Chem.*, 2019;119:e25971. https://doi.org/10.1002/qua.25971

[5] PAN, Y.; LIU, C.; LI, J. Kirchhoff Indices and Numbers of Spanning Trees of Molecular Graphs Derived from Linear Crossed Polyomino Chain. *Polycyclic Aromatic Compounds*, 2020. DOI: 10.1080/10406638.2020.1725898

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Arc-disjoint Branching Flows: a study of necessary and sufficient conditions

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Introduction

In this work, we investigate a conjecture by [1] that aims to characterize networks admitting k arc-disjoint s-branching flows, generalizing a result from [2] that provides such characterization when all arcs have capacity $n-1$, based on Edmonds' branching theorem [3].

- **Network**: $\mathcal{N}=(D,c)$, where $D=(V,A)$ is a digraph and $c: A(D) \to \mathbb{Z}_+$ is the *capacity function*. For an integer $\lambda \geq 0$, we write $c \equiv \lambda$ to state that $c(a)=\lambda,\ \forall a \in A(D)$. For an arc $a \in A(D)$ with tail u and head v, we may refer to a as uv.
- A **flow** f on a network $\mathcal{N}$ is a function $f: A(D) \to \mathbb{Z}_+$ such that $f(a) \leq c(a),\ \forall a \in A(D)$.
  Two flows f_1 and f_2 on a network $\mathcal{N}$ are **arc-disjoint flows** if $f_1(a) \times f_2(a) = 0,\ \forall a \in A(D)$.
- The **balance** of a vertex v with respect to a flow f is $bal_f(v) = \sum_{vu \in A(D)} f(vu) - \sum_{uv \in A(D)} f(uv)$. That is, $bal_f(v)$ is the sum of the flow leaving v minus the sum of the flow entering v.
- **s-branching flow**: flow f such that $bal_f(s) = n-1$ and $bal_f(v) = -1$ for all $v \in V(D) \setminus \{s\}$.

The hardness of the problem of finding k arc-disjoint s-branching flows in a network $\mathcal{N}=(D,c)$ where $c \equiv \lambda$, in general, depends on the choice of λ. Table 1 summarizes those results.

λ	Hardness
$\lambda \geq n-\ell$	Poly-time solvable for fixed ℓ [5]
$(\log n)^{1+\varepsilon} \leq \lambda \leq n-(\log n)^{1+\varepsilon}$	No poly-time algorithm (unless ETH[1]fails) [1,5]
$\lambda \leq \ell$	$\mathcal{NP}$-complete [5]

Table 1: *Summary of known hardness and algorithmic results for the problem of finding k arc-disjoint s-branching flows in a network $\mathcal{N}=(D,c)$ with $c \equiv \lambda$. Here, ℓ is a non-negative integer, $\varepsilon > 0$, and $n = |V(D)|$.*

In [1], the authors showed that the following property is a necessary condition satisfied by any network admitting k arc-disjoint s-branching flows.

$$d^-_D(X) \geq k \cdot \left\lceil \frac{|X|}{\lambda} \right\rceil, \forall X \subseteq V(D) \setminus \{s\}. \qquad \text{(Property 1)}$$

They also conjectured that Property 1 is a sufficient condition for the existence of k arc-disjoint s-branching flows in a network $\mathcal{N}=(D,c)$ with $c \equiv \lambda$, for any choices of k, λ, and s. In this work, we prove that their conjecture is true for some graphs, but false in general. An *out-branching with root* r is a digraph where $d^-_D(r)=0$ and $d^-_D(v)=1$ for every $v \in V(D) \setminus \{r\}$. Let a *multi out-branching with root* r be a digraph D formed by adding parallel arcs to an out-branching with root r. Observe that the underlying simple graph of D, constructed by discarding the orientation of the edges of D and removing parallel edges, is a tree. See Figure 1 for an example of a multi out-branching with root r and its underlying simple graph.

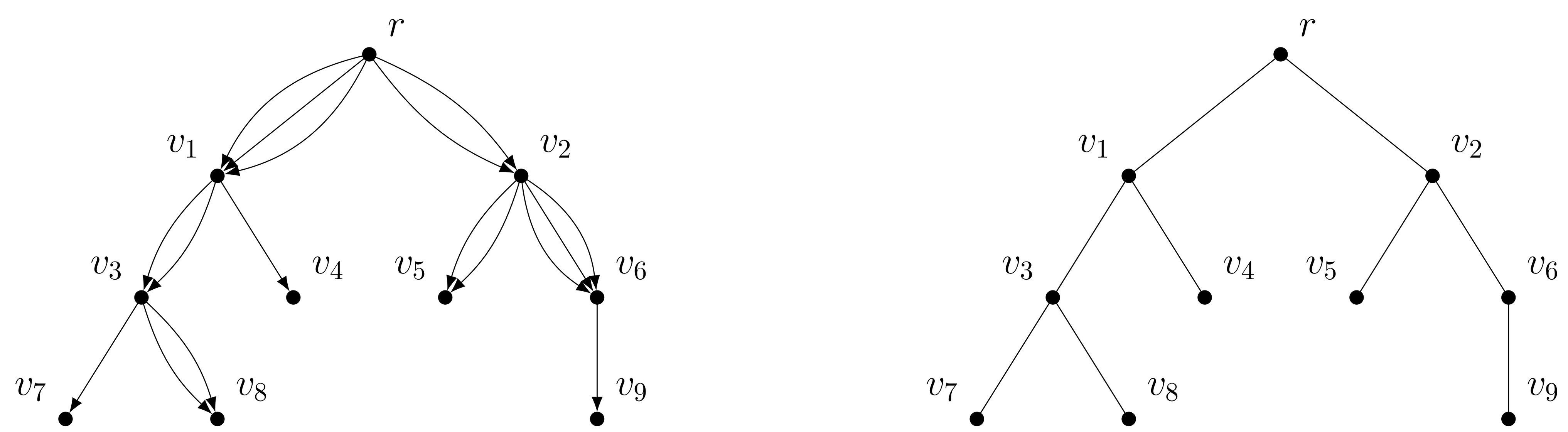

Figure 1: *Example of a multi out-branching with root r and its underlying simple graph.*

[1] Exponential Time Hypothesis [4].

Arc-disjoint branching flows on networks satisfying Property 1

We now state our results.

Theorem 1. *Let $\mathcal{N}=(D,c)$ be a network, where D is a multi out-branching with root s and $c \equiv \lambda$. If Property 1 holds for D with respect to k, λ and s then $\mathcal{N}$ admits k arc-disjoint s-branching flows.*

Figure 2 shows a network satisfying Property 1 for $k=\lambda=2$ that does not contain 2 arc-disjoint s-branching flows. This statement is formalized by Theorem 2.

Theorem 2. *Let D be the digraph shown in Figure 2 and $\mathcal{N}=(D,c)$ be a network with $c \equiv 2$. Then Property 1 holds for $\mathcal{N}$ with respect to $\lambda=2$, s,and $k=2$, and there are no 2 arc-disjoint s-branching flows in $\mathcal{N}$.*

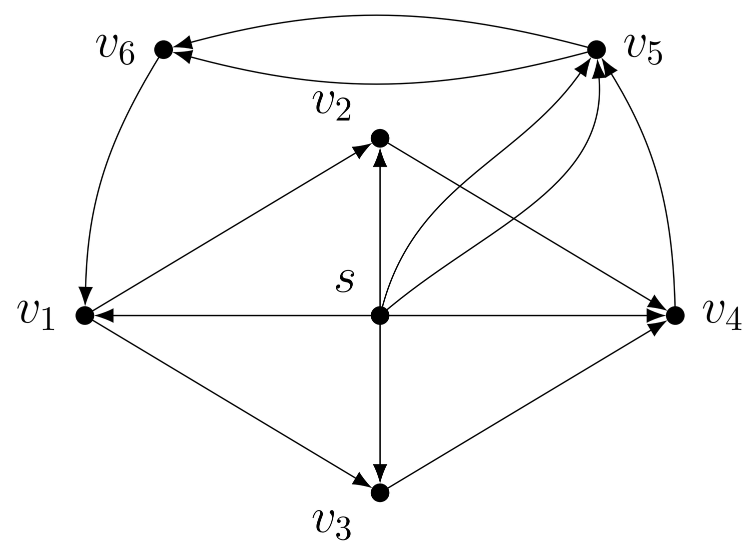

Figure 2: *A network for which Property 1 holds with respect to $k=\lambda=2$ and the vertex s, but not containing 2 arc-disjoint s-branching flows.*

Future works

In future works, it will be interesting to consider whether there is a version of Theorem 1 for larger classes of digraphs, or whether there is a stronger necessary and sufficient condition that applies to all cases. We remark that, by the results shown in Table 1, we do not expect this condition to be easily verifiable in a given digraph.
In [5] the authors left open the question of whether the problem of finding k arc-disjoint s branching flows in a network $\mathcal{N}=(D,c)$ with $c \equiv n-\ell$ is *fixed-parameter tractable* with respect to ℓ. To our knowledge, this question remains open.

References

[1] J. Costa, C. L. Sales, R. Lopes, and A. K. Maia, "Um estudo de redes com fluxos ramificados arco-disjuntos," *Matemática Contemporânea*, vol. 46, pp. 230–238, 2019.

[2] J. Bang-Jensen and S. Bessy, "(Arc-)disjoint flows in networks," *Theoretical Computer Science*, vol. 526, pp. 28–40, 2014.

[3] J. Edmonds, "Edge-disjoint branchings," *Combinatorial Algorithms*, 1973.

[4] R. Impagliazzo, R. Paturi, and F. Zane, "Which problems have strongly exponential complexity?" *Journal of Computer and System Sciences*, vol. 63, no. 4, pp. 512–530, 2001.

[5] J. Bang-Jensen, F. Havet, and A. Yeo, "The complexity of finding arc-disjoint branching flows," *Discrete Applied Mathematics*, vol. 209, pp. 16–26, 2016.

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Maximal Independent Sets in Graphs of Girth at Least 6

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Introduction

For a graph G we denote by $\alpha(G)$ the maximum size of an independent set in G and by $i(G)$ the minimum size of a maximal independent set in G. The independence gap of a graph G, denoted by $\mu_\alpha(G)$ is the difference $\alpha(G) - i(G)$. Well-covered graphs have independence gap zero. We present characterizations of some graphs with independence gap at least 1 that are of girth at least 6, including graphs with independent gap $r-1$, for $r \geq 2$, with r distinct and consecutive sizes of maximal independent sets.

Finbow et al. [3] define the set $\mathcal{M}_r$, for every positive integer r, to be the set of graphs that have maximal independent sets of exactly r different sizes. If the r different sizes of its maximal independent sets are consecutive, then it is also a member of $\mathcal{I}_r$, defined by Barbosa and Hartnell [1].

We present results related to the number of trees with specific maximum and minimum sizes of maximal independent sets (MIS). For a graph G, $\text{miss}(G) = \{|I| : I \text{ is a MIS of } G\}$. See Figure 1. A vertex is said to be of *type* r if it is adjacent to exactly r leaves.

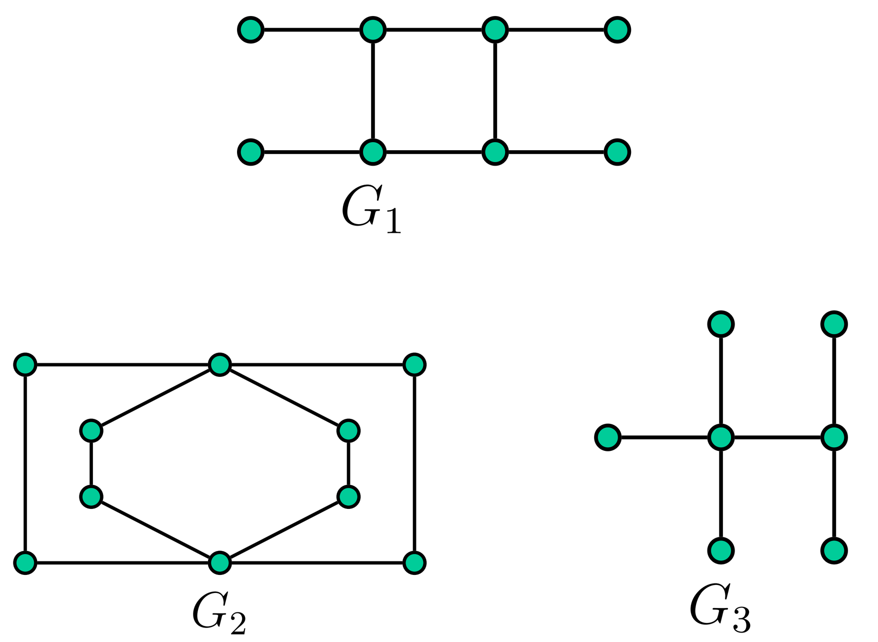

Figure 1: Graph G_1 is well-covered, with $\text{miss}(G_1) = \{4\}$, and $\mu_\alpha(G_1) = 0$; $G_2 \in \mathcal{M}_3$, but $G_2 \notin \mathcal{I}_3$, with $\text{miss}(G_2) = \{2, 4, 5\}$, and $\mu_\alpha(G_2) = 3$; $G_3 \in \mathcal{I}_3$, therefore $G_3 \in \mathcal{M}_3$, with $\text{miss}(G_3) = \{3, 4, 5\}$, and $\mu_\alpha(G_3) = 2$.

Results

Before we show some results regarding trees, we present in Table 1 the distribution in the set $\mathcal{I}_r$ of trees with n vertices, where $6 \leq n \leq 20$. Not all trees in $\mathcal{M}_r$ belong to $\mathcal{I}_r$. The data were obtained via a computational program.

In Theorem 1, we show the number of non-isomorphic trees having specific sizes of MIS and prove that there are exactly $\lceil \frac{n}{2} \rceil - 1$ non-isomorphic trees T with n vertices having $\mu_\alpha(T) = n - 4$.

		Vertices														
		6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
r	1	1		2		3		6		11		23		47		106
	2	2	5	4	12	14	31	40	78	122	202	351	522	1018	1370	2890
	3	1	2	7	12	32	59	129	262	500	1063	1877	4069	6837	14817	24298
	4				7	15	52	130	319	806	1737	4354	8812	21397	42069	98236
	5					4	14	63	191	579	1654	4200	11561	27109	71181	160724
	6						1	9	57	244	813	2856	7822	24781	63028	183301
	7								4	55	266	1066	4206	12977	44759	125465
	8									1	41	241	1206	5536	18954	72259
	9											24	219	1282	6878	25945
	10												10	184	1212	8079
	11													3	134	1177
	12															77

Table 1: Quantity of Trees of a given order in $\mathcal{I}_r$.

Theorem 1

Let $n \geq 3$ and T be a tree with n vertices.
1. There are exactly $n-3$ trees with $\alpha(T) = n-2$.
2. There are exactly $n-3$ trees with $i(T) = 2$.
3. There are exactly $\lceil \frac{n}{2} \rceil - 1$ trees $\mu_\alpha(T) = n-4$.

Next result is a generalization of a result in [2] for graphs G of girth at least 6 with $\mu_\alpha(G) = 1$. We adapt their proof considering $\mu_\alpha(G) \geq 1$. Additionally, we present the different sizes of MIS of G. Its proof gives a polynomial-time algorithm and it has some consequences to the class $\mathcal{I}_r$. In the following cases the sizes of MIS of G are not consecutive: if $r \geq 3$ and the girth of G is at least 7, and if $r \geq 4$ and the girth of G is at least 6. We summarize these conditions in Corollary 3. We denote G_i the subgraph of G induced by internal vertices of G that are type i.

Theorem 2

Let $r \geq 2$ and G be a connected graph of girth at least 6, with exactly two vertices u_1 and u_2 of type r, and with no type k vertices for $k \geq r+1$. Then $\mu_\alpha(G) = r-1$ if and only if u_1 and u_2 are adjacent, any other support vertex of G is type 1, and one of the following two conditions holds:
1. $V(G_0) = \emptyset$;
2. $G_0 \cong K_2$, neither of u_1 and u_2 has a neighbor in G_0, and the two vertices of G_0 are of degree 2 in G and are contained in an induced 6-cycle containing u_1 and u_2.

Moreover, if $V(G_0) = \emptyset$, then $\text{miss}(G) = \{|V(G_1)| + r + 1, |V(G_1)| + 2r\}$ otherwise $\text{miss}(G) = \{|V(G_1)| + r + 2, |V(G_1)| + 2r, |V(G_1)| + 2r + 1\}$.

Proof 1: (Sketch)

Let F_1 and F_2 be the sets of leaves, respectively, of vertices u_1 and u_2. Suppose $\mu_\alpha(G) = r-1$. We claim that the other neighbors of vertices u_1 and u_2 are vertices of type 1, and u_1 and u_2 are adjacent. Suppose $V(G_0) \neq \emptyset$; Let $L_1 = N_G(u_1) - (F_1 \cup \{u_2\})$ and $L_2 = N_G(u_2) - (F_2 \cup \{u_1\})$. Let L'_i the set of leaves adjacent to vertices of L_i, $i = 1, 2$. Now, let $I = F_1 \cup F_2 \cup L'_1 \cup L'_2$ and let $G' = G - N_G[I]$. See Figure 2. We also claim that: 1) graph G' is well-covered and has a perfect matching formed by its pendant edges. 2) G_0 has only one component that is isomorphic to K_2 and their vertices are under a 6-cycle containing u_1 and u_2. For the converse, we show all possible sizes of MIS considering the two cases: $V(G_0) = \emptyset$ and $V(G_0) \neq \emptyset$. If $V(G_0) = \emptyset$, then $\text{miss}(G) = \{|V(G_1)| + r + 1, |V(G_1)| + 2r\}$ otherwise $\text{miss}(G) = \{|V(G_1)| + r + 2, |V(G_1)| + 2r, |V(G_1)| + 2r + 1\}$. Therefore $\mu_\alpha(G) = r-1$.

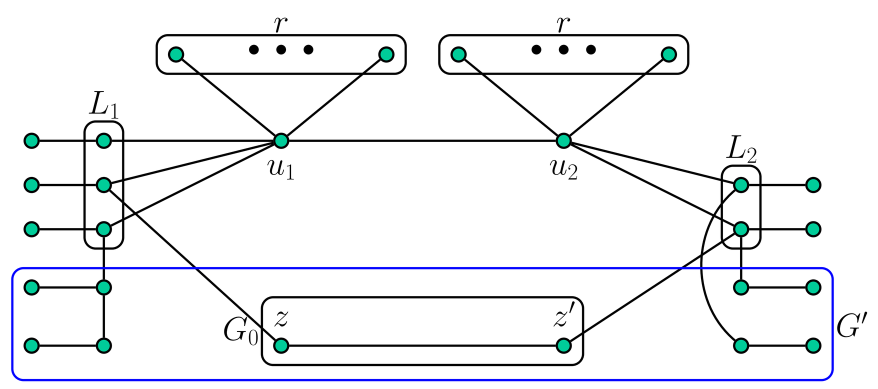

Figure 2: Graph G of girth 6 and two vertices of type r.

Corollary 3

Let $r \geq 3$ and let G be a graph of girth at least 6 with $\mu_\alpha(G) = r-1$ such that G contains exactly two vertices of type r. Then, $G \in \mathcal{I}_r$ only if $r = 3$ and the girth of G is exactly 6.

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References

[1] R. Barbosa and B. Hartnell. Some problems based on the relative sizes of the maximal independent sets in a graph. *Congressus Numerantium*, 131:115–121, 01 1998.
[2] T. Ekim, D. Gözüpek, A. Hujdurović, and M. Milanič. On Almost Well-Covered Graphs of Girth at Least 6. *Discrete Mathematics & Theoretical Computer Science*, 20(2), November 2018.
[3] A. Finbow, B. Hartnell, and C. Whitehead. A characterization of graphs of girth eight or more with exactly two sizes of maximal independent sets. *Discrete Mathematics*, 125(1):153 – 167, 1994.

The 3-flow conjecture for almost even graphs with up to six odd vertices

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1. Integer flows

Let $G = (V(G), E(G))$ be an undirected graph. Let D be an orientation for $E(G)$, and f an assignment of non-negative integer weights to each edge of $E(G)$. We say that (D, f) is a *k-flow* for G if:

1. $0 < f(e) < k$, for each $e \in E(G)$;
2. the *flow balance* $\sum_{e\in\partial^+(v)} f(e) - \sum_{e\in\partial^-(v)} f(e) = 0$, for each $v \in V(G)$,

where $\partial^+(v)$ ($\partial^-(v)$) is the set of edges leaving (entering) vertex v.

In a *mod-k flow*, the flow balance at each vertex v is $\sum_{e\in\partial^+(v)} f(e) - \sum_{e\in\partial^-(v)} f(e) \equiv 0 \pmod{k}$. Figure 1 shows two graphs that admit a mod-3 flow.

A graph G admits a k-flow if and only if it admits a mod-k flow. Also, if G admits a mod-k flow, then it admits a mod-k flow for any given orientation. See [1], [2] and [3] for more on k-flows.

Figure 1: Examples of mod-3 flows for graphs $K_{3,3}$ and K_4 plus an edge. In both cases, all weights are equal to 1.

2. Tutte's 3-flow Conjecture and equivalent formulations

A *3-cut* is an edge cut of size three. A *bridge* is an edge cut of size one. Tutte's 3-flow conjecture is

Conjecture (Tutte's 3-flow conjecture)

Every bridgeless graph with no 3-cuts admits a 3-flow.

Two equivalent forms of this conjecture are:

- Every bridgeless 5-regular graph with no 3-cuts admits a 3-flow.
- Every bridgeless graph with at most three 3-cuts admits a 3-flow.

3. Objective

In this work, our objective is to characterize classes of graphs with up to four 3-cuts that admit a 3-flow. K_4, the complete graph on four vertices, is the smallest bridgeless graph that does not admit a 3-flow. We focus on *essentially 4-edge connected graphs*, i.e., whose edge cuts of size less than four are associated with vertices of degree three (3-vertices). Also, our graphs are *almost even*, i.e., having at most six odd vertices.

We obtain a characterization for such graphs with up to four odd vertices. We also obtain a partial characterization for graphs with up to four 3-vertices and two odd vertices of higher degree.

4. Motivation

Our motivation is to provide tools for a possible inductive approach to prove Tutte's 3-flow conjecture.

5. Graphs with exactly four vertices of odd degree

Let G be an essentially 4-edge connected, almost even, graph having at most four odd vertices, with S its set of odd vertices. We say that G has a *forbidden configuration* if: (i) the vertices of S all have degree three; (ii) $G[S]$ contains $K_{1,3}$; and (iii) every even-degree vertex v of G is separated from S by an edge cut of size at most four. We abuse this definition by saying that K_4 has a forbidden configuration.

Theorem 1

An essentially 4-edge connected, almost even, graph G with at most four odd-degree vertices admits a 3-flow, if and only if G does not have a forbidden configuration.

6. Graphs with exactly six vertices of odd degree

We give a partial characterization of almost even graphs with six odd-degree vertices that admit a 3-flow.
By using the same definition of forbidden configuration to graphs with four 3-vertices and two odd-degree vertices of degree greater than 3, we obtain

Theorem 2

Let G be an essentially 4-edge-connected, almost even, graph with four 3-vertices and two other odd vertices of degree greater than 3, and assume G has a forbidden configuration. Then, G admits a 3-flow if and only if there are no 4-cuts separating the 3-vertices from the remaining odd vertices.

Sketch of proof: (if) We contract a set X that contains the two odd vertices with degree higher that three, and having an associated edge-cut of size six (e.g. V(G) minus the vertices of degree three). By Theorem 1, the resulting graph admits a 3-flow, that can be extended to $G/\overline{X}$. This is a 3-flow for G.

(only if) We contract a set X that contains the two odd vertices of degree higher than three, with an associated edge-cut of size four. By the previous theorem, G/X does not admit a 3-flow, and so neither does G.

References

[1] M. Kochol.
Superposition and constructions of graphs without nowhere-zero k-flows.
European Journal of Combinatorics, 23(3):281–306, 2002.

[2] D. Younger.
Integer flows.
Journal of graph theory, 7(3):349–357, 1983.

[3] C.-Q. Zhang.
Integer flows and cycle covers of graphs, volume 205.
CRC Press, 1997.

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2-Independent Sets in Complementary Prisms

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Introduction

We consider a generalization of the concepts of domination and independence in graphs. For a positive integer k, a subset S of vertices in a graph $G=(V,E)$ is *k-dominating* if every vertex of $V-S$ is adjacent to at least k vertices in S. The subset S is *k-independent* if the maximum degree of the subgraph induced by the vertices of S is at most $k-1$. Thus for $k=1$, the 1-independent and 1-dominating sets are the classical independent and dominating sets. The minimum and maximum sizes of a maximal k-independent set in G are denoted $i_k(G)$ and $\alpha_k(G)$, respectively. The minimum and maximum sizes of a minimal k-dominating set in G are denoted γ_k and Γ_k, respectively.

The *complementary prism* of a graph G, denoted by $G\overline{G}$, is a graph obtained by the disjoint union of G and its complement $\overline{G}$ by adding edges of a perfect matching between the corresponding vertices. The Petersen graph is a complementary prism of the cycle on 5 vertices, as shown in Figure 1.

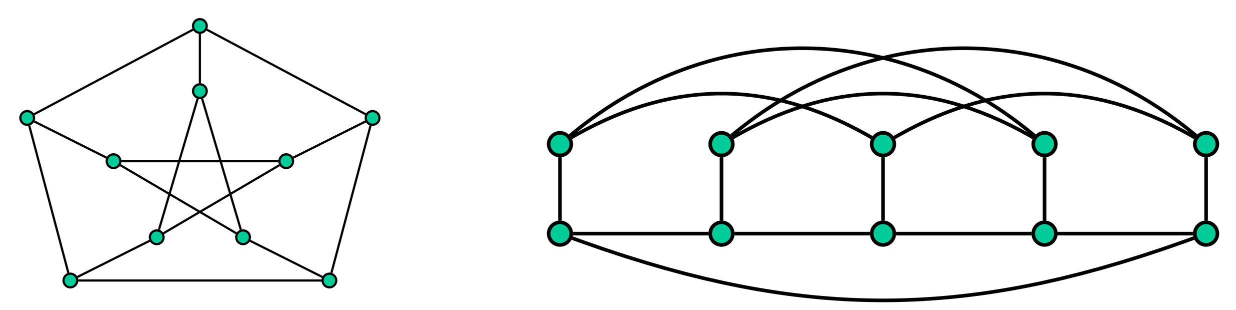

Figure 1: Two representations of the Petersen graph, the $C_5\bar{C}_5$ graph.

Haynes et al. [4] show upper and lower bounds for the maximum cardinality of 1-independent sets and for the minimum cardinality of 1-dominating sets. For a graph G, Chellali et al. [1] present a survey with relations and bounds between $\alpha(G)$, $i(G)$, $\gamma(G)$ and $\Gamma(G)$. Duarte et al. [2] prove that finding α_1 of a complementary prism $G\overline{G}$ is an NP-complete problem.

We present sharp lower and upper bounds for maximum 2-independent sets in complementary prism of any graph, characterize the graphs for which the upper and lower bound holds, and present closed formulas for the complementary prism of paths, cycles and complete graphs.

Relationships between the parameters

Since every set which is both 1-independent and 1-dominating is a minimal 1-dominating set of G, it is easy to see that

$$\gamma_1(G) \le i_1(G) \le \alpha_1(G) \le \Gamma_1(G)$$

for any graph G.

Favaron [3] shows that, for any graph G and positive integer k, $\gamma_k(G) \le \alpha_k(G)$ and $i_k(G) \le \Gamma_k(G)$.

Some general properties:

- Every k-dominating set of a graph G contains at least k vertices and all vertices of degree less than k; so $\gamma_k(G) \ge k$ when $n \ge k$.
- Every set with k vertices is k-independent; so $i_k(G) \ge k$ when $n \ge k$.
- Every set S that is both k-independent and k-dominating is a maximal k-independent set and a minimal k-dominating set.
- Every $(k+1)$-dominating set is also a k-dominating set.
- Every k-independent set is also a $(k+1)$-independent set.

Results on 2-independent sets in complementary prisms

Haynes et al. [4] show that, for any graph G, $\alpha_1(G)+\alpha_1(\overline{G})-1 \le \alpha_1(G\overline{G}) \le \alpha_1(G)+\alpha_1(\overline{G})$, and both these bounds are sharp. In Theorem 1, we generalize this result for $\alpha_2(G\overline{G})$.

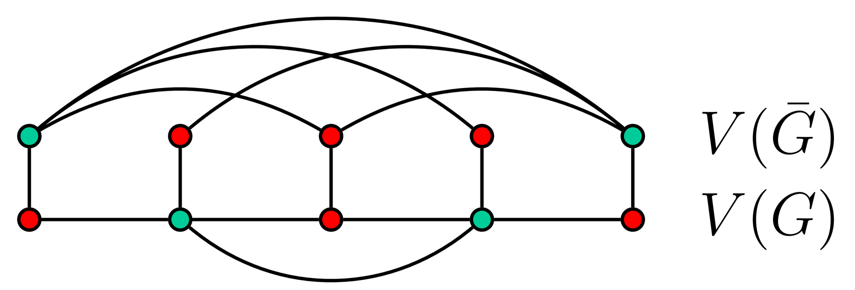

Figure 2: Graph with a maximum 2-independent set highlighted (red vertices) with $\alpha_2(G\overline{G}) = \alpha_1(G)+\alpha_1(\overline{G})$.

Theorem 1

For any graph G,

$$\alpha_1(G)+\alpha_1(\overline{G}) \le \alpha_2(G\overline{G}) \le \alpha_2(G)+\alpha_2(\overline{G}),$$

and both these bounds are sharp.

The graph G whose complementary prism $G\overline{G}$ is shown in Figure 2 attains the lower bound of Theorem 1, and the graph C_5 attains the upper bound. In the following result, we characterize the graphs for which the upper bound holds.

Theorem 2

A graph G has $\alpha_2(G\overline{G}) = \alpha_2(G)+\alpha_2(\overline{G})$ if and only if there exist disjoint vertex sets S and T in $V(G)$ such that S is $\alpha_2(\overline{G})$-set and T induces a maximum multipartite graph in G such that every partition has size at most two.

Now we show exact values for α_2 for some particular graph classes.

Theorem 3

Let $n \ge 5$. Then, $\alpha_2(K_n\overline{K}_n) = n+1$,

$$\alpha_2(P_n\overline{P}_n) = \begin{cases} 2\lfloor n/3 \rfloor + 4, n \equiv 2 \pmod 3, \\ 2\lfloor n/3 \rfloor + 3, \text{otherwise,} \end{cases}$$

$$\alpha_2(C_n\overline{C}_n) = \begin{cases} 2\lfloor n/3 \rfloor + 3, n \equiv 2 \pmod 3, \\ 2\lfloor n/3 \rfloor + 2, \text{otherwise.} \end{cases}$$

Future work

As future work, we plan to characterize graphs attaining the lower bound on Theorem 1; to extend the presented results for α_k, for $k \ge 3$; and to study k-dominating sets in complementary prisms.

References

[1] M. Chellali, O. Favaron, A. Hansberg, and L. Volkmann. K-domination and k-independence in graphs: A survey. *Graphs and Combinatorics*, 28:1–55, 01 2012.

[2] M. A. Duarte, L. Penso, D. Rautenbach, and U. Santos S. Complexity properties of complementary prisms. *J. Comb. Optim.*, 33(2):365–372, February 2017.

[3] O. Favaron. On a conjecture of fink and jacobson concerning k-domination and k-dependence. *Journal of Combinatorial Theory, Series B*, 39(1):101 – 102, 1985.

[4] T. W. Haynes, M. A.Henning, P. J. Slater, and L. C. Merwe. The complementary product of two graphs. *Bulletin of the Institute of Combinatorics and its Applications*, 51, 2007.

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LEXICOGRAPHIC BREADTH-FIRST SEARCH AND EXACT ALGORITHMS FOR THE MAXIMUM CLIQUE PROBLEM

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INTRODUCTION

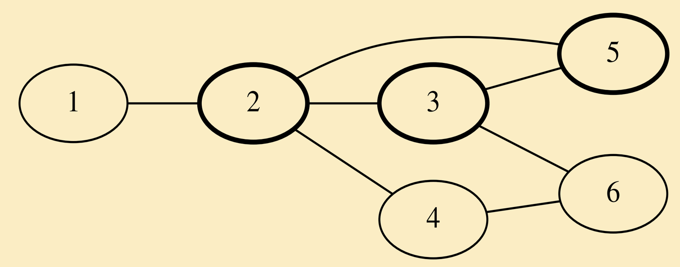
In this work five exact algorithms for the maximum clique problem (MC) were modified with Lexicographic Breadth-first Search (**LexBFS**) algorithm. Also, Experimental Analysis of Algorithms and hypothesis test were used to evalutate the changes.

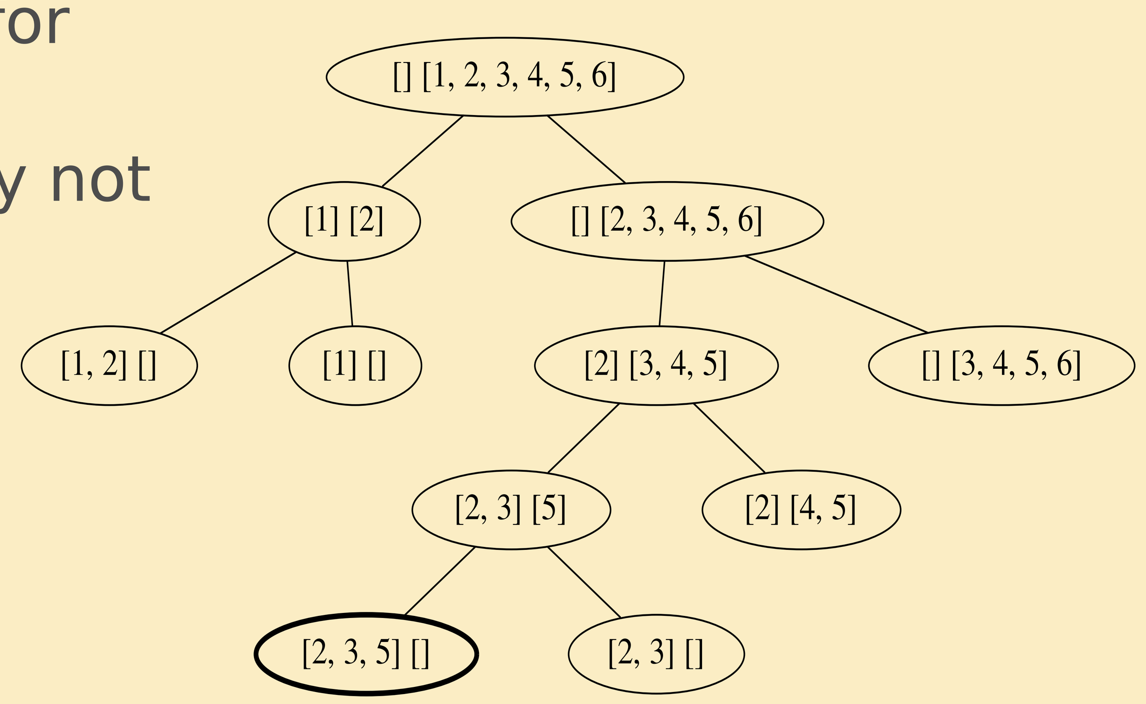
OBJECTIVES

1. Study **branch and bound** algorithms that use vertex coloring to solve MC modified with the **LexBFS** algorithm.
2. Apply Experimental Analysis of Algorithms.
3. Apply Statical Inference Theory.

MAXIMUM CLIQUE ALGORITHMS

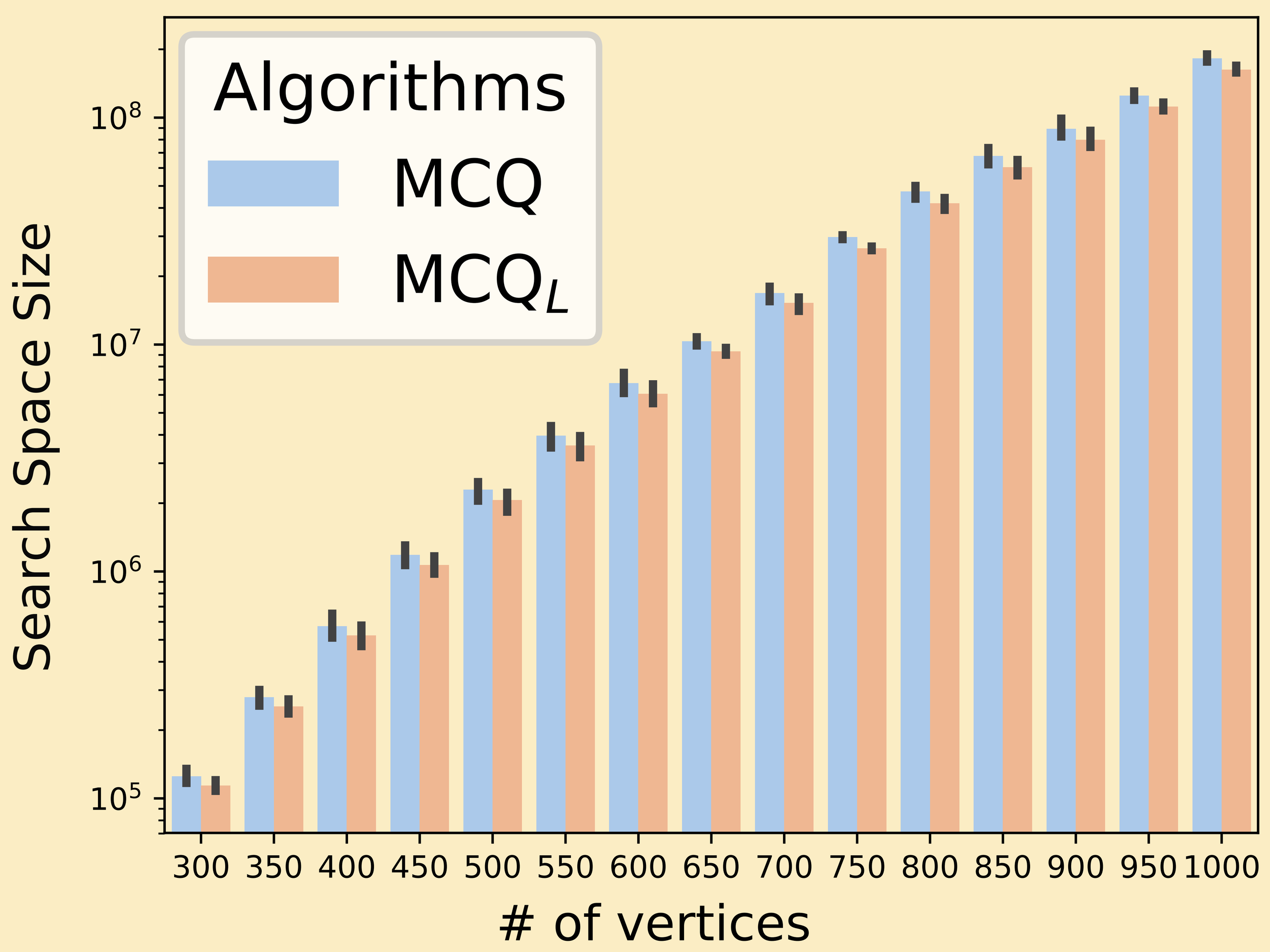
Branch and bound algorithms for MC evaluate a search space. A small search space may or may not result in shorter runtime.

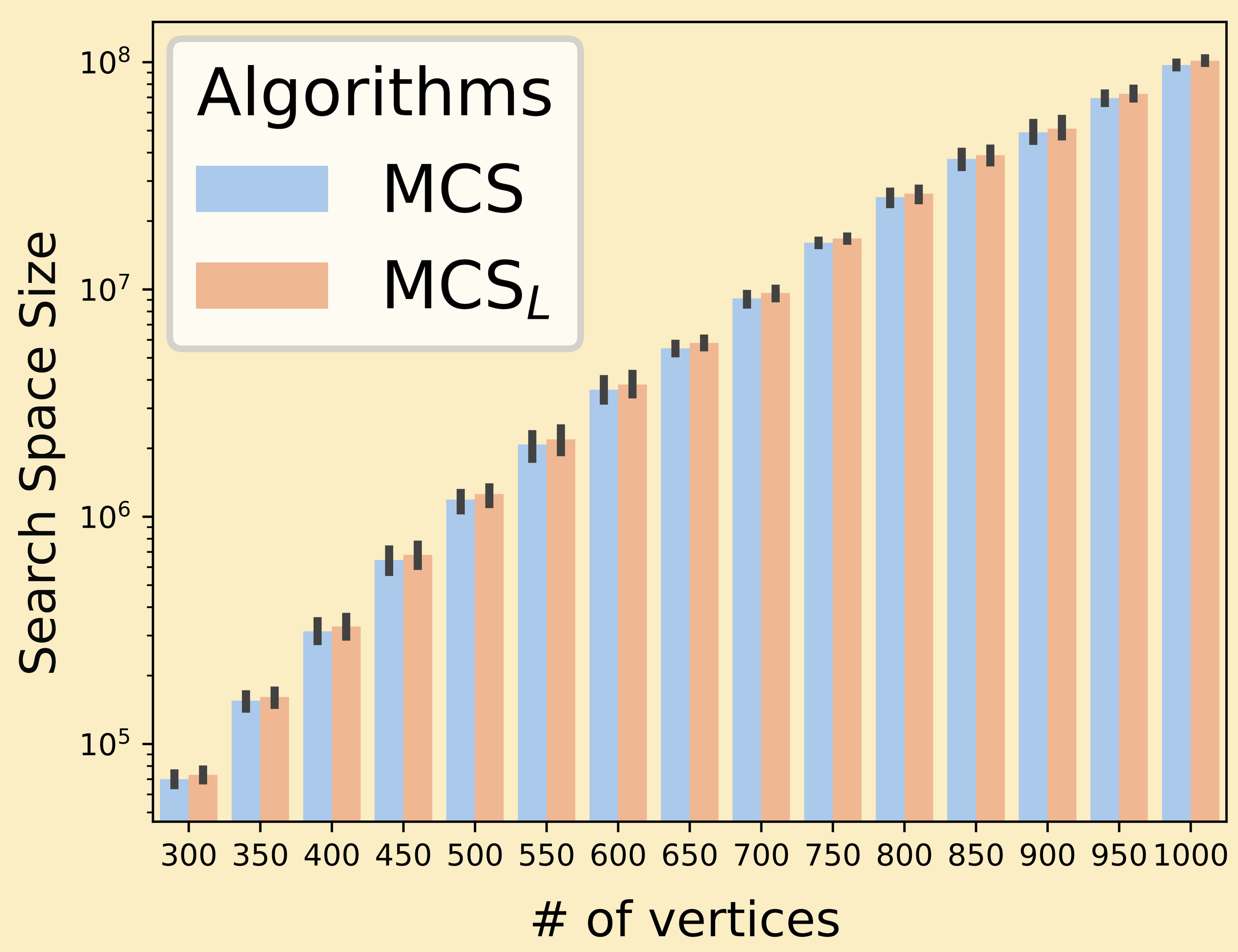




RESULTS

- 80 instances from *DIMACS Second Implementation Challenge*
- 150 instances: random graphs, chordal graphs and cographs
- C++ implementation in https://gitlab.c3sl.ufpr.br/apzuge/maxcliquebb





Comparison of the search space evaluated by two algorithms and their respective modifications. For each vertex set size there is a sample size of 10.

CONCLUSION

With algorithms modified by LexBFS the search space is significative smaller, but the runtime was significative longer.

References

McGeoch, Catherine C. A guide to experimental algorithmics. Cambridge University Press, 2012.

Tomita, Etsuji, and Tomokazu Seki. "An efficient branch-and-bound algorithm for finding a maximum clique." International conference on discrete mathematics and theoretical computer science. Springer, 2003. https://doi.org/10.1007/3-540-45066-1_22.

Tomita E., Seki T. (2003) An Efficient Branch-and-Bound Algorithm for Finding a Maximum Clique. In: Calude C.S., Dinneen M.J., Vajnovszki V. (eds) Discrete Mathematics and Theoretical Computer Science. DMTCS 2003. Lecture Notes in Computer Science, vol 2731. Springer, Berlin, Heidelberg. https://doi.org/10.1007/3-540-45066-1_22.

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Kernelization lower bounds for Multicolored Independent Set

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1. Introduction

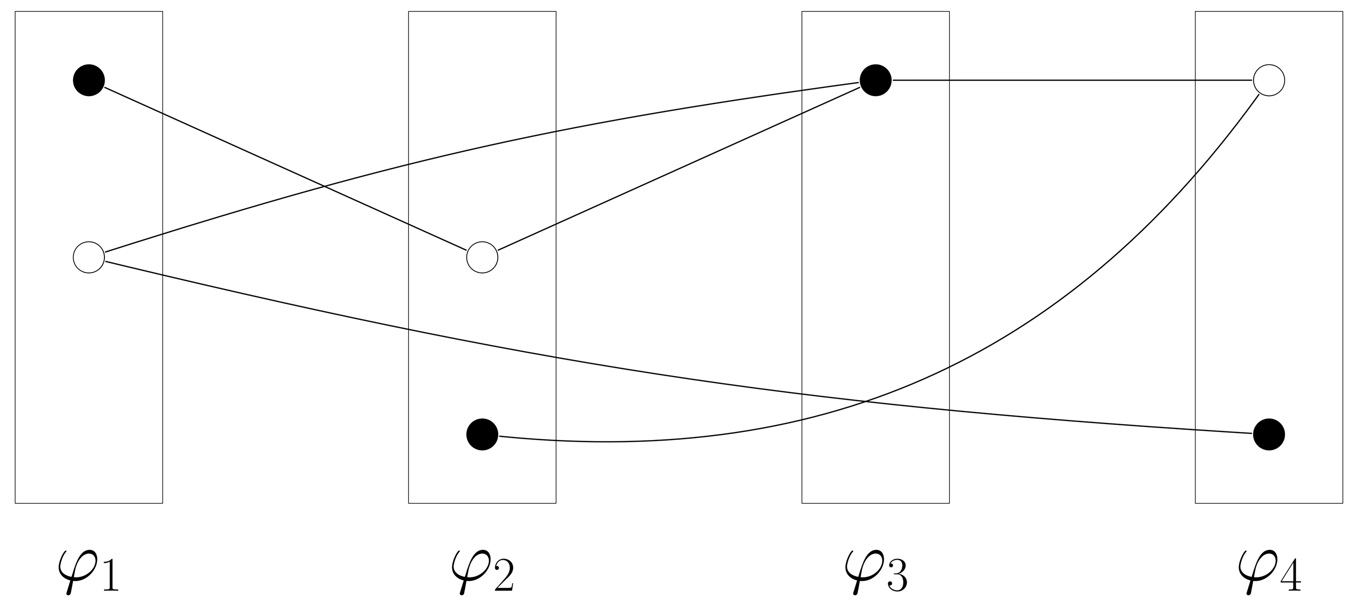
Pre-processing algorithms are frequently employed when solving large problems and are often fundamental to do so. Until recently, however, these algorithms were designed without theoretical guarantees, and measuring their effectiveness was a completely empirical process. Parameterized complexity offers a sound theoretical framework that allows us to prove lower and upper bounds for these **kernelization** algorithms, as they came to be known in the community [2]. Given an instance (x, k) of a parameterized problem Π, we say that Π admits a kernel of size $g(k)$ when parameterized by k if we can build an equivalent Π instance of size at most $g(k)$. Motivated by the fact that Multicolored Independent Set is a central problem in parameterized complexity, we prove the following theorem, where a class $\mathcal{G}$ is non-trivial if, for every $t \in \mathbb{N}$, $\mathcal{G}$ contains a graph on t vertices; we point out that Independent Set does admit a polynomial kernel [3] under vertex cover.

2. The theorem

For every fixed non-trivial graph class $\mathcal{G}$, Multicolored Independent Set does not admit a polynomial kernel when jointly parameterized by vertex deletion distance to $\mathcal{G}$ and size of the solution, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.

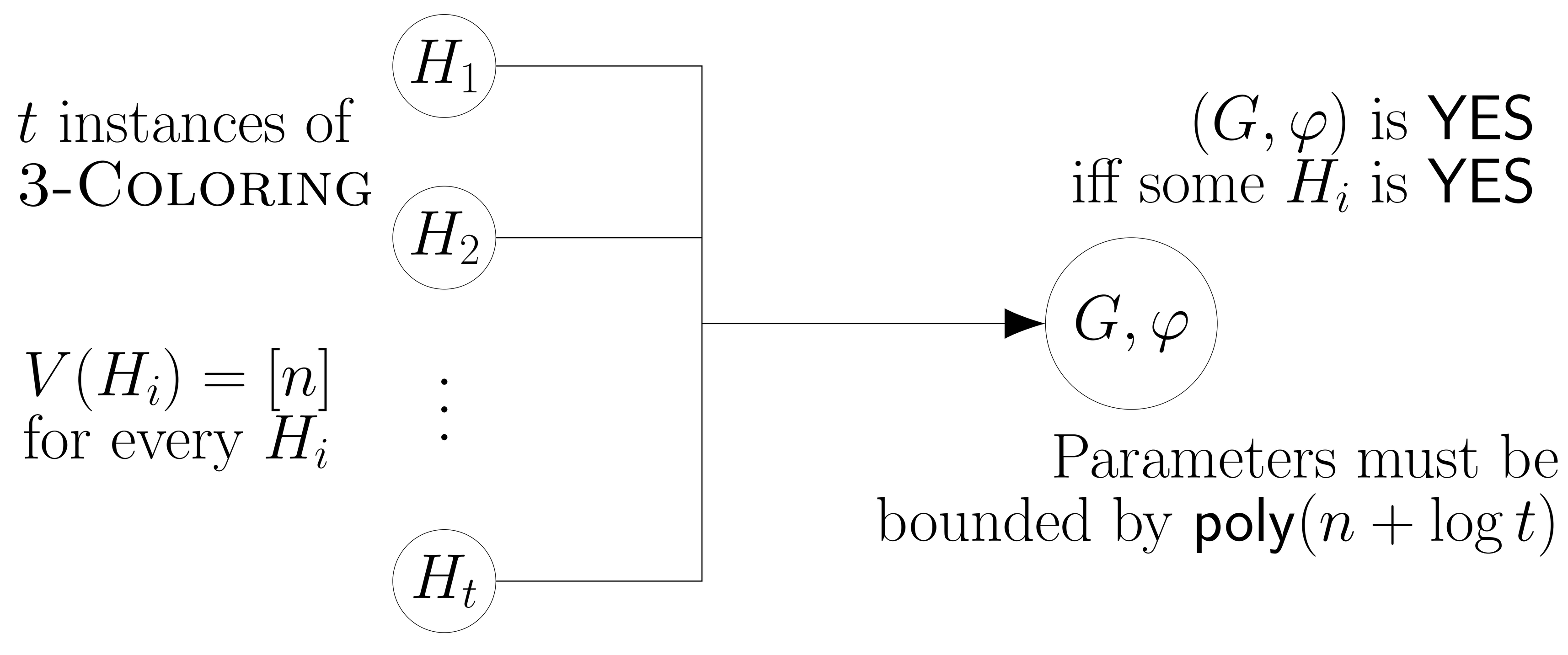
3. Multicolored Independent Set

An instance is a pair (G, φ) where G is a graph, φ is a partition of $V(G)$, and the goal is to find an independent set of G that hits each part of φ exactly once.



4. Cross-composition

We use the cross-composition framework of Bodlaender et al. [1] to show that 3-Coloring OR-cross-composes into Multicolored Independent Set parameterized by distance to $\mathcal{G}$ and size of the solution. That is, it is a many to one reduction with the following constraints:

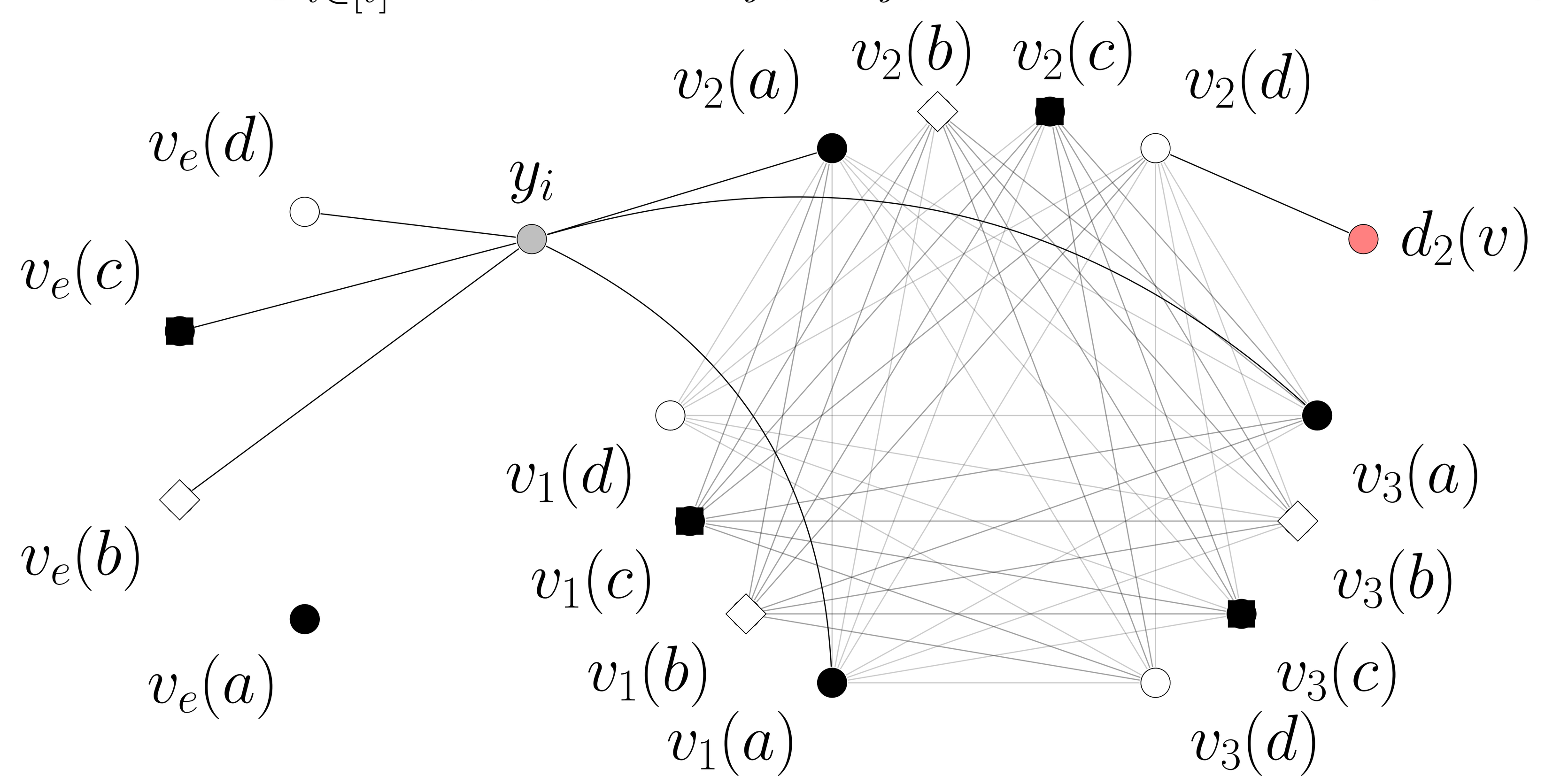


5. Instance Selector Gadget

Begin by adding to G a set $Y = \{y_1, \ldots, y_t\}$ that induces a graph of $\mathcal{G}$, and add Y as a part of φ.

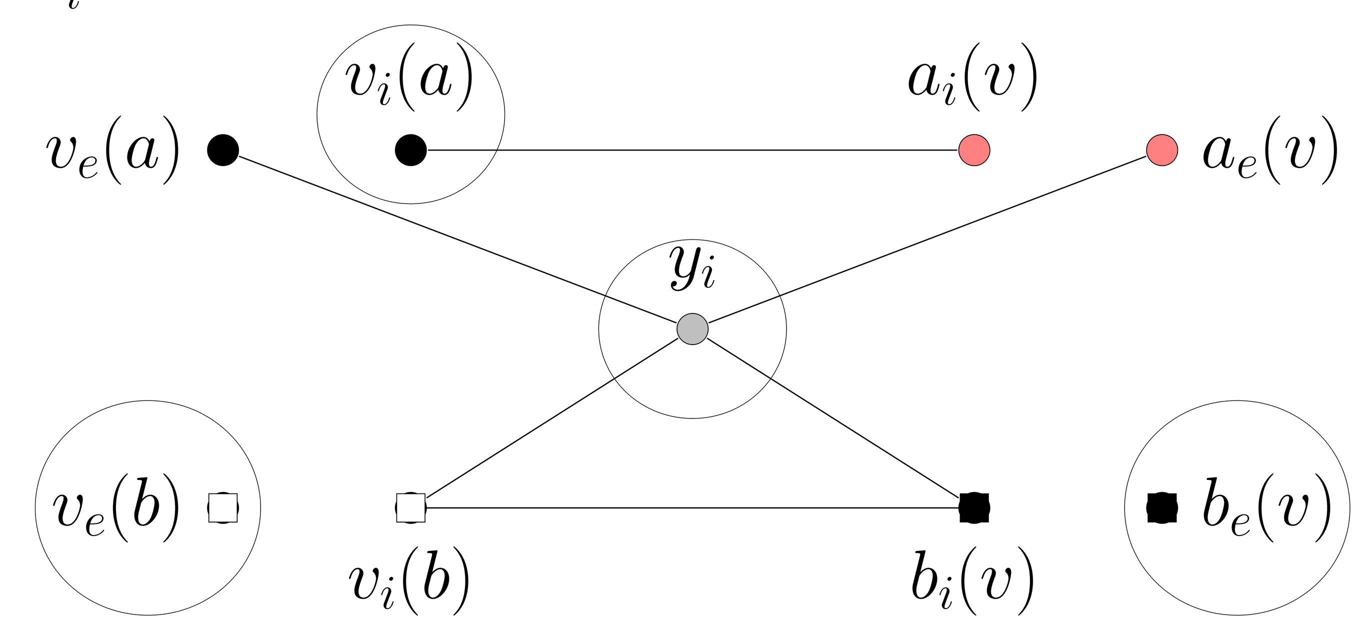
6. Vertex Gadget

For each $v \in [n]$, add to G a gadget G_v containing an independent set $A(v) = \{v_e(a) \mid a \in [n] \setminus \{v\}\}$ and a copy $K(v)$ of the complete tripartite graph $K_{n-1,n-1,n-1}$. For each G_v, add parts $\{p(v, a) \mid a \in [n] \setminus \{v\}\}$ to φ; each $p(v, a)$ contains $v_e(a)$ and three non-adjacent vertices $v_1(a), v_2(a), v_3(a)$ of $K(v)$. For each H_i, if $av \notin E(H_i)$, add edges $\{y_i v_j(a)\}_{j \in [3]}$ and $\{y_i a_j(v)\}_{j \in [3]}$, otherwise add edges $y_i v_e(a)$ and $y_i a_e(v)$. For each $av \in \bigcup_{i \in [t]} E(H_i)$, add $v_j(a) a_j(v)$ for every $j \in \{1, 2, 3\}$.



7. Intuition

- For each $v \in [n]$ we can only choose vertices of one color class of $K(v) \Rightarrow v_i(\cdot)$ is in the solution $\mathcal{I}$ if and only if we color $v \in [n]$ with color i.
- If $v_i(a) \in \mathcal{I}$, then $a_i(v) \notin \mathcal{I} \Rightarrow v$ and a cannot have the same color.
- If $v_e(a) \in \mathcal{I}$, we can ignore edge $av \Rightarrow$ when $v_e(a) \in \mathcal{I}$, v and a can have the same color.
- There is a unique $y_i \in \mathcal{I}$ and, for every $av \in E(H_i)$, $v_e(a) \notin \mathcal{I}$, so some $v_i(a)$ must be in $\mathcal{I}$ and $a_i(v)$ must not $\Rightarrow$ if $y_i \in \mathcal{I}$, vertices that are adjacent in H_i cannot have the same color.



References

[1] Hans L. Bodlaender, Bart M. P. Jansen, and Stefan Kratsch. "Cross-Composition: A New Technique for Kernelization Lower Bounds". In: *Proc. of the 28th International Symposium on Theoretical Aspects of Computer Science (STACS)*. Vol. 9. LIPIcs. 2011, pp. 165–176.

[2] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. 1st. Springer Publishing Company, Incorporated, 2015. ISBN: 3319212745.

[3] Fedor V. Fomin, Bart M.P. Jansen, and Michał Pilipczuk. "Preprocessing subgraph and minor problems: When does a small vertex cover help?" In: *Journal of Computer and System Sciences* 80.2 (2014), pp. 468 –495. ISSN: 0022-0000.

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Partitioning Graphs into Monochromatic Trees

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UFMG

Introduction

This work presents complexity results about the NP-Completeness of Partition edge-coloured Graphs into vertex disjoint Monochromatic Trees (**PGMT**) when we restrict the frequency with each color occurs at the edges of the graph. Jin and Li [3] defined the the PGMT problem as follows:

THE PGMT PROBLEM
Instance: An edge-coloured graph G and a positive integer k.
Question: Are there k or less vertex disjoint monochromatic trees which cover the vertices of the graph G?

Figure 1 shows an example of a graph partitioned into monochromatic trees; even in the colored graph with 3 colors, only two trees are sufficient to cover the vertices.

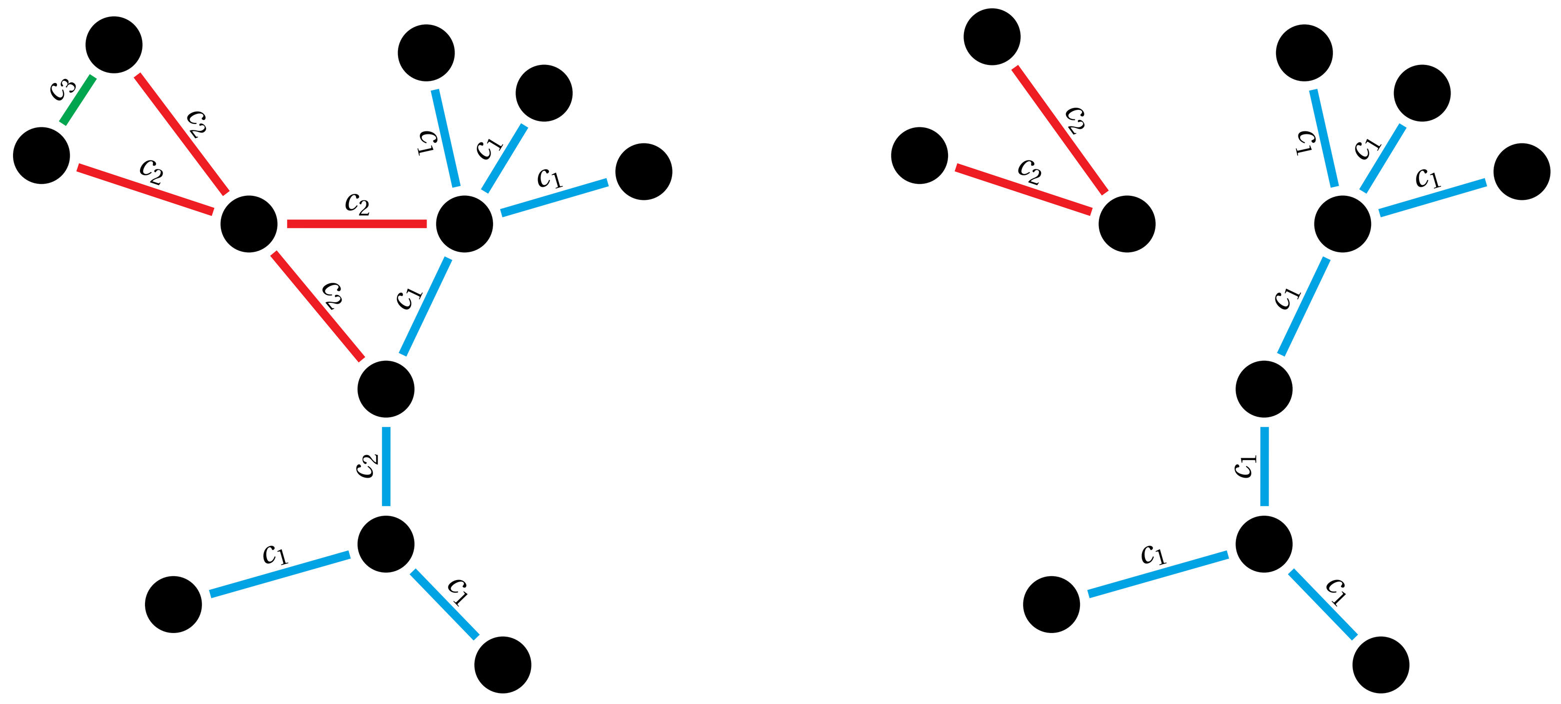


Figure 1

Related Works

- In their work, Jin and Li [3] showed that **PGMT** is *NP-Complete* and there is no constant factor approximation algorithm.
- Jin and Li [4], defined a more restricted version of the problem. In this version, the number of distinct colors of G is fixed, and this version is known as **r-PGMT**, where r is the number of colors. For all $r \geq 5$, they showed that **r-PGMT** is also *NP-Complete*.
- Jin et al. [2] showed that, for $r = 2$, **r-PGMT** is also *NP-Complete* for bipartite graphs. For complete bipartite and complete multipartite graphs, however, they presented algorithms that solve the problem in polynomial time.

The f_{MAX}-PGMT Problem

Jin and Li [4] considered a version of **PGMT** where the number of different colors of the graph is fixed. In this work we consider another kind of restriction to the input graph. In this version, instead of fixing the number of different colors, we only guaranteed that each color appears at most f times. We define this version as follows:

THE f_{MAX}-PGMT PROBLEM
Instance: An edge-coloured graph G, where each color occurs at most f times, and a positive integer k.
Question: Are there k or less vertex disjoint monochromatic trees which cover the vertices of the graph G?

NP-Completeness Results

We now show that f_{MAX}-PGMT is *NP-Complete*, when $f = 3$, by reducing from *Exact Cover by 3-Sets* - **X3C** [1], which is defined as follows:

The X3C Problem
Instance: An set $\mathcal{X} = \{v_1, ..., v_n\}$, $|\mathcal{X}| = 3k$; an family of subsets $\mathcal{F} = \{S_1, S_2, ..., S_m\}$, $S_i \subseteq \mathcal{X}$ e $|S_i| = 3$, $i \in \{1, 2, ..., |\mathcal{F}|\}$.
Question: Is there $\mathcal{F}' \subseteq \mathcal{F}$, such that $\bigcup_{S \in \mathcal{F}'} S = \mathcal{X}$?

We build an instance $(G, k + m - 2)$ of f_{MAX}**-PGMT** that is equivalent to an instance $(\mathcal{X}, \mathcal{F})$ of **X3C** as follows: The set of vertices is $V(G) = \{v_1, ..., v_n, S1, ..., S_m, z_1, ..., z_{m-2}\}$. The set of edges is

$$E(G) = \begin{cases} v_i S_j, \ \ if\ v_i \in S_j \\ z_i S_p \end{cases} \tag{1}$$

for all $i \in \{1, ..., n\},\ j \in \{1, ..., m\},\ p \in \{i, i+1, i+2\}$. And coloring the edges as follow:

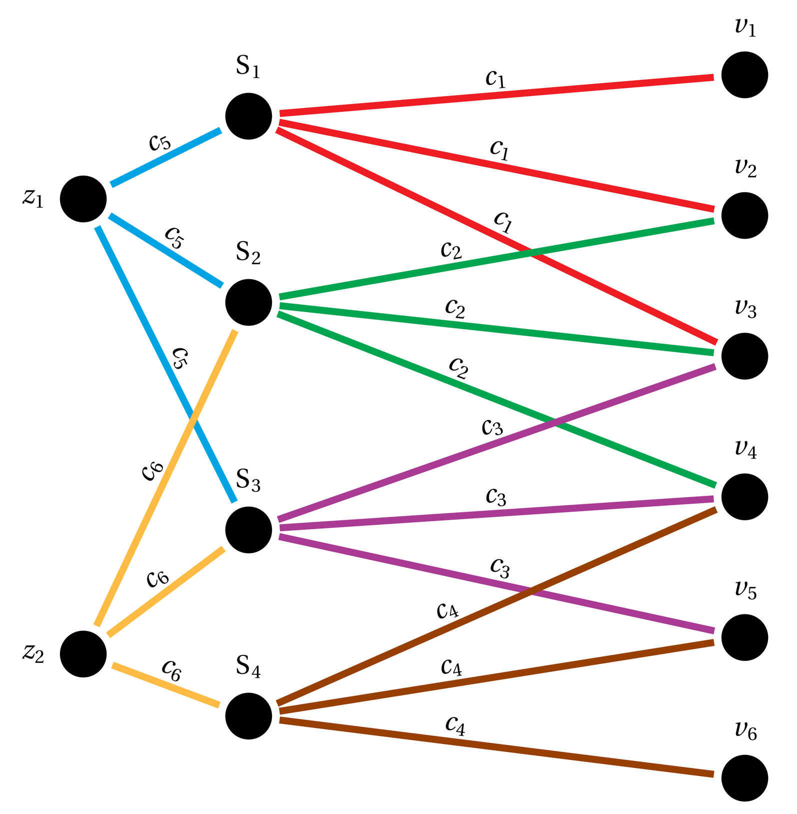
$$c(e) = \begin{cases} c_j \quad\ ,\ if\ e = v_i S_j \\ c_{m+p}\ ,\ if\ e = z_p S_q \end{cases} \tag{2}$$

for all $e \in E(G),\ i \in \{1, ..., n\},\ j \in \{1, ..., m\},\ p \in \{1, ..., m-2\},\ q \in \{p, p+1, p+2\}$. Figure 2 shows an example of the transformation described: (a) **X3C** instance and (b) colored graph from that instance.

$$\mathcal{X} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

$$\mathcal{F} = \begin{Bmatrix} S_1 = \{v_1, v_2, v_3\}, \\ S_2 = \{v_2, v_3, v_4\}, \\ S_3 = \{v_3, v_4, v_5\}, \\ S_4 = \{v_4, v_5, v_6\} \end{Bmatrix}$$

(a)



(b)

Figure 2

References

[1] M. R. Garey and D. S. Johnson.
Computers and Intractability: A Guide to the Theory of NP-Completeness.
W. H. Freeman, United States, New York, 1979.
Series of Books in the Mathematical Sciences.

[2] Z. Jin, M. Kano, X. Li, and B. Wei.
Partitioning 2-edge-colored complete multipartite graphs into monochromatic cycle, paths and trees.
Journal of Combinatorial Optimization, 11(4):445–454, 2006.

[3] Z. Jin and X. Li.
The complexity for partitioning grapgs by monochromatic trees, cycles and paths.
International Journal of Computer Mathematics, 81(11):1357–1362, 2004.

[4] Z. Jin and X. Li.
Vertex partitions of r-edge-colored graphs.
Applied Mathematics-A Journal of Chinese Universities, 23(1):120–126, 2008.

A_α-Spectrum of some Matrogenic Graphs

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A_α-Spectral Theory

Definition 1.([3]) Let $G = G(V, E)$ be a simple graph. The matrix $A_\alpha(G)$ is defined by

$$A_\alpha(G) = \alpha \cdot D(G) + (1 - \alpha) \cdot A(G),\ \alpha \in [0, 1],$$

where $A(G)$ denotes the adjacency matrix of G and $D(G) = (d_{ij})$, is a matrix of order n, where $d_{ij} = 0$, if $i \neq j$ and $d_{ij} = d(v_i)$, if $i = j$.

Matrogenic Graphs

Definition 2. Let $G = G(V, E)$ be a graph. Given $u, v \in V$, we say that u *dominates* v if $N_G(v) - \{u\} \subseteq N_G(u) - \{v\}$. When neither u dominates v nor v dominates u, then u and v are called *incomparable*.

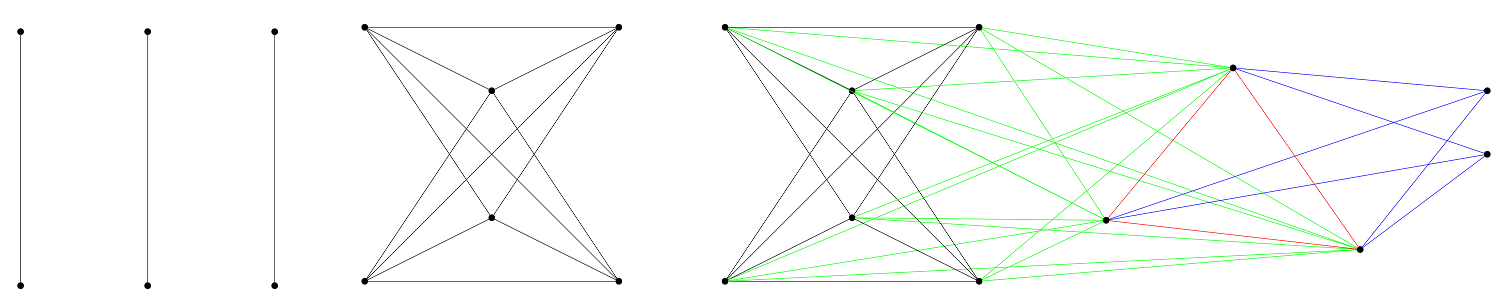
Definition 3. A graph G is *matrogenic* if for any two vertices u and v, incomparable in G, we have $|(N_G(u) - \{v\}) \oplus (N_G(v) - \{u\})| = 2$, where the symbol $\oplus$ denotes the symmetric difference.

Definition 4. A split graph $S(r, s)$ is a graph whose vertices can be partitioned into a clique of size r, and a independent set of size s. A split graph is called complete if every vertex in the independent set is adjacent to every vertex in the clique; it is denoted by $CS(s, r)$.

Definition 5. A graph G is threshold if for $u, v \in V(G)$, either u dominates v or v dominates u.

Properties of Matrogenic Graphs

Definition 6. A perfect matching, tK_2, is the union of t copies of K_2 and a cocktail party graph, $CP(2t)$, is the complement of a perfect matching.



$3K_2$ $\quad$ $CP(6)$ $\quad$ $G_{11}(CS(3, 2), CP(6))$

Some properties of the matrogenic graphs: all induced subgraphs of a matrogenic graph are matrogenic; the complement of a matrogenic graph is matrogenic and the class of matrogenic graphs contains the class of threshold graphs. In particular, as the split complete graph is threshold, it is matrogenic.

Theorem 1.([2]) A graph $G = G(V, E)$ of order n is matrogenic if and only if V can be partitioned into three distinct sets K, S, and C such that
(i) $K \cup S$ induces a matrogenic split subgraph in which K is a clique and S is a independent set;
(ii) C induces a perfect matching, or a cocktail party, or a C_5;
(iii) every vertex of C is adjacent to every vertex of K and to no vertex in S.

Theorem 1 gives us a way to characterize matrogenic graphs from a partition of its vertex set V. Thus, we can denote every matrogenic graph as $G_n([K \cup S], [C])$. In the previous figure we show the matrogenic graph $G_{11}(CS(3, 2), CP(6))$.

A_α-Spectrum

In this work, we analyze the A_α-spectrum of a subclass of matrogenic graphs.

Theorem 2. If $H = G_n(CS(k, s), CP(2t))$ then A_α-characteristic polynomial of H is given by

$$P_{A_\alpha(H)}(x) = f(x)[x - \alpha(2t + k) + 2]^{t-1}(x - \alpha n + 1)^{k-1}(x - \alpha k)^{s-1}[x - \alpha(2t + k - 2)]^t,$$

where $f(x) = det(xI - \overline{A_\alpha}(H))$,

$$\overline{A_\alpha}(H) = \begin{pmatrix} \alpha(k + 2t - 2) + (1 - \alpha)(2t - 2) & (1 - \alpha)k & 0 \\ (1 - \alpha)2t & \alpha(k - 1 + s + 2t) + (1 - \alpha)(k - 1) & (1 - \alpha)s \\ 0 & (1 - \alpha)k & \alpha k \end{pmatrix}.$$

Sketch of proof. There is a labeling of the vertices of the graph H, so that the matrix A_α can be written

$$A_\alpha(H) = \begin{pmatrix} B_\alpha & (1 - \alpha)J_{2t\times k} & 0_{2t\times s} \\ (1 - \alpha)J_{k\times 2t} & C_\alpha & (1 - \alpha)J_{k\times s} \\ 0_{s\times 2t} & (1 - \alpha)J_{s\times k} & \alpha k I_s \end{pmatrix},$$

where we denote the all-ones matrix by J, the all-zeros matrix by 0, the identity matrix by I, $B_\alpha = \alpha(k + 2t - 2)I_{2t} + (1 - \alpha)(J_{2t} - I_{2t} - A(tK_2))$ and $C_\alpha = \alpha(k - 1 + s + 2t)I_k + (1 - \alpha)(J_k - I_k)$. Denote by e_k the vector with $2t$ coordinates whose k-th entry is equal to 1 and the others entries are zero. For each j, ℓ and i, with $1 \leq j \leq t$, $2 \leq \ell \leq k$ and $2 \leq i \leq s$, consider the vectors $z_j = (e_{2j-1} - e_{2j}|0|0)^T$, $w_\ell = (0|e_{2t+k+1} - e_{2t+k+\ell}|0)^T$ and $v_i = (0|0|e_{2t+k+1} - e_{2t+k+i}|0)^T$. We have,

$$A_\alpha(H)z_j = \alpha(2t + k - 2)z_j, \quad A_\alpha(H)w_\ell = (\alpha n - 1)w_\ell \quad \text{and} \quad A_\alpha(H)v_i = \alpha k v_i.$$

Now, consider the vector $v^{(i)} = e_{2i-1} + e_{2i}$. Some calculations show that the $t - 1$ vectors of the form $\left(v^{(1)} - v^{(i)}|0|0\right)^T$, $2 \leq i \leq t$, are the eigenvectors of $A_\alpha(H)$ associated with the eigenvalue $\alpha(k + 2t) - 2$. The other eigenvalues are the roots of the polynomial $f(x)$, which follows from the matrix reduction technique (see Theorem 1.3.14 of [1]).

Conclusion

As it was claimed in [3], the matrix A_α can underpin a unified theory of the spectral study of the adjacency and singless Laplacian matrices of a graph. In this work, we obtain a partial factorization of the A_α-characteristic polynomial of a subfamily of matrogenic graphs which explicitly gives some eigenvalues of the graph.

References

[1] D. Cvetković, P. Rowlinson, and S. Simić. *An Introduction to the Theory of Graph Spectra.* Cambridge University Press. Cambridge. 2010.

[2] S. Foldes and P. Hammer. *On a class of matroid-producing graphs.* Colloq. Math. Soc. J. Bolyai (Combinatorics). 18: 331-352. 1978.

[3] V. Nikiforov. *Merging the A- and Q-spectral theories.* Applicable Analysis and Discrete Mathematics. 11: 81-107. 2017.

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Decomposing cubic graphs into locally irregular subgraphs

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Introduction

A **decomposition** of a graph G is a set $\mathcal{D} = \{H_1, \dots, H_k\}$ of edge-disjoint subgraphs of G such that $\cup_{i=1}^{k} E(H_i) = E(G)$. A **locally irregular** graph is a graph in which adjacent vertices have distinct degrees.

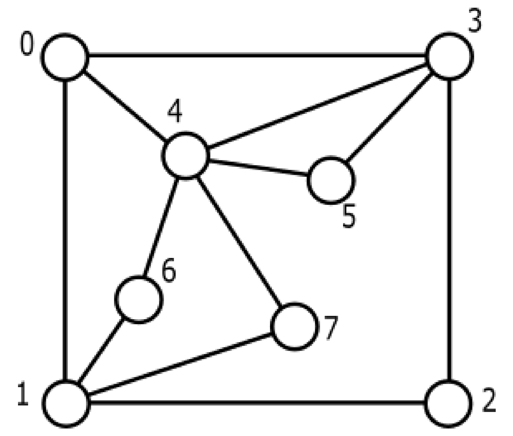


Figure 1: A locally irregular graph

A **locally irregular decomposition** (or **locally irregular coloring**) of a graph G is a decomposition in which every element is locally irregular. We say that G is **decomposable** if it admits a locally irregular decomposition. Equivalently, a locally irregular decomposition is a coloring of $E(G)$ in which every color class induces a locally irregular subgraph in G. If k colors are used, then we say **locally irregular k-edge-coloring** or **k-LIC** for short.

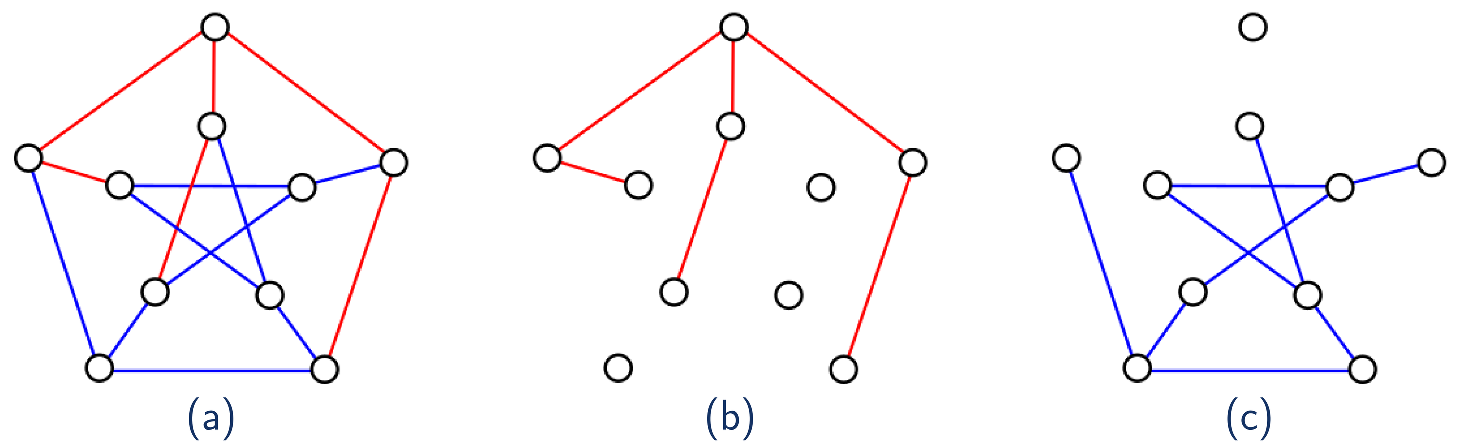


Figure 2: (a) A 2-LIC of G. (b) An induced subgraph of G using the edges with color red. (c) An induced subgraph of G using the edges with color blue.

Given a decomposable graph G, the **irregular chromatic index** of G is the smallest number k for which G admits a k-LIC. We denote the irregular chromatic index of G by $\chi'_{irr}(G)$. The problem of computing the irregular chromatic index was proven to be an NP-complete problem [2].

In this work we explore the following conjecture posed by Baudon et al. [1].

Conjecture 1 (O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak, 2015)**.** For every decomposable graph G, we have $\chi_{irr}(G) \leq 3$.

Results toward confirming Conjecture 1 include that graphs whose set of vertices can be partitioned into a clique and an independent set admit a 3-LID [3] and graphs with maximum degree at most 3 admit a 4-LID [4]. We explore Conjecture 1 for graphs in which all vertices have degree 3, which are called **cubic graphs**.

Contribution

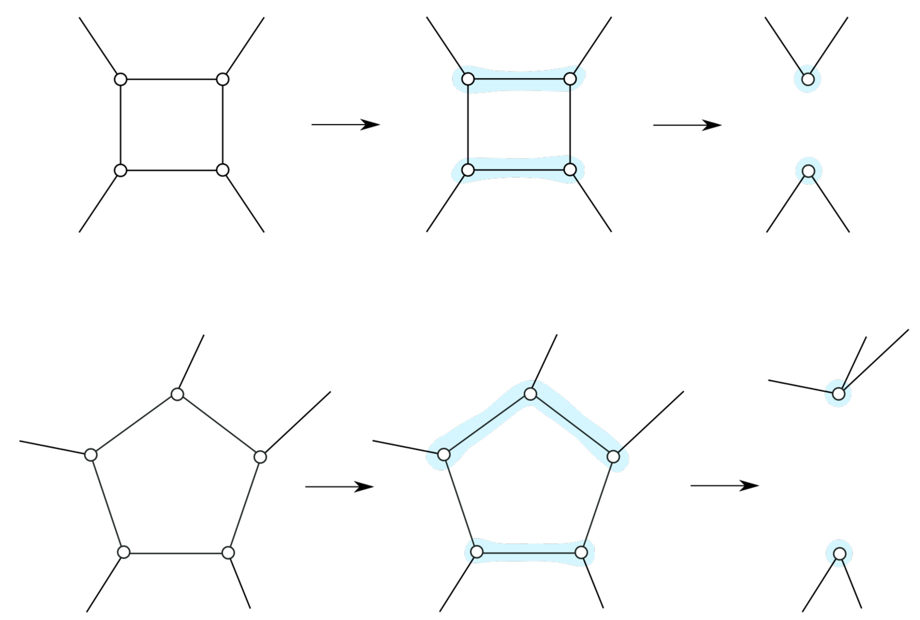
In this poster we verify Conjecture 1 for a class of cubic graphs; and we present a condition for a graph not to be 2-LIC.

Locally irregular coloring of some cubic graphs

A **proper edge-coloring** of a graph G is an assignment of colors to the edges of G in which edges that share a vertex are colored with different colors. A P_2-decomposition of a graph G is a decomposition of G into paths of length 2. Let G be a cubic graph, and let $\mathcal{P}$ be a P_2-decomposition of G. Given a vertex $v \in V(G)$, let $\mathcal{P}(v)$ denote the number of paths $P \in \mathcal{P}$ for which $d_P(v) = 1$, and let $V_i^{\mathcal{P}}$ be the set of vertices v of G for which $\mathcal{P}(v) = i$.

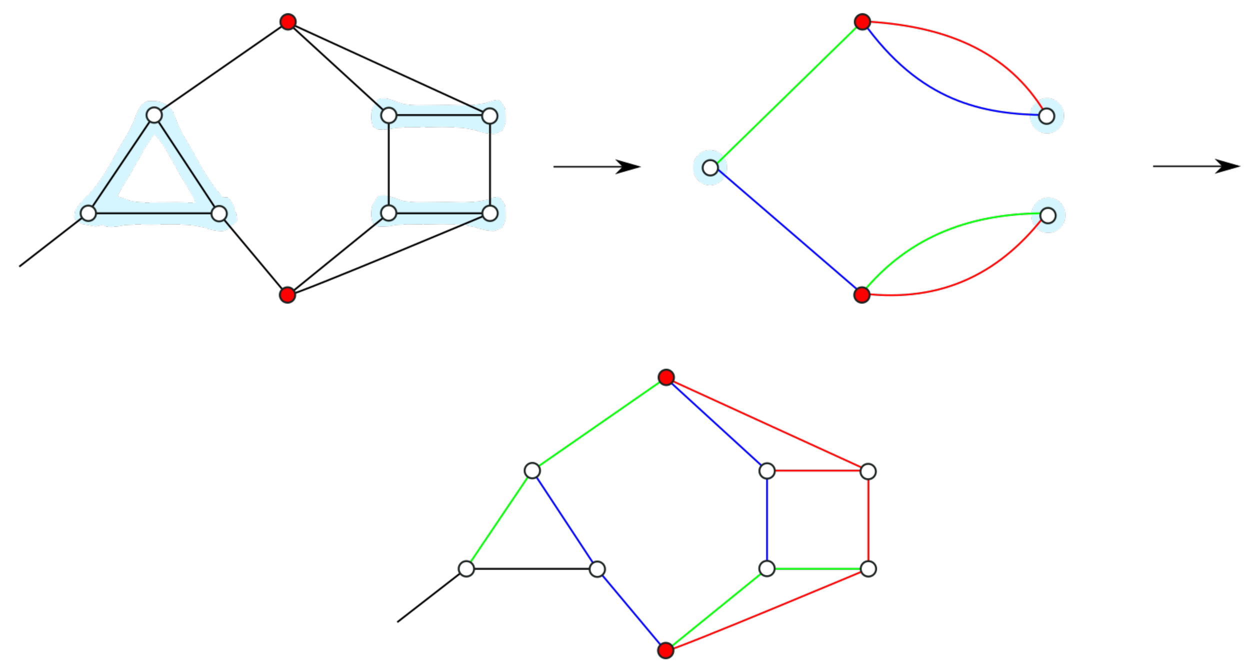
***Theorem* 1.** *If G is a cubic graph that admits a P_2-decomposition $\mathcal{P}$ for which $G[V_1^{\mathcal{P}}]$ is a set of vertex-disjoint cycles, then $\chi'_{irr}(G) \leq 3$.*

Proof: First note that $\mathcal{P}(v) \in \{1, 3\}$ for every $v \in V(G)$. In particular every vertex of $V_1^{\mathcal{P}}$ is the interior vertex of precisely one path of $\mathcal{P}$. Since $G[V_1^{\mathcal{P}}]$ is a set of vertex-disjoint cycles, every vertex in $V_1^{\mathcal{P}}$ is adjacent to precisely one vertex of $V_3^{\mathcal{P}}$ and two vertices of $V_1^{\mathcal{P}}$. Given a cycle $C \in G[V_1^{\mathcal{P}}]$, we partition the vertex set of C into pairs and at most one triple of consecutive vertices.

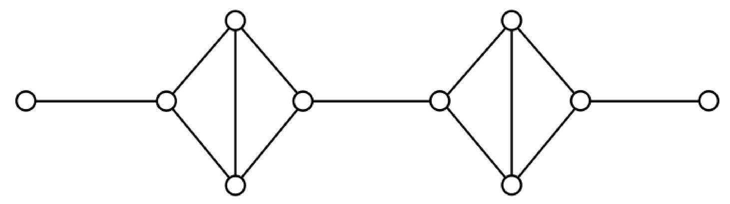


Let H be the graph obtained from $G \setminus E(G[V_1^{\mathcal{P}}])$ by identifying vertices in the same pair or triple, and keeping parallel edges. Note that every path of $\mathcal{P}$ has exactly one edge in H. The graph H is a bipartite graph with maximum degree exactly 3. It is not hard to prove that G admits a proper edge-coloring with three colors.

Now, we use the the proper edge-coloring above to obtain a locally irregular coloring of $E(G)$. By construction every path in $\mathcal{P}$ has precisely one edge already colored in H, and we color its remaining edge (which is in a cycle of $G[V_1^{\mathcal{P}}]$) with the same color. Since each vertex of $G[V_1^{\mathcal{P}}]$ is in the same pair or triple of at least one of its neighbors in $G[V_1^{\mathcal{P}}]$, each path of $\mathcal{P}$ is colored with the same color of at most one path with which it shares a vertex. Therefore each color consists of vertex-disjoint paths of length 2 and trees with four edges and one vertex of degree 3, and hence, is a locally irregular graph. □

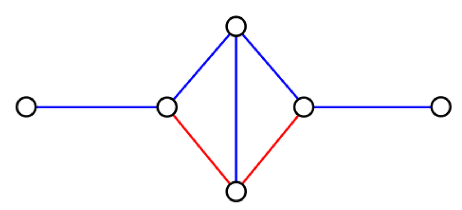


In order to prove that some cubic graphs have locally irregular chromatic index at least 3, we define the gadget below which we call a **strip**. So we have the following theorem.



***Theorem* 2.** *If G has a strip S whose vertices with degree 3 are not adjacent to vertices in $V(G) \setminus S$, then $\chi'_{irr}(G) > 2$.*

Proof: The proof follows from the fact that any 2-LIC of a "half strip" must be as in the figure below, and then the two "half strips" of the same strip cannot be colored in a compatible manner.



By replacing one edge by strip, we can prove that there are infinitely many graphs that do not admit a 2-LIC. In particular, there are an infinite number of cubic graphs with chromatic index 4 and planar graphs that do not admit an 2-LIC, and hence the upper bound of Conjecture 1 is tight for these classes of graphs.

References

[1] O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak.
On decomposing regular graphs into locally irregular subgraphs.
European J. Combin., 49:90–104, 2015.

[2] O. Baudon, J. Bensmail, and É. Sopena.
On the complexity of determining the irregular chromatic index of a graph.
Journal of Discrete Algorithms, 30:113–127, 2015.

[3] C. N. Lintzmayer, G. O. Mota, and M. Sambinelli.
Decomposing split graphs into locally irregular graphs.
Electronic Notes in Theoretical Computer Science, 346:603–612, 2019.

[4] B. Lužar, J. Przybyło, and R. Soták.
New bounds for locally irregular chromatic index of bipartite and subcubic graphs.
J. Comb. Optim., 36(4):1425–1438, 2018.

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Some results on Vertex Separator Reconfiguration

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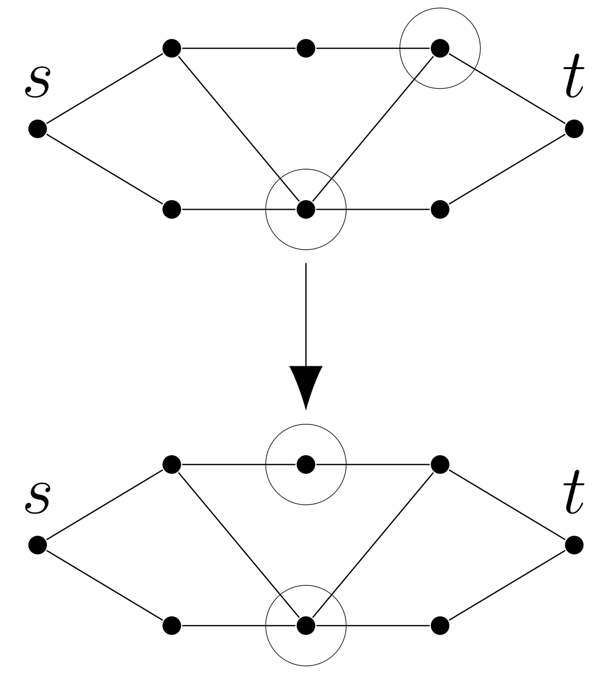
1. Introduction

Graph reconfiguration problems have been studied extensively in the literature, with Independent Set Reconfiguration [3] being by far the favorite research topic. Nevertheless, reconfiguration problems of other graph structures, such as vertex covers [4] and vertex colorings [1], have also been investigated. No previous work, however, has dealt with the reconfiguration of vertex separators. In this work, we begin this study in the form of the Vertex Separator Reconfiguration problem. In this problem, we are given a graph G and two st-separators S_a and S_b of G, and the goal is to reconfigure S_a into S_b. We prove its complexity on a subclass of bipartite graph under the three most common reconfiguration rules: token sliding (TS), token jumping (TJ), and token addition/removal (TAR); being **PSPACE-hard** under TS and **NP-hard** under the other two. We also show that TS and TAR computationally equivalent.

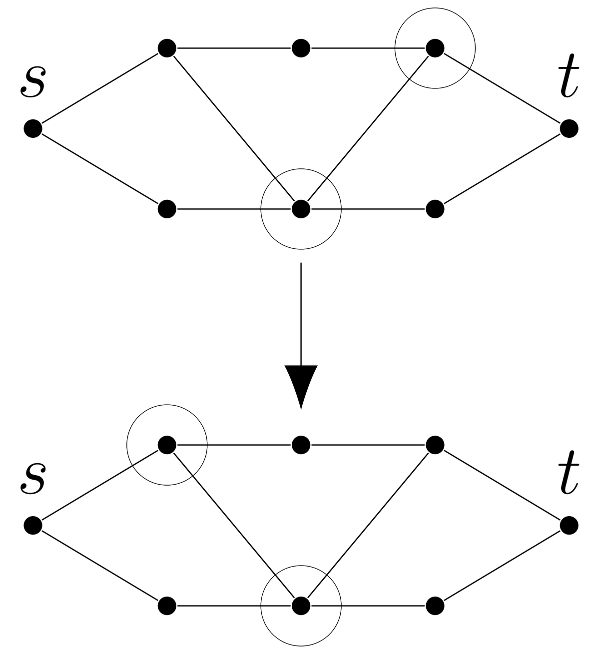
2. Token Sliding

Replace $v \in A$ with some vertex in $N(v) \setminus A$.



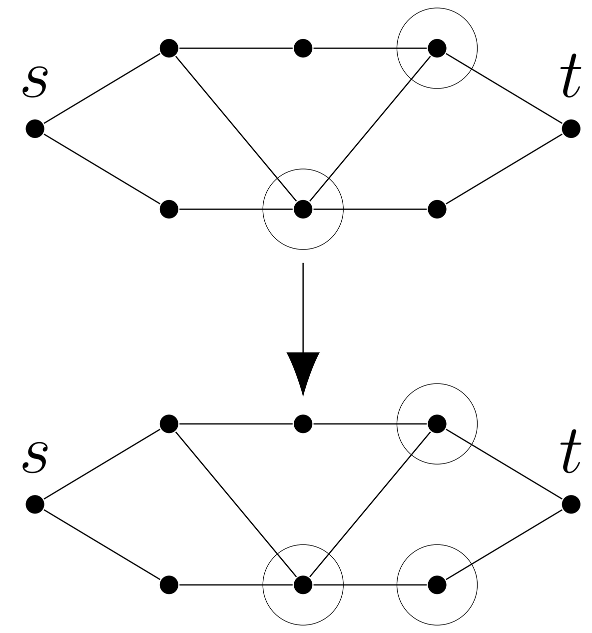
3. Token Jumping

Replace a vertex of A with any other vertex of G not in A.



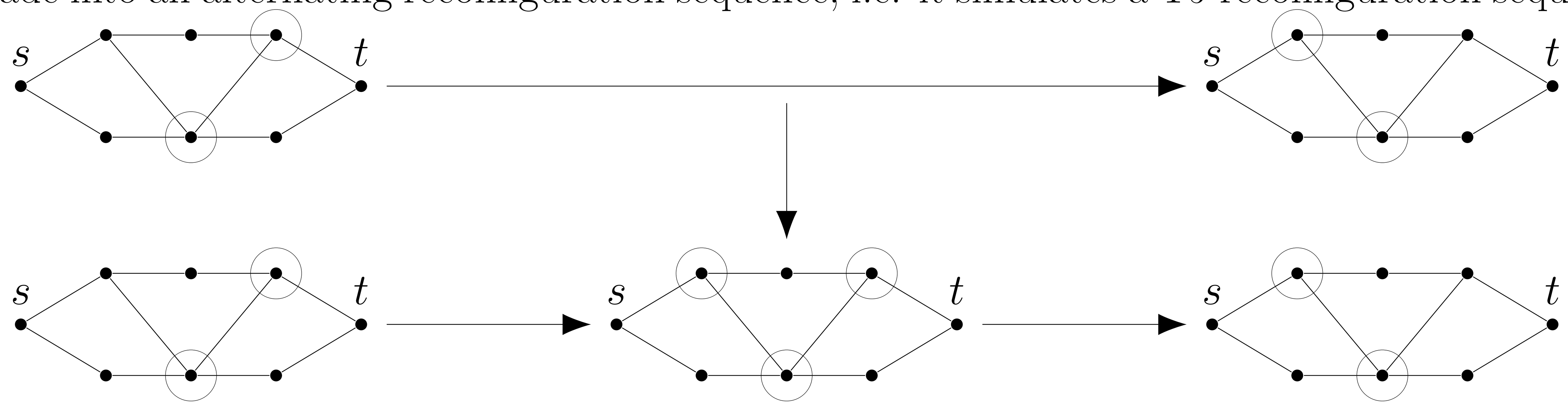
4. k-Token Add./Rem.

Add/remove a vertex from A, so long as the resulting A' satisfies $|A'| \leq k$.



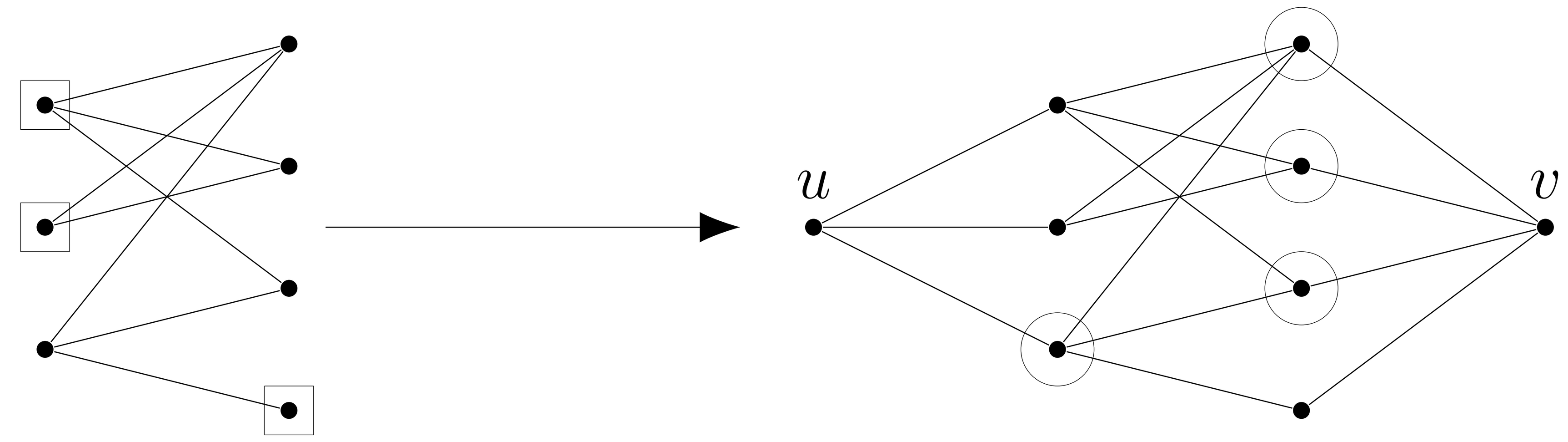
5. TAR/TJ are equivalent

Let us assume that $|S_b| \geq |S_a|$ and $S_a \neq S_b$. We can easily simulate a TJ instance (G, S_a, S_b): just create the TAR instance $(G, S_a, S_b, k+1)$, where $k = \max\{|S_a|, |S_b|\}$. For the converse, given a TAR instance (G, S_a, S_b, k), if $|S_b| = k$ and S_b is minimal, then we answer negatively. Otherwise, pick any two st-separators $S'_a \subseteq S_a$ and $S'_b \subseteq S_b$ of same size and with at most $k-1$ vertices; it follows that (G, S_a, S_b, k) is equivalent to (G, S'_a, S'_b, k) and that we can reconfigure S'_a and S'_b into S_a and S_b, respectively. We can also show that any reconfiguration sequence between S'_a and S'_b can be made into an alternating reconfiguration sequence, i.e. it simulates a TJ reconfiguration sequence.



6. Complexity on bipartite graphs

Let G be a bipartite graph with bipartition A, B and H the bipartite graph obtained by adding to G two vertices u, v such that u is adjacent to every vertex of A and v to every vertex of B. Our reduction is from Independent Set Reconfiguration which is **NP-complete** on bipartite graphs under TJ and **PSPACE-hard** under TS [3]. Its correctness follows from a simple but powerful observation: *A set $I \subseteq V(G)$ is independent in G if and only if $V(G) \setminus I$ is an uv-separator of H.* Formally, if (G, I_a, I_b) is the Independent Set Reconfiguration instance, we construct the equivalent Vertex Separator Reconfiguration $(H, V(G) \setminus I_a, V(G) \setminus I_b)$ where H is defined as before.



7. Final Remarks

We investigated the complexity of the reconfiguration of vertex separators under three commonly studied reconfiguration rules. We also showed that TAR and TJ are computationally equivalent. In the arXiv version of this work [2], we also presented polynomial time algorithms for various classes, including series-parallel graphs and graphs with a polynomial number of minimal separators, which have been omitted here.

References

[1] Luis Cereceda, Jan van den Heuvel, and Matthew Johnson. "Finding Paths between 3-Colorings". In: *J. Graph Theory* 67.1 (May 2011), 69–82.
[2] Guilherme C. M. Gomes, Sérgio H. Nogueira, and Vinicius F. dos Santos. *Some results on Vertex Separator Reconfiguration*. 2020. arXiv: `2004.10873 [cs.CC]`.
[3] Robert A. Hearn and Erik D. Demaine. "PSPACE-completeness of sliding-block puzzles and other problems through the nondeterministic constraint logic model of computation". In: *Theoretical Computer Science* 343.1-2 (2005), 72–96.
[4] Daniel Lokshtanov and Amer E. Mouawad. "The Complexity of Independent Set Reconfiguration on Bipartite Graphs". In: *ACM Trans. Algorithms* 15.1 (Oct. 2018). issn: 1549-6325. doi: `10.1145/3280825`.

Generalized Packing Functions of Graphs with few P_4's

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Abstract

We introduce a concept of packing of graphs which generalizes all those previously defined in the literature and we study the computational complexity of computing the associated parameter, the generalized packing number of the graph. We find that this new packing parameter comes to be much more complicated to handle than those previously defined, even on particular graph classes as spider and quasi-spider graphs. Nevertheless, we prove that the associated optmization problem can be solved in linear time for some graph classes with few P_4's.

General packing functions

Let $G=(V,E)$ be a graph and $\mathbf{k},\boldsymbol{\ell},\mathbf{u}\in\mathbb{Z}_+^V$ with $\boldsymbol{\ell}\le\mathbf{u}$. A $(\mathbf{k},\boldsymbol{\ell},\mathbf{u})$-*packing function of* G is a function $f:V\to\mathbb{Z}_+$ satisfying the following conditions for all $v\in V$: $\ell(v)\le f(v)\le u(v)$ and $f(N[v])\le k(v)$. In addition, we define $\mathcal{L}_{\mathbf{k},\boldsymbol{\ell},\mathbf{u}}(G)=\{f : f \text{ is a } (\mathbf{k},\boldsymbol{\ell},\mathbf{u})-\text{packing function of } G\}$. Then, the $(\mathbf{k},\boldsymbol{\ell},\mathbf{u})$- generalized packing number of G is

$$L_{\mathbf{k},\boldsymbol{\ell},\mathbf{u}}(G)=max\{f(V): f\in\mathcal{L}_{\mathbf{k},\boldsymbol{\ell},\mathbf{u}}(G)\}.$$

Reduction to instances with $\boldsymbol{\ell}=0$: $L_{\mathbf{k},\boldsymbol{\ell},\mathbf{u}}(G)=\boldsymbol{\ell}(V)+L_{\tilde{\mathbf{k}},0,\tilde{\mathbf{u}}}(G)$ for $\tilde{\mathbf{k}}(v)=\mathbf{k}(v)-\boldsymbol{\ell}(N[v])$, $\tilde{\mathbf{u}}(v)=\mathbf{u}(v)-\boldsymbol{\ell}(v)$

$$L_{\mathbf{k},\ell,\mathbf{u}}(G)\to L_{\tilde{\mathbf{k}},0,\tilde{\mathbf{u}}}(G)\to L_{\tilde{\mathbf{k}},\tilde{\mathbf{u}}}(G).$$

Given $f\in\mathcal{L}_{\mathbf{k},\mathbf{u}}(G)$ such that $f(V)=L_{\mathbf{k},\mathbf{u}}(G)$ we say that f is an *optimal* $(\mathbf{k},\mathbf{u})$-packing function of G.
The *Packing Function Problem* (PFP) has a graph G and vectors $\mathbf{k},\mathbf{u}\in\mathbb{Z}_+^{V(G)}$ as input and the objective is to obtain an optimal $(\mathbf{k},\mathbf{u})$-packing function of G.

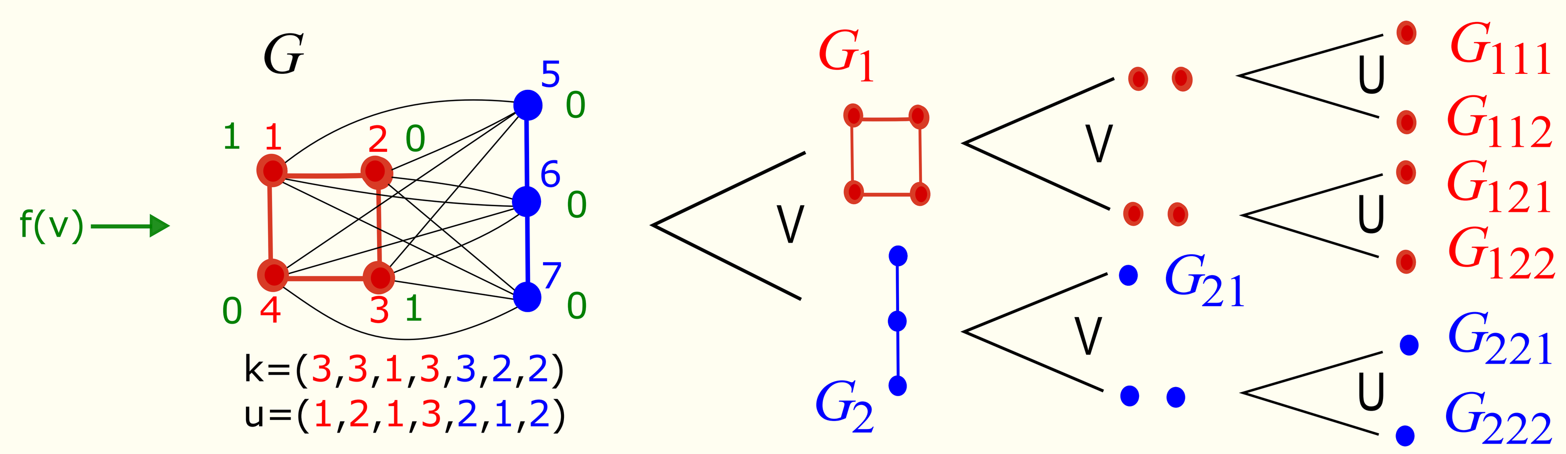
Modular decomposition

The modular decomposition of graphs involves two graph operations, *union* ($\cup$) and *join* ($\vee$). If a graph is not connected, it is the union of two graphs and if the complement of a graph is not connected, the graph is the join of two graphs (figure below). A graph is *modular* if it is connected and its complement is connected.

The parameter for these two operations can be computed as follows. Let $\mathbf{k}=(\mathbf{k}_1,\mathbf{k}_2)$, $\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2)$, $k_i^*=\min\{k_i(u): u\in V_i\}$ for $i=1,2$, $\ell_1(r)=\min\{L_{\mathbf{k}_1-r.\mathbf{1},\mathbf{u}_1}(G_1),k_2^*\}$, $\ell_2(r)=\min\{L_{\mathbf{k}_2-r.\mathbf{1},\mathbf{u}_2}(G_2),k_1^*\}$, and $\Delta(s)=\min\{\Delta: s\le\ell_1(\ell_2(s)-\Delta),\ \Delta\in[0,\ell_2(s)]\}$. Then,

$$L_{\mathbf{k},\mathbf{u}}(G_1\cup G_2)=L_{\mathbf{k}_1,\mathbf{u}_1}(G_1)+L_{\mathbf{k}_2,\mathbf{u}_2}(G_2),$$
$$L_{\mathbf{k},\mathbf{u}}(G_1\vee G_2)=\max\{s+\ell_2(s)-\Delta(s): s\in[0,\ell_1(0)]\}.$$

Example:



$$L_{\mathbf{k},\mathbf{u}}(G)=L_{\mathbf{k},\mathbf{u}}(G_1\vee G_2)=L_{\mathbf{k},\mathbf{u}}(((G_{111}\cup G_{112})\vee(G_{121}\cup G_{122}))\vee(G_{21}\vee(G_{221}\cup G_{222}))=2$$

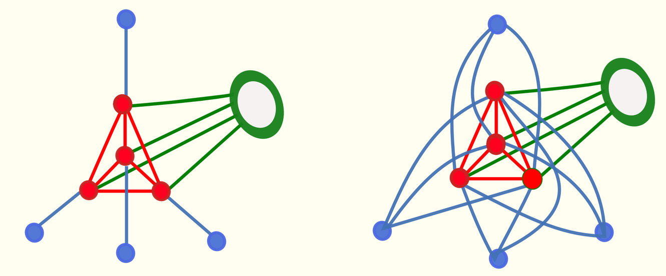
For a graph class $\mathcal{F}$, we denote by $M(\mathcal{F})$ the class of graphs in $\mathcal{F}$ which are modular. From the previous formulas we have the following result.

Lemma 1. *Let $\mathcal{F}$ be a hereditary family of graphs such that the PFP can be solved in polynomial (resp. linear) time for graphs in $M(\mathcal{F})$. Then, the PFP can be solved in polynomial (resp. linear) time for every graph in $\mathcal{F}$.*

Thus, let us study the graphs in $\mathcal{F}$ for graph classes with *few P_4's*, such as *P_4-sparse graphs* and *P_4-tidy graphs*. Formally, a graph is P_4-sparse if every set of five vertices contains at most one induced path on four vertices.

Spider graphs and P_4-sparse graphs

A *spider* is a graph $G=(V,E)$ such that V is partitioned into sets S, C and H, where $S=\{s_j : j\in[n]\}$ is a stable set, $C=\{c_j : j\in[n]\}$ is a clique, $n\ge 2$, and the *head* H is allowed to be empty. Moreover, all vertices in H are adjacent to all vertices in C and no vertex in S. Besides, in a *thin spider* graph, s_i is adjacent to c_j if and only if $i=j$, and in a *thick spider* graph, s_i is adjacent to c_j if and only if $i\ne j$. We denote a spider by (C,S,H).



Lemma 2. *Let $G=(C,S,H)$ be a thin spider graph. Then*

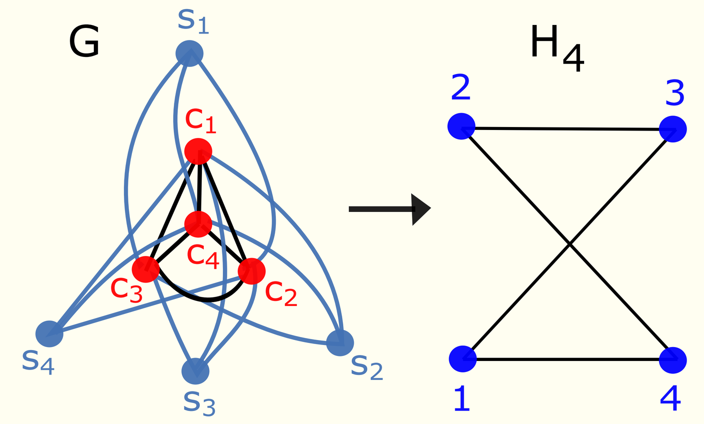
$$L_{\mathbf{k},\mathbf{u}}(G)=u(S)+L_{\tilde{\mathbf{k}},\tilde{\mathbf{u}}}(G[C]\vee G[H]).$$

where $\tilde{k}(h)=k(h)$ and $\tilde{u}(h)=u(h)$ for all $h\in H$, $\tilde{k}(c_i)=k(c_i)-u(s_i)$ and $\tilde{u}(c_i)=min\{u(c_i),k(s_i)-u(s_i)\}$.

The approach to study the problem in thick spiders is based on technical lemmas. They allow us to reduce the general problem to a particular instance ($\mathbf{k}=\mathbf{u}$) in thick spiders with empty head. Then, we apply a transformation from a thick spider with empty head to a particular graph H_n, which has even order and the edges missing form a perfect matching as is shown in the next example.

For a spider graph G and $\mathbf{k}=(6,3,5,4,6,7,5,3)$ we have

$$\tilde{\mathbf{k}}=(7,6,3,5),\ \mathbf{u}=(6,3,5,4)\Rightarrow\ L_{\mathbf{k},\mathbf{k}}(G)=L_{\tilde{\mathbf{k}},\mathbf{u}}(H_4)$$



Finally, this process derives the following result.

Proposition 1. *If the PFP is polynomial (linear) time solvable on a graph family $\mathcal{F}$, then the PFP can be solved in polynomial (linear) time on spider graphs such that the graph induced by the head is in $\mathcal{F}$.*

From the decomposition results [1, 2], if $\mathcal{F}$ is the class of P_4-sparse graphs, we know that the graphs in $M(\mathcal{F})$ are the trivial graph and spider graphs such that the graph induced by the head is P_4-sparse.
Lastly, considering the previous results and Lemma 1, we obtain the next theorem.

Theorem 1. *The PFP is linear time solvable for P_4-sparse graphs.*

Particular case in P_4-tidy graphs

A partner of a path P on four vertices in a graph G is a vertex $v\in V(G)\setminus V(P)$ such that the subgraph induced by $V(P)\cup\{v\}$ has at least two paths on four vertices. A graph G is a P_4-tidy if every path on four vertices has at most one partner.

Considering the problem, a particular case of the PFP is obtained when $\mathbf{u}(v)=\mathbf{k}(v)=k\ \forall v\in V$, for $k\in\mathbb{Z}_+$ fixed. In this case the problem is denoted $\{k\}$-PFP.

Concerning this restricted problem, the linearity result can be settled in P_4-tidy graphs, a graph class larger than P_4-sparse. Regarding modular decomposition, it is known that a P_4-tidy graph G is modular if and only if G is the trivial graph, C_5, P_5, $\overline{P_5}$, or a quasi-spider graph such that the graph induced by the head is P_4-tidy. A quasi-spider graph is a graph obtained from a spider by replacing at most one vertex in $S\cup C$ by a K_2 or a S_2.
Based on modular decomposition and applying the results obtained for the mentioned P_4-tidy modular graphs, we derive the following.

Theorem 2. *The $\{k\}$-PFP is linear time solvable for P_4-tidy graphs.*

References

[1] Hoang, C. T.: *Perfect graphs*, Phd. Thesis, School of Computer Science, McGill University (1985).
[2] Jamison, B., Olariu, S.: *A tree representation of P_4-sparse graphs*, Discrete Applied Mathematics **35** (1992), 115–129.

O número de dominação total e independência aberta para produto lexicográfico de grafos

remote 9th LAWCG and MDA
November 25th, 2020

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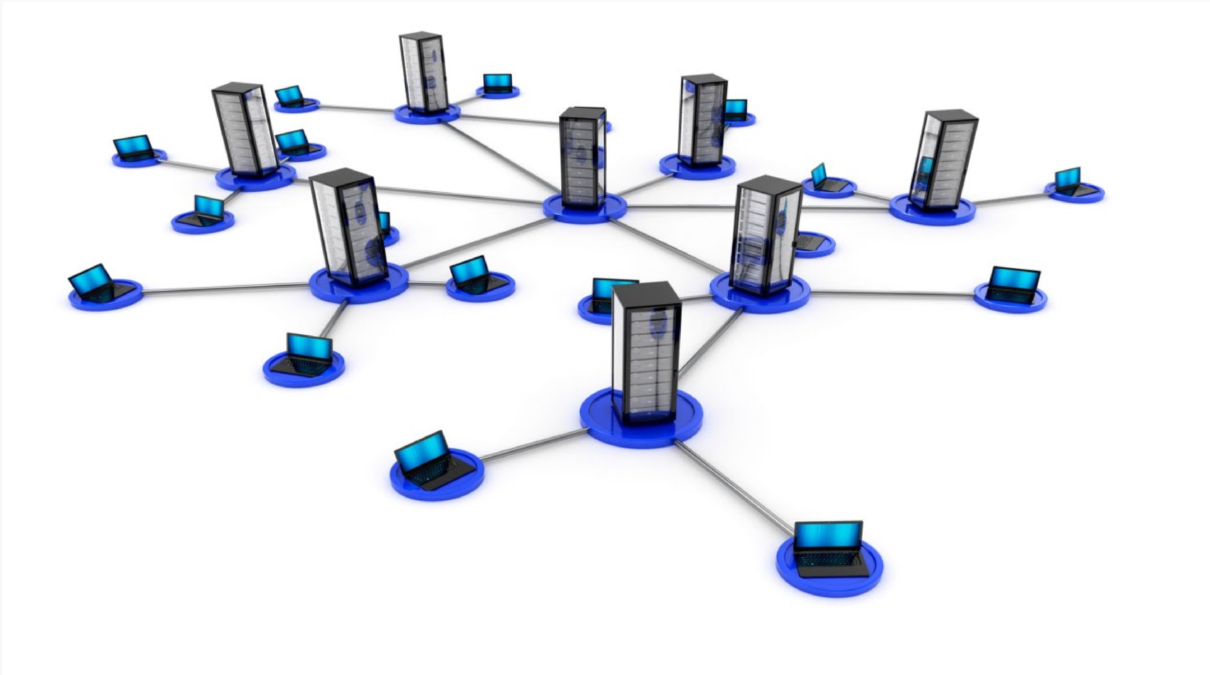
Introdução

Neste trabalho, consideraremos grafos simples, finitos e não direcionados. Considere a situação em que um grafo G modela uma rede de multiprocessadores com dispositivos de detecção colocados em vértices escolhidos de G. O objetivo desses dispositivos é detectar e identificar com precisão a localização de um processador defeituoso que pode estar presente em qualquer vértice. Às vezes, esse dispositivo pode determinar se um processador defeituoso está em sua vizinhança, mas não pode detectar se o defeito está em sua própria localização. Como é caro instalar e manter dispositivos de detecção, nessa rede cada detector terá no máximo um dispositivo detector em sua vizinhança. Então, queremos determinar a localização do número mínimo de dispositivos que podem, entre eles, determinar com precisão uma falha em qualquer local. Essa situação é uma aplicação direta do conceito de conjuntos dominantes e independentes abertos.
O principal objetivo desse trabalho é a determinação do número de dominação total e independência aberta, γ^{OP}_{OIND}, para o produto lexicográfico de grafos.



Conceitos Básicos

- **Produto Lexicográfico**
  Sejam os grafos G e H, onde os conjuntos de vértices é dado por $V(G) = \{g_1, g_2, ..., g_n\}$ e $V(H) = \{h_1, h_2, ..., h_m\}$. O *produto lexicográfico*, [1, 3], desses dois grafos, representado por $G \circ H$, terá o conjunto de vértices, $V(G \circ H) = V(G) \times V(H)$ e o conjunto de arestas, $E(G \circ H) = \{(g_i, h_j)(g_k, h_l) \mid g_i g_k \in E(G) \text{ ou } g_i = g_k \text{ e } h_j h_l \in E(H)\}$.
- **Conjuntos Dominantes Totais e Independentes Abertos**
  Um *conjunto dominante* de um grafo $G = (V, E)$, é um subconjunto D de $V(G)$ onde cada vértice que não pertence a D é adjacente a pelo menos um vértice de D. O conjunto D é *dominante total* se $\cup_{x \in D} N(x) = V(G)$, ou seja, se $|N(v) \cap D| \geqslant 1$, para todo $v \in V(G)$.
  Um *conjunto independente* de um grafo G é um conjunto S de vértices de G, tal que não existem dois vértices adjacentes contidos em S. O conjunto $S \subseteq V(G)$ é *independente aberto* se para cada $v \in S$, $|N(v) \cap S| \leqslant 1$.
  Denotaremos por $\gamma^{OP}_{OIND}(G)$ a cardinalidade mínima de um conjunto dominante total e independente aberto de um grafo G, quando existir.

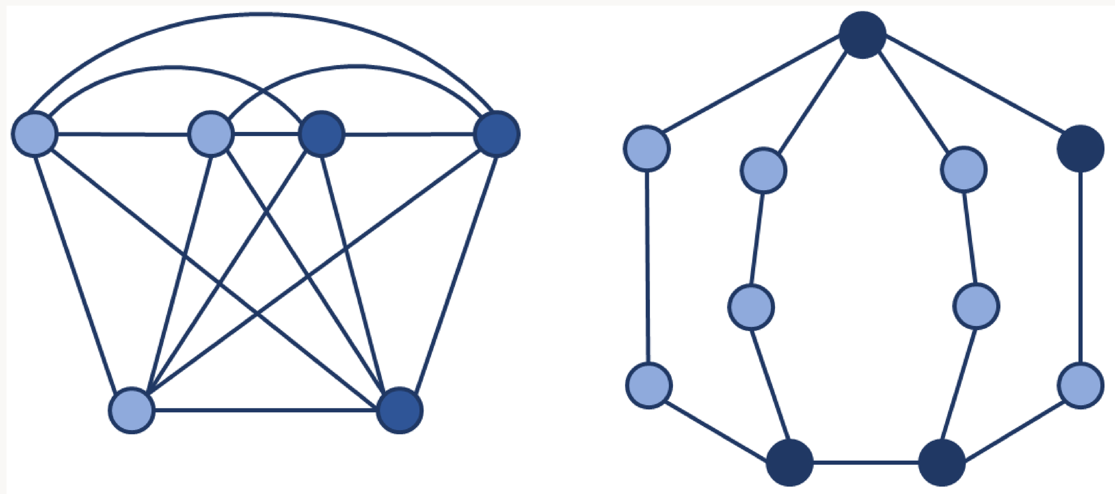


Figura 2: Produto lexicográfico $P_2 \circ C_3$ e um grafo G com $\gamma^{OP}_{OIND}(G) = 4$.

Trabalhos Relacionados

Seo e Slater [2] definem conjuntos dominantes totais e independentes abertos e consideram propriedades adicionais para esse parâmetro em classes específicas de grafos. Em [4], os autores apresentam resultados de conjuntos dominantes totais para produtos lexicográficos e produtos lexicográficos generalizados. Samodivkin, em [5], mostrou resultados de conjuntos dominantes emparelhados, que são conjuntos dominates S em que o subgrafo induzido por S contém um emparelhamento perfeito. Samodivkin provou que o problema de decisão associado a conjuntos dominantes emparelhados é NP-completo mesmo para grafos bipartidos.
Motivadas pelos trabalhos citados nós determinados o valor do número de dominação total e independência aberta para classes simples e para o produto lexicográfico de dois grafos. A Figura 2 exibe grafos e conjuntos independentes, dominantes e dominantes total e independente aberto.

Alguns resultados básicos

Proposição 1 *Seja K_n um grafo completo, para $n \geqslant 2$, então $\gamma^{OP}_{OIND}(K_n) = 2$.*

Proposição 2 *Seja P_n um grafo caminho, para $n \geqslant 2$, então $\gamma^{OP}_{OIND}(P_n) = 2 \cdot \lceil \frac{n}{4} \rceil$.*

Proposição 3 *Seja C_n um grafo ciclo, com $n \geqslant 2$ e $n \neq 5$, então $\gamma^{OP}_{OIND}(C_n) = 2 \cdot \lceil \frac{n}{4} \rceil$.*

Produtos Lexicográficos

Teorema 4 *Sejam G e H dois grafos quaisquer. Se G admite um conjunto dominante total e independente aberto, então $\gamma^{OP}_{OIND}(G \circ H) = \gamma^{OP}_{OIND}(G)$.*

Ideia da prova: Seja G um grafo qualquer com n vértices e com conjunto dominante total e independente aberto D. Considere $V(G) = \{g_1, g_2, \cdots, g_n\}$, $V(H) = \{h_1, h_2, \cdots, h_m\}$, $X_i = \{(g_i, h_j) \in V(G \circ H) : 1 \leqslant j \leqslant m\}$ e a componente $G_i = G \circ H[X_i]$ .. Seja $D' = \{(g_i, h_j) \in V(G_i) | g_i \in D \text{ e } j \in \{1, 2, \cdots, m\}\}$. É possível verificar que D' é um conjunto dominante total e independente aberto em $G \circ H$ e logo $\gamma^{OP}_{OIND}(G \circ H) \leqslant \gamma^{OP}_{OIND}(G)$.
Agora, seja D' um conjunto dominante total e independente aberto em $G \circ H$. Seja $D = \{g_i | (g_i, h_j) \in D'\}$. É possível verificar que D é um conjunto dominante total e independente aberto em G. O que implica em $\gamma^{OP}_{OIND}(G \circ H) \geqslant \gamma^{OP}_{OIND}(G)$.

Corolário 5 Seja K_n o grafo completo, C_n o grafo ciclo e P_n o grafo caminho com n vértices. Então,
- $\gamma^{OP}_{OIND}(K_n \circ H) = 2$, para $n \geqslant 2$.
- $\gamma^{OP}_{OIND}(C_n \circ H) = 2 \cdot \lceil \frac{n}{4} \rceil$, para $n \geqslant 2$ e $n \neq 5$.
- $\gamma^{OP}_{OIND}(P_n \circ H) = 2 \cdot \lceil \frac{n}{4} \rceil$, para $n \geqslant 2$.

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Referências

[1] S. U. Maheswari, M. S. Parvathi., *Independent Dominating Sets of Lexicographic Product Graphs of Cayley Graphs with Arithmetic Graphs, International Journal of Advanced in Management, Technology and Engineering Sciences*, 7(12)(2017), 160-166.

[2] S.J. Seo e P.J. Slater, *Open-independent, open-locating-dominating sets. Elec. Journal of Graph Theory and Applications*, 5 (2) (2017), 179-193.

[3] T. Sitthiwirattham , *DOMINATION ON LEXICOGRAPHICAL PRODUCT OF COMPLETE GRAPHS. International Journal of Pure and Applied Mathematics*, 85(4)(2013), 745-750.

[4] D.S. Studer, T.W. Haynes, and L.M. Lawson, *Induced-Paired Domination in Graphs, Ars Combinatoria* 57 (2000), 111–128.

[5] Samodivkin, Vladimir, *Domination related parameters in the generalized lexicographic product of graphs*, (2020).

Mutually Included Biclique Graphs of Interval Containment Bigraphs and Interval Bigraphs

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Abstract

The recognition of biclique graphs in general is still open. Recently, Groshaus and Guedes introduced the mutually included biclique graph as an intermediate operator to define the biclique graph. Also, we previously studied the biclique graph of interval bigraphs and proper interval bigraphs. In this work, we extend the results to a superclass, the interval containment bigraphs, in the context of the mutually included biclique graphs.

Introduction

Given a graph G, its ***biclique graph*** ($KB(G)$) is the intersection graph of the bicliques of G. It was introduced by Groshaus and Szwarcfiter in 2010 [5]. They presented a characterization of biclique graphs and a characterization of biclique graphs of bipartite graphs, but the time complexity of the problem of recognizing biclique graphs remains open.

Bicliques in graphs have applications in various fields, for example, biology: protein-protein interaction networks, social networks: web community discovery, genetics, medicine, information theory. More applications (including some of these) can be found in the work of Liu, Sim, and Li [9].

In 2018 Groshaus and Guedes introduced the ***mutually included biclique graph*** ($KB_m(G)$) [3, 4] as a spanning subgraph of $KB(G)$. They proved that $KB(G) = (KB_m(G))^2$ for any K_3-free graph G, and that $KB_m(\text{bipartite}) \subset$ comparability graphs.

In this work, we present some results about biclique graphs and mutually included biclique graphs of interval bigraph, interval containment bigraph, and bipartite graphs in general.

Classes Studied

- CGI: Containment graph of intervals [2]
  A graph G is a ***containment graph of intervals*** if its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if one of the corresponding intervals contains the other. Call that family of intervals a ***interval containment model*** of G.
- ICB: Interval containment bigraphs [8]
  A bipartite graph G is an ***interval containment bigraph*** if its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if they are of different parts and one of the corresponding intervals contains the other. Call that family of intervals a ***bipartite interval containment model*** of G.
- IBG: Interval bigraphs ($IBG \subseteq ICB$) [6]
  A bipartite graph G is an ***interval bigraph*** if its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if they are of different parts and the corresponding intervals intersect. Call that family of intervals a ***bipartite interval model*** of G.
- PG: Permutation Graphs = CGI = comparability $\cap$ co-comparability [1]

Definitions

- Two bicliques B_1, B_2 are ***vertex-intersecting*** if they intersect and $G[B_1 \cap B_2]$ has no edges.
- Two bicliques B_1, B_2 are ***edge-intersecting*** if $G[B_1 \cap B_2]$ has at least an edge.
- Two bicliques B_1, B_2 are ***mutually included*** if one part of B_1 is properly include in one part of B_2, and the other part of B_2 is properly included in the other part of B_2 [3, 4]. See Figure 1.
- The ***mutually included biclique graph*** of a graph G (denoted $KB_m(G)$) is the graph which each vertex corresponds to a biclique of G and two vertices are adjacent if the corresponding bicliques are mutually included [3, 4]. Note that the binary relation of being mutually include is not transitive.

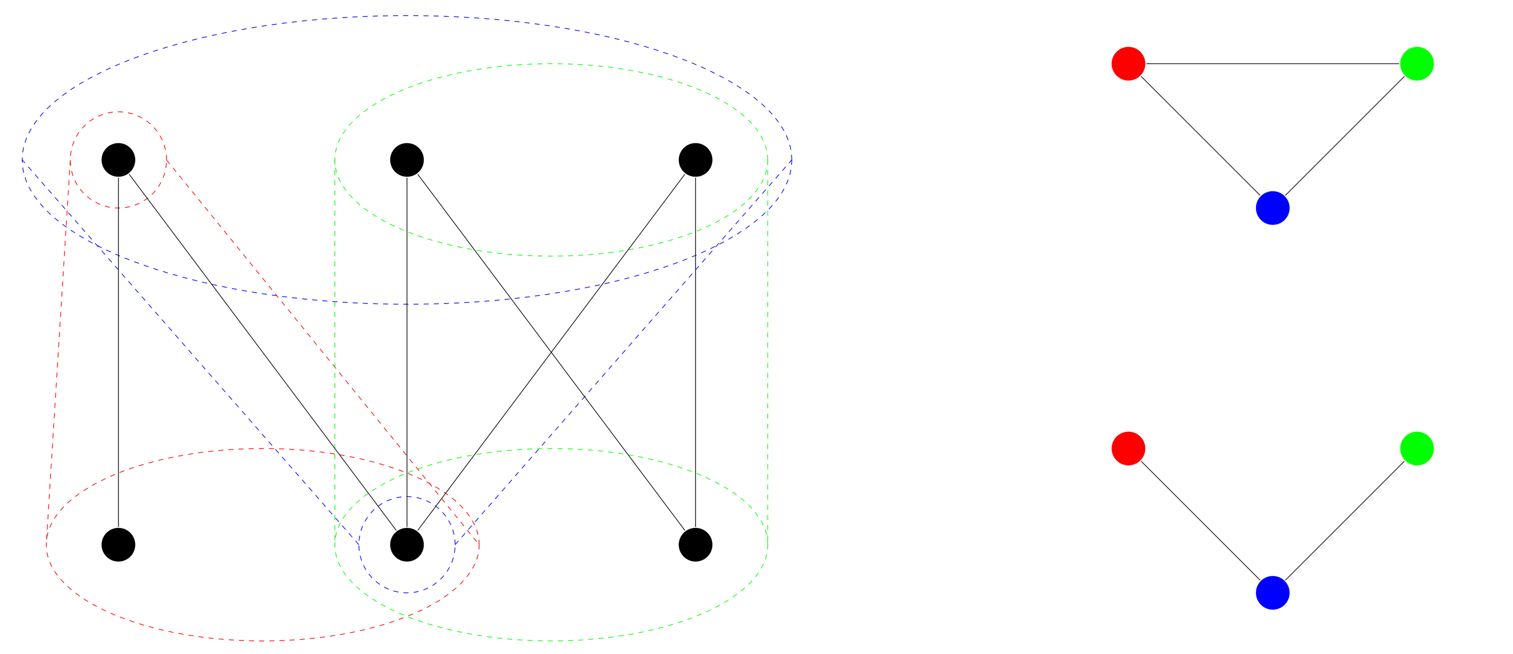

Figure 1: Graph G with 3 bicliques, red, blue and green. The blue biclique is mutually included with both red and green, but the red and green bicliques are not mutually included with each other. On the right, the graph above is $KB(G)$ and the one below is $KB_m(G)$.

- An ***asteroidal triple*** (AT) is an independent set with 3 vertices such that for every pair of vertices there is a path connecting them while avoiding the neighbors of the third vertex [7]. See Figure 2 (right) for an example of an AT.
- A ***bi-asteroidal triple*** ($biAT$) is an asteroidal triple such that the path between each pair of vertices is not adjacent to the neighborhood of the third vertex [6]. See Figure 2 (left) for an example of an $biAT$.

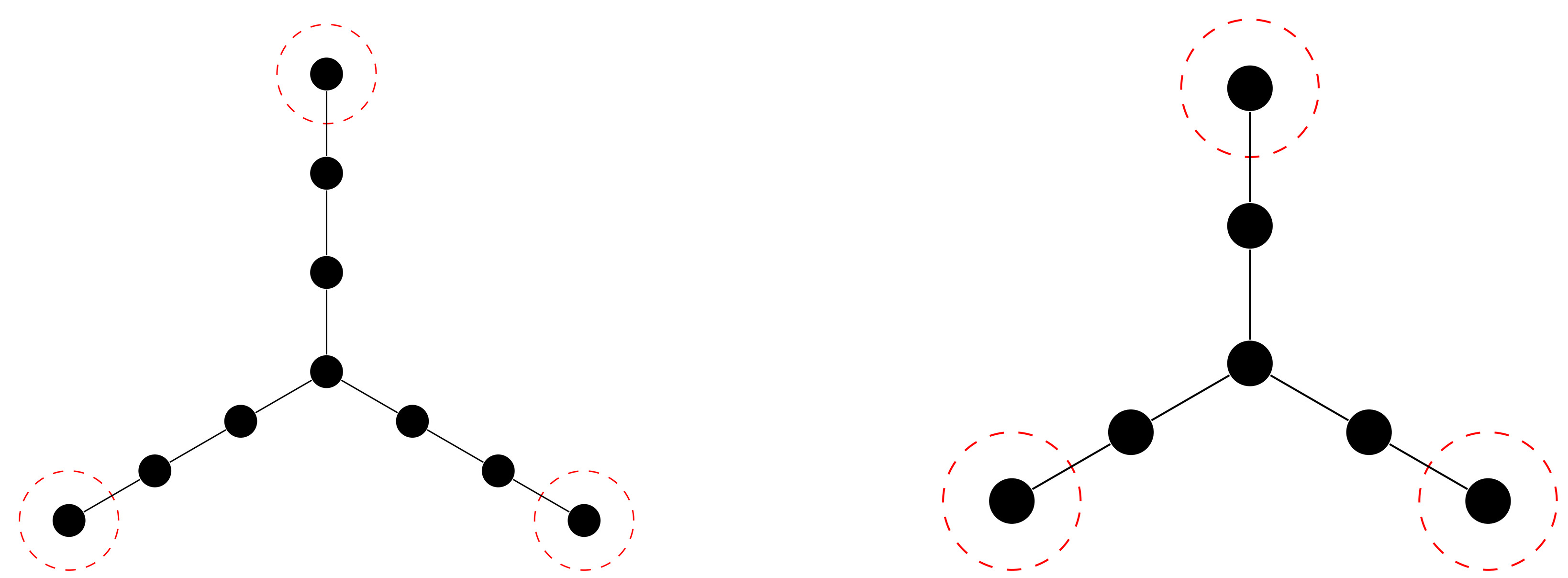

Figure 2: Example of a bi-asteroidal triple (left) and an asteroidal-triple (right). Note that the graph on the right is the same as the KB_m graph of the graph on the left.

Results

KB and KB_m of ICB and IBG

- $\mathbf{KB_m(ICB) \subset PG}$.
  Proof idea: Find a partial order $\leq_1$ such that, for $B_1 \neq B_2$, $\{B_1, B_2\} \in E(KB_m(G))$ if and only if $B_1 \leq_1 B_2$ or $B_2 \leq_1 B_1$, and a partial order $\leq_2$ such that $\{B_1, B_2\} \notin E(KB_m(G))$ if and only if $B_1 \leq_2 B_2$ or $B_2 \leq_2 B_1$. That is, prove that $KB_m(G)$ is a comparability and a co-comparability graph.
- $\mathbf{KB(ICB) \subseteq PG^2}$.
  Proof idea: Corollary of previous item and the fact that $KB(G) = (KB_m(G))^2$ [4].
- **For every** $\mathbf{H \in PG}$**, there is a** $\mathbf{G \in IBG}$ **such that** $\mathbf{H \subseteq KB_m(G)}$.
  Proof idea: Construct an interval bigraph G (constructing a bipartite interval model) from an interval containment model C of H (as H is also an CGI) such that $H \subseteq KB_m(G)$.

Bipartite Graphs

Let G be a bipartite graph, then:

- **If** $\mathbf{KB_m(G)}$ **is** AT**-free then G is** $biAT$**-free.**
  Consequently, for some AT-free graph class $\mathcal{A}$, $KB_m^{-1}(\mathcal{A}) \cap$ bipartite is $biAT$-free.
  Proof idea: By construction, proving that if $\mathbf{G}$ has a bi-asteroidal triple then $\mathbf{KB_m(G)}$ has an asteroidal triple.
- $\mathbf{G}$ **is** $\mathbf{P_4}$**-free if and only if** $\mathbf{E(KB_m(G)) = \emptyset}$.
  Proof idea: By inspection of the possibilities of mutually included bicliques. If there is a pair of mutually included bicliques then there is a P_4 in G. See Figure 3 (left).
- **If G is** $\mathbf{P_5}$**-free then there is no pair of vertex-intersecting bicliques.**
  Proof idea: By inspection of the possibilities of vertex-intersecting bicliques. If there is a pair of vertex-intersecting bicliques then there is a P_5 in G. See Figure 3 (middle).
- **G is domino-free if and only if every pair of edge-intersecting bicliques are mutually included.**
  Proof idea: By inspection of the possibilities of edge-intersecting bicliques that are not mutually included. If there is a pair of not mutually included edge-intersecting bicliques then there is a domino in G. See Figure 3 (right).

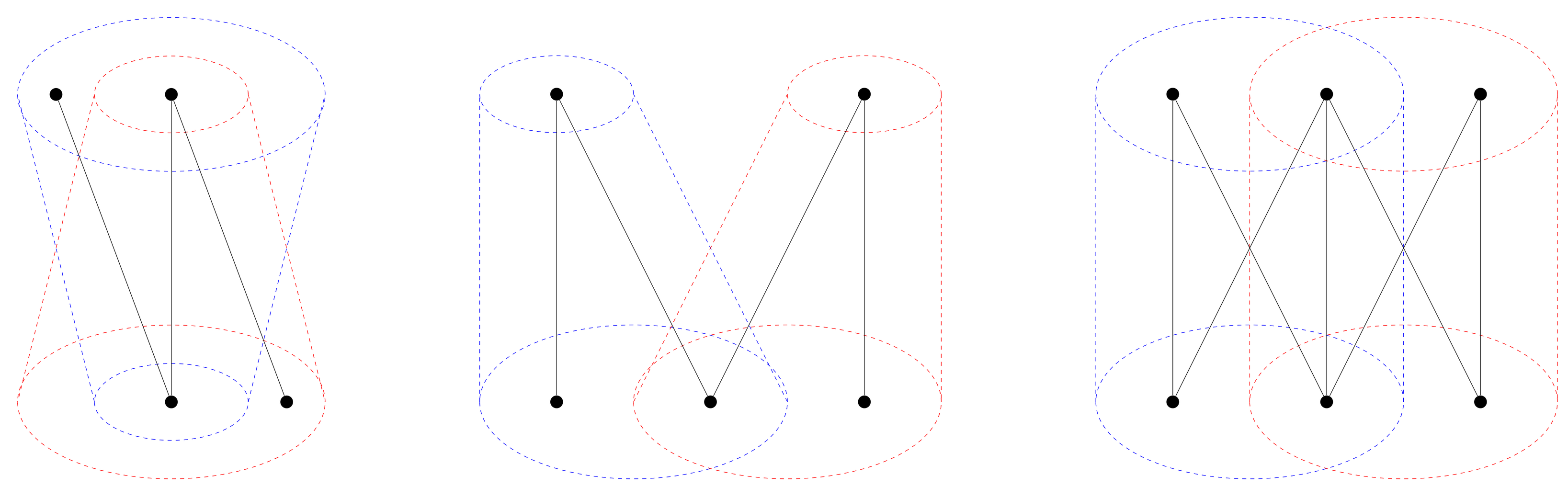

Figure 3: Edge types of bicliques that are: mutually included and a P_4 (left), vertex-intersecting and a P_5 (middle), and edge-intersecting not mutually included and a domino (right).

References

[1] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: a Survey*. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[2] Martin Charles. Golumbic. *Algorithmic graph theory and perfect graphs / Martin Charles Golumbic*. Academic Press New York, 1980.

[3] Marina Groshaus and André L. P. Guedes. The biclique graph of K_3-free graphs are the square of some graph. In *Proc. VIII Latin American Workshop on Cliques in Graphs*, page 25, Rio de Janeiro, 2018.

[4] Marina Groshaus and André Luiz Pires Guedes. Biclique graphs of k_3-free graphs and bipartite graphs. *CoRR*, abs/2006.00040, 2020.

[5] Marina Groshaus and Jayme L. Szwarcfiter. Biclique graphs and biclique matrices. *Journal of Graph Theory*, 63(1):1–16, 2010.

[6] Frank Harary, Jerald A. Kabell, and Frederick R. McMorris. Bipartite intersection graphs. *Commentationes Mathematicae Universitatis Carolinae*, 023(4):739–745, 1982.

[7] C. Lekkeikerker and J. Boland. Representation of a finite graph by a set of intervals on the real line. *Fundamenta Mathematicae*, 51(1):45–64, 1962.

[8] Min Chih Lin and Jayme L. Szwarcfiter. Characterizations and recognition of circular-arc graphs and subclasses: A survey. *Discrete Mathematics*, 309(18):5618–5635, 2009. Combinatorics 2006, A Meeting in Celebration of Pavol Hell's 60th Birthday (May 1–5, 2006).

[9] Guimei Liu, Kelvin Sim, and Jinyan Li. Efficient Mining of Large Maximal Bicliques. In *Proceedings of the 8th International Conference on Data Warehousing and Knowledge Discovery*, DaWaK'06, pages 437–448, Berlin, Heidelberg, 2006. Springer-Verlag.

Swapping Tokens on Graphs

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Introduction to the Token Swap (TS) Problem

Let $G = (V, E)$ be a graph with $|V|$ vertices and $|E|$ edges, with distinct tokens placed on it's vertices. The objective is to reconfigure this initial token placement called $f_0 : V \mapsto V$ into the identity token placement f_i, that maps every node to itself, through a sequence of pairs of adjacent graph vertices that swap the tokens between these vertices. The aim is to know if it is possible to have a swap sequence S that achieve the objective in k or less swaps, with $k \in \mathbb{N}$.
Applications of the TS problem encompass a wide range of fields. From computing efficient interconnection network structures, [1], computational biology [2, 3], modelling Wireless Sensor Networks (WSS) [4], protection routing [5] to qubit allocation for quantum computers [6, 7].

Token Swap on Specific Graph Classes

A Conflict Graph $CG_f := (V(G), E_{CG})$ is a digraph that, for a token placement f of a graph G, an edge $(u, v) \in E_{CG}$ if and only if $f(u) = v$. Each node has outdegre 1 and the digraph may contain self-loops.

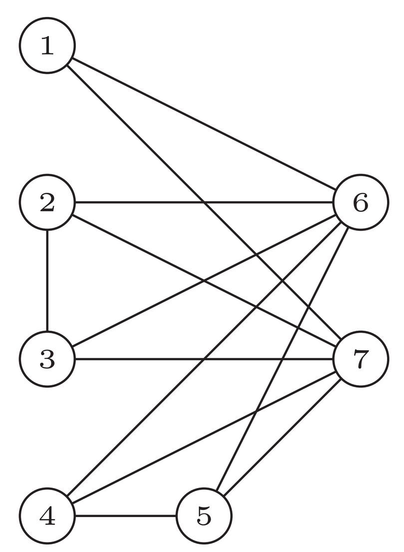

Figure 1: Example of a cograph.

A cograph is defined recursively as follows: a graph on a single vertice is a cograph; if $G_1, G_2, \ldots, G_k$ are cographs, then so is their disjoint union; if G is a cograph, then so is its complement $\overline{G}$. A cotree $T(G)$ of a cograph $G = (V, E)$ is a rooted tree representing it's structure. The leaves of $T(G)$ are exactly V and each internal node is either a 0-node and 1-node. The children of an 1-node are 0-nodes or leaves and the children of a 0-node are 1-nodes or leaves. Two vertices are adjacent in a cograph if and only if their lowest common ancestor is an 1-node.
We define $CS(CG_f) = \{C_1, C_2, \ldots, C_k\}$ as the set of permutation cycles of CG for f. Let $C^1 \subseteq CS$ be the set of cycles that have a lowest common ancestor of all vertice pairs of $V(C)$ as an 1-node in the cotree or is a cycle of size one and let $C^0 = CS \backslash C^1$. The *Cycle Matching Graph* H of a cograph G has each cycle on C^0 as vertice set and two vertices are adjacent if the lowest common ancestor of all vertice pairs in the vertice union in $T(G)$ is an 1-node. Let $\mu(H)$ be the maximum matching in this graph. The following theorem implies the polynomial time solvability of `Token Swap` for cographs.

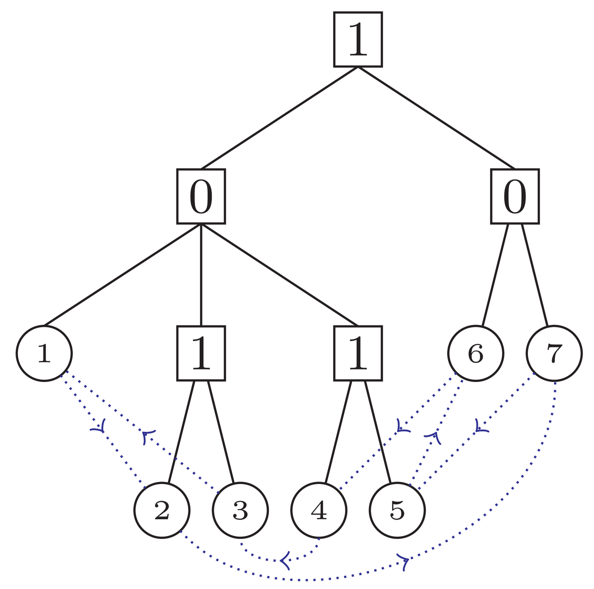

Figure 2: Cotree and conflict graph joint representation.

Theorem. *Let G be a cograph with an initial token placement f_0. The minimum number of required swaps is given by $|V(G)| + |C^0| - |C^1| - 2 \times |\mu(H)|$.*

This result came from two observations: Each independent cycle $C \in CS$ can be solved in $|C| + 1$ or $|C| - 1$ swaps depending on whether this cycle is part of C^0 or C^1, respectively. Also, it is possible to show that cycle interaction is restricted in the best-case scenario and the best improvement on swaps can be calculated on the value of the maximum matching of the cycle matching graph H. This behavior is also being used to find more efficient algorithms in other graph classes like bipartite chain, wheel and gear.

Token Swap as a Integer Linear Program

The novel TS problem model given by Formulation (1)-(10) tests all possible configurations of the problem with a given upper-bound in the number of swaps T, allowing at maximum one swap per step $t \in [T]$. Each step is composed of a set of variables that describe the current configuration, which swap is being selected and Equation 8 checks if a swap sequence solves the current instance. The constant T can be calculated by using any of the best approximation algorithms, or by using the trivial upper-bound $O(n^2)$ on the size of an optimal swap sequence. Binary variables x_{iut} determine if a token i is at node u in step t. The binary variables y_{uvt} flags if a swap happened between nodes u and v in step t.

$$
\begin{aligned}
\min \quad & \sum\nolimits_{\forall_{uv \in E, u <_M v}, \forall_{t \in [T]}} y_{uvt} && && (1)\\
\text{s.t.} \quad & \sum\nolimits_{\forall_{u \in V}} x_{iut} = 1 && \forall_{t \in [T]}, \forall_{i \in V} && (2)\\
& \sum\nolimits_{\forall_{i \in V}} x_{iut} = 1 && \forall_{t \in [T]}, \forall_{u \in V} && (3)\\
& x_{iut} + x_{ivt+1} \leq y_{uvt} + 1 && \forall_{i \in V}, \forall_{t \in [T-1]}, \forall_{uv \in E}, u <_M v && (4)\\
& x_{ivt} + x_{iut+1} \leq y_{uvt} + 1 && \forall_{i \in V}, \forall_{t \in [T-1]}, \forall_{uv \in E}, u <_M v && (5)\\
& x_{iut} + x_{ivt+1} \leq 1 && \forall_{i \in V}, \forall_{t \in [T-1]}, \forall_{uv \notin E} && (6)\\
& \sum\nolimits_{\forall_{u,v \in V}, u <_M v} y_{uvt} \leq 1 && \forall_{t \in [T]} && (7)\\
& x_{iiT} = 1 && \forall_{i \in V} && (8)\\
& y_{uvt} \in \{0, 1\} && \forall_{t \in [T]}, \forall_{uv \in E} && (9)\\
& x_{iut} \in \{0, 1\} && \forall_{i \in V}, \forall_{u \in V}, \forall_{t \in [T]} && (10)
\end{aligned}
$$

Some techniques are being used in this model to try to achieve a better overall performance, and they will be explained in detail in future papers. The performance measurement, improvements and other models for the problems of `Colored Token Swap` and `Parallel Colored Token Swap` are all planned for future research. Note that the problem of `Parallel Token Swap` currently has a model, but it was omitted here for the sake of conciseness. These models differentiate from the usual TS problem by allowing swaps to be done in parallel or by removing the uniqueness property of a token, assigning a color for a set of tokens instead of a single label.

References

[1] Fred Annexstein, Marc Baumslag, and Arnold L. Rosenberg. Group action graphs and parallel architectures. *SIAM Journal on Computing*, 19:544–569, 1990.

[2] Vineet Bafna and Pavel A. Pevzner. Sorting by transpositions. *SIAM J. Discret. Math.*, 11:224–240, 1998.

[3] Lenwood Heath and John Vergara. Sorting by short swaps. *Journal of computational biology : a journal of computational molecular cell biology*, 10:775–89, 2003.

[4] Lei Wang and K. W. Tang. The cayley graph implementation in tinyos for dense wireless sensor networks. In *2007 Wireless Telecommunications Symposium*, pages 1–7, Italy, 2007. 2007 Thyrrenian International Workshop on Digital Communication.

[5] Kung-Jui Pai, Ruay-Shiung Chang, and Jou-Ming Chang. Constructing dual-cists of pancake graphs and performance assessment of protection routings on some cayley networks. *The Journal of Supercomputing*, 76:124546, 2020.

[6] Marcos Yukio Siraichi, Vinícius Fernandes dos Santos, Sylvain Collange, and Fernando Magno Quintao Pereira. Qubit allocation. In *Proceedings of the 2018 International Symposium on Code Generation and Optimization*, pages 113–125, New York, NY, USA, 2018. ACM.

[7] Marcos Yukio Siraichi, Vinícius Fernandes dos Santos, Caroline Collange, and Fernando Magno Quintão Pereira. Qubit allocation as a combination of subgraph isomorphism and token swapping. *Proc. ACM Program. Lang.*, 3:1–29, 2019.

On the edge-P_3 and edge-P_3*-convexity of grid graphs

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Introduction

Consider a graph $G = (V, E)$ and a subset C of V. The *P_3-convex hull* (resp. *P_3*-convex hull*) of C is obtained by iteratively adding vertices with at least two neighbors in C (resp. two non-adjacent neighbors in C). A subset S of V is *P_3-Helly-independent* (resp. *P_3*-Helly-independent*) when the intersection of the P_3-convex hulls (resp. P_3*-convex hulls) of $S \setminus \{v\}$ ($\forall\ v \in S$) is empty. The *P_3-Helly number* (resp. *P_3*-Helly number*) is the size of a maximum P_3-Helly-independent (resp. P_3*-Helly-independent).

The *line graph* L(G) of a graph G is the intersection graph of the edges of G, i.e.,$V(L(G)) = E(G)$ and there is an edge between two vertices in $L(G)$ if the edges they represent in G share a common endpoint. The edge counterparts of P_3-Helly-independent and P_3*-Helly-independent of a graph follow the same restrictions applied to its edges instead of its vertices, i.e., the edge P_3-convexity (resp. edge P_3*-convexity) of a graph G is described by P_3-convexity (resp. P_3*-convexity) of its line graph $L(G)$.

Each vertex of a *grid graph Gpxq* are related to a pair (x, y) that defines its position on the grid with $1 \leqslant x \leqslant p$, $1 \leqslant y \leqslant q$. There is an edge between two vertices of a grid graph if they share a same coordinate x (or y) and the other coordinate differ only by one unit.

In this work, we established the edge P_3*-Helly number of grid graphs *Gpxq* when both p and q are equal or larger than 16. Moreover, we give partial results on forbidden configurations of the edge P_3-Helly independent sets of these grid graphs.

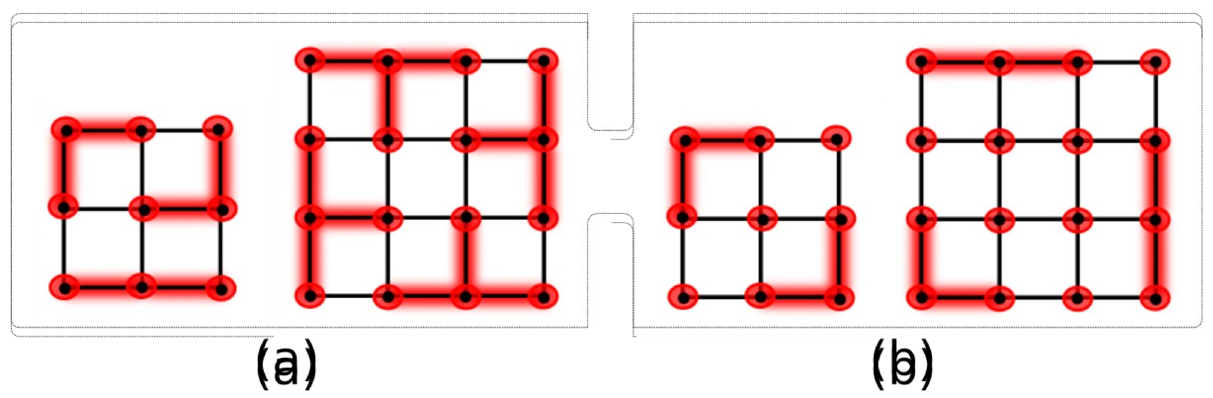

Figure 1: The red edges are an (a) edge P_3*-Helly independent (resp. (b) edge P_3-Helly-independent) set of a G3x3 and a G4x4 grid graphs

Motivations and Related Works

There are many applications for graph convexity on distributed systems[7], social networks, and marketing strategies[6]. Moreover, the excelent survey [5] describes several results of the Helly property on graphs. The problem we address considers the Helly property on the edge P_3-convexity and edge P_3*-convexity of grid graphs.

The first results about the P_3-Helly number on grid graphs appeared in [1]. Later, several results about P_3-Helly number, P_3*-Helly number and their edge counterparts were established in [2] and [3].

Results

In [3] the authors established that the edge P_3*-Helly number of a graph occurs between $|V(G)|- \iota\ (G)$ and $|V(G)|-\gamma\ (G)$, where $\iota\ (G)$ is the minimum independent dominating set and $\gamma\ (G)$ is the minimum dominating set of a graph G.

For grid graphs *Gpxq* with $p \geq 16$ and $q \geq 16$, we have that $\iota\ (G) = \gamma\ (G)$. Moreover, the value of $\gamma\ (G)$ is given by $\lfloor (p+2)(q+2)/5 \rfloor -4$ [4]. Therefore, we know the P_3*-Helly number of grid graphs *Gpxq* with $p \geq 16$ and $q \geq 16$. The other values for the cases when $p < 16$ and $q < 16$ are computationally obtained. Now, we aim to use these values to obtain the edge P_3*-Helly numbers when $p < 16$ and $q \geq 16$ or $p \geq 16$ and $q < 16$.

We also consider the edge P_3-Helly independent of grid graphs by establishing several forbidden configurations that allow us to reduce the patterns of possible optimal solutions. Note that in both cases (a) and (b), the edge P_3 convex hull of the red edges without the horizontal edge contains this edge.

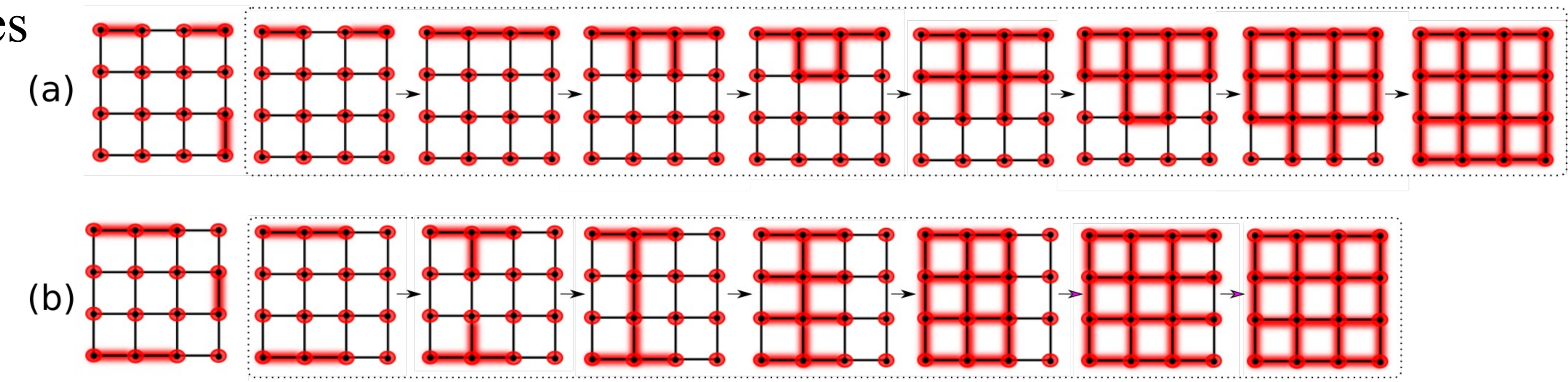

Figure 2: Two forbidden configurations for edge P_3-Helly independent sets.

Ongoing works

We are currently trying to establish the edge P_3-Helly number of grid graphs. Moreover, we want to extend this study to include the other two types of regular grids: the triangular grids and the hexagonal grids.

References

[1] M. CARVALHO. O Número de Helly na Convexidade Geodética em Grafos, Tese de Doutorado, PESC/COPPE-UFRJ, 2016.

[2] M. CARVALHO; S. DANTAS; M. DOURADO; J. SZWARCFITER; D. POSNER. On the computational complexity of the Helly number in the P_3 and related convexities for (q, q-4) graphs, Matemática Contemporânea Vol. 46, 138–146, 2019

[3] M. CARVALHO; S. DANTAS; M. DOURADO; J. SZWARCFITER; D. POSNER. On the computational complexity of the Helly number in the {P_3} and related convexities, Electronic Notes in Theoretical Computer Science, Vol. 346, 285-297, 2019

[4] S. CREVALS,; P. OSTERGARD. Independent domination of grids, Discrete Mathematics, Vol. 338, 1379-1384, 2015

[5] L M. C. DOURADO; F. PROTTI; J.L. SZWARCFITER. Complexity Aspects of the Helly Property: Graphs and Hypergraphs, The Eletronic Journal of Combonatorics DS17, 2009 1998.

[6] D. KEMP; J. KLEINBERG; E. Tardos. Maximizing the spread of influence through a social network, Proceedings of t SIGKDD, 2003

[7] D. PELEG. Local majorities, coalitions and monopolies in graphs, Theoretical Computer Science 282, 213--257, 2002

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Recognition of Biclique Graphs: What we know so far

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Abstract

The recognition of biclique graphs in general is still open. In recent years we presented some result on the characterization of biclique graphs of graphs of certain graph classes, along with the complexity associated to the recognition problem. Those results introduced some intermediate operators, which we call now as "functors". In this work we summarize all those results and organize the different approaches using the functors.

Introduction

The *biclique graph* of a graph G, denoted by $KB(G)$, is the intersection graph of the bicliques of G. The biclique graph was introduced by Groshaus and Szwarcfiter [8], based on the concept of clique graphs. They gave a characterization of biclique graphs (in general) and a characterization of biclique graphs of bipartite graphs. The time complexity of the problem of recognizing biclique graphs remains open.

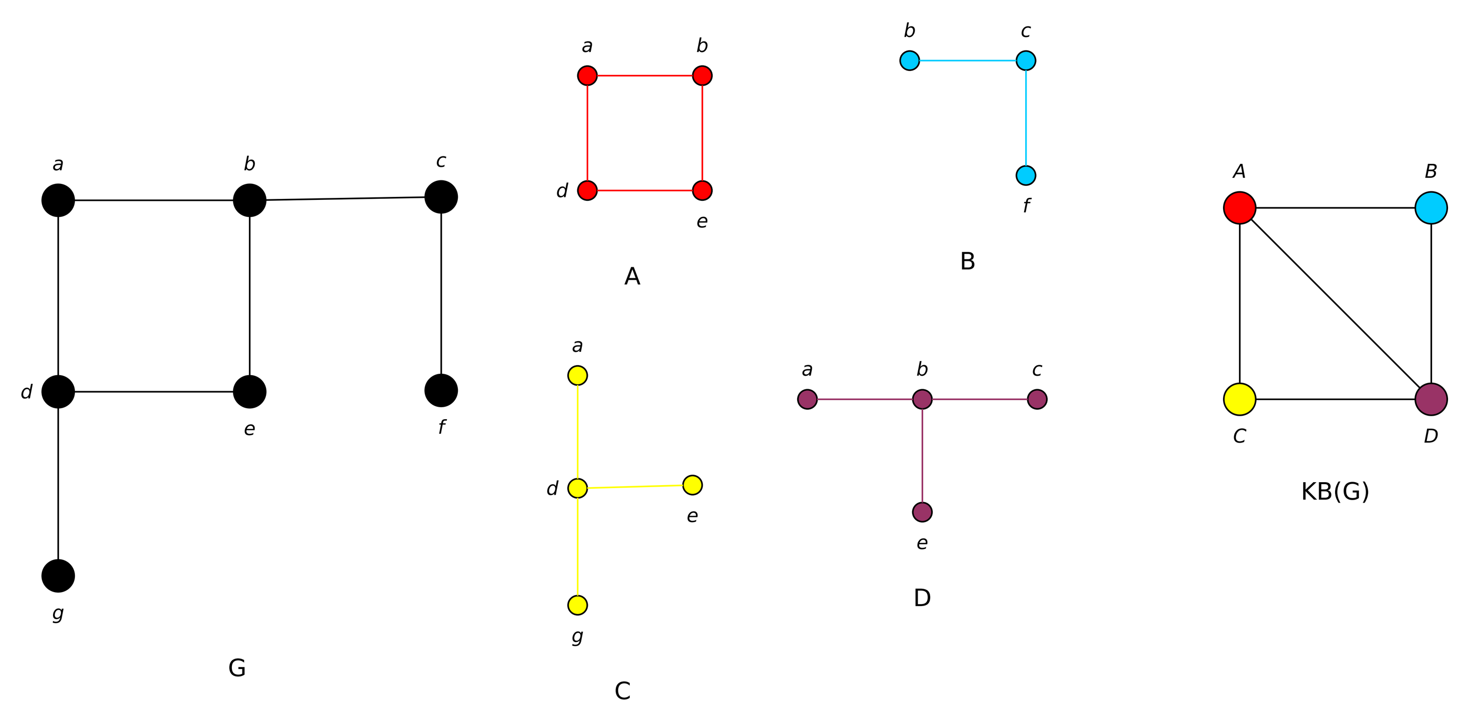

Figure 1: Example of a biclique graph.

Bicliques in graphs have applications in various fields, for example, biology: protein-protein interaction networks, social networks: web community discovery, genetics, medicine, information theory. More applications (including some of these) can be found in the work of Liu, Sim, and Li [10].

The efforts since the definition of the problem of recognizing biclique graphs, similarly to what have been done for others graph operators, are mainly focused on understanding the class $KB(\mathcal{A})$, for some graph class $\mathcal{A}$.

In this work we summarize what is known about recognition of $KB(\mathcal{A})$ for a collection of graph classes.

Classes Studied

- $\mathcal{G}$: All graphs
- $\mathcal{G}_k$: Graphs with girth at least k
- P_n: Path with n vertices
- C_n: Cycle with n vertices
- K_n: Complete graph of order n
- co-CG: Co-comparability graphs
- IIC-comparability: Interval intersection closed comparability graphs [6, 7]
- IIC-PG: IIC-Permutation Graphs = IIC-comparability $\cap$ co-CG [6, 7]
- IBG: Interval bigraphs
- HIB: Helly interval bigraphs [4]
- PIB: Proper interval bigraphs
- PIB-ASG: Proper interval bigraphs having acyclic simplification graph [1]
- PIG: Proper interval graphs
- 1-PIG: 1-Proper interval graphs [1]
- $BBHGD$: Bipartite biclique-Helly graphs with no dominated vertices [9]
- $CHBDI$: Clique independent Helly-bicovered with no dominated vertices graphs [9]
- $NSSG$: Nested separable split graphs [3, 5]

Operators and Functions

- G^2: Square of graph G
- $K(G)$: Clique graph of graph G
- $KB_m(G)$: Mutually included biclique graph of graph G [6, 7]
- $L(G)$: Line graph of graph G
- $leaves(G)$: Set of leaves (vertices of degree 1) of graph G
- $S(G)$: Simplification graph of graph G [1]

Functors

The idea behind the techniques used in most of the results on $KB(\mathcal{A})$ is to characterize $KB(\mathcal{A})$ using some other operator (or a composition of operators). That is, $KB(G) = \mathcal{F}(G)$, for $G \in \mathcal{A}$ and some operator $\mathcal{F}$.

We say that such scheme with more than one way to compute an operator is a "functor".

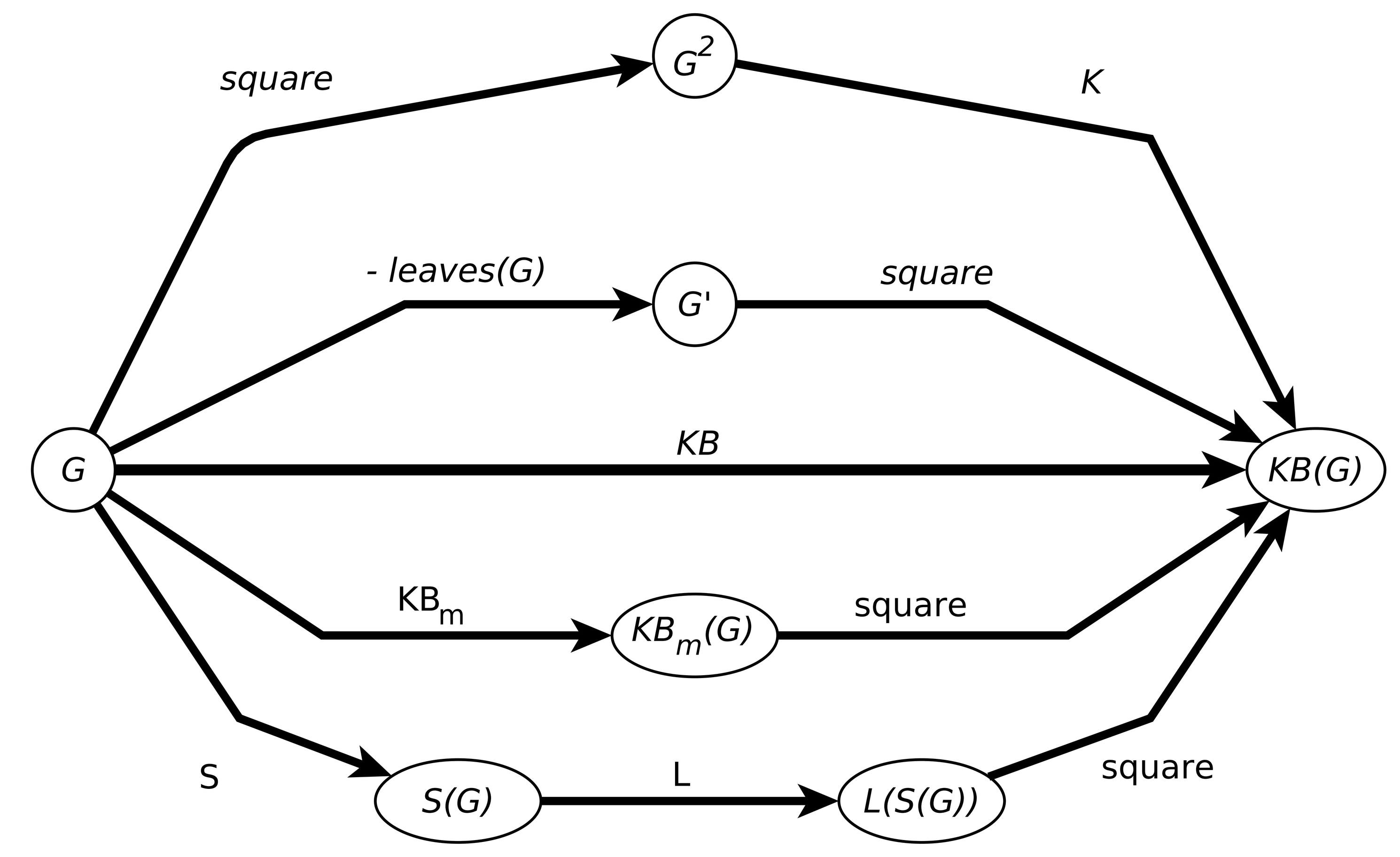

Figure 2: Summary of known functors for KB.

- **$\{K_3, C_5, C_6\}$-free**: $KB(G) = K(G^2)$ [9]
  $\implies KB(\{K_3, C_5, C_6\}\text{-free}) = K((\{K_3, C_5, C_6\}\text{-free})^2)$
  Proof idea: Open neighborhood Helly, so there is a bijection between bicliques of G and cliques of G^2.
- **girth at least k ≥ 5**: $KB(G) = (G - leaves(G))^2$ [1]
  $\implies KB(\mathcal{G}_k) = (\mathcal{G}_k)^2$, with $k \geq 5$
  Proof idea: Every bicliques is a maximal star, every vertex that is not a leaf is the center of a maximal star (biclique) and to remove the leaves does not affect the girth.
- **K_3-free**: $KB(G) = (KB_m(G))^2$ [6, 7]
  $\implies KB(K_3\text{-free}) = (KB_m(K_3\text{-free}))^2$
  Proof idea: Note that $KB_m(G) \subseteq KB(G)$ (same vertex set) and every pair of intersecting bicliques in G are at distance at most 2 in $KB_m(G)$.
- **PIB**: $KB(G) = (L(S(G)))^2$ [1]
  $\implies KB(PIB) = (L(PIB))^2$
  Proof idea: $S(PIB) = PIB$, the edges of $S(G)$ are bicliques of G, and every pair of intersecting bicliques in G are at distance at most 2 in $L(S(G))$.

Table 1 summarize the results about recognition of biclique graphs of some graph classes.

class $\mathcal{A}$	$KB(G)$, $G \in \mathcal{A}$	**class $KB(\mathcal{A})$**	**complexity**
complete [1]	$L(G)$	L(complete)	$\mathcal{P}$
tree [1]	$(G - leaves(G))^2$	(tree)^2	$\mathcal{P}$ (linear)
path (P_n) [1]	$\emptyset$, for $n = 1$ K_1, for $n = 2$ $(P_{n-2})^2$, for $n > 2$	(path)^2	$\mathcal{P}$ (linear)
caterpillar (tree)	$(G - leaves(G))^2$	(path)^2	$\mathcal{P}$ (linear)
cycle (C_n) [1]	K_1, for $n = 4$ $(C_n)^2$, for $n \neq 4$	(cycle)$^2 - K_4 + K_1$	$\mathcal{P}$
$\mathcal{G}_k$, for $k \geq 5$ [1] (*) [2]	$(G - leaves(G))^2$	$(\mathcal{G}_k)^2$, for $k \geq 5$	$\mathcal{P}$, for $k \geq 6$ $\mathcal{NP}$-**complete**, for $k = 5$
IBG [1, 6, 7]	**OPEN**	$\subset$ (IIC-PG)^2 $\subset K_{1,4}$-free co-CG	**OPEN**
PIB [1]	$(L(S(G))^2$	$(L(PIB))^2$	**OPEN**
PIB-ASG [1]	$(L(S(G))^2$	1-PIG	$\mathcal{P}$
HIB [9, 4]	$K(G^2)$	$\subset PIG \cap (L(PIB))^2$	**OPEN**
$\{K_3, C_5, C_6\}$-free [9]	$K(G^2)$	**OPEN**	**OPEN**
$BBHGD$ [9]	**OPEN**	$CHBDI$	**OPEN**
$NSSG$ [5]	**OPEN**	**OPEN**	$\mathcal{P}$
threshold [5]	**OPEN**	**OPEN**	$\mathcal{P}$
K_3-free [6, 7]	$(KB_m(G))^2$	$\subset \mathcal{G}^2$	**OPEN**
bipartite [6, 7]	$(KB_m(G))^2$	(IIC-comparability)^2	**OPEN**
$\mathcal{G}$ [9]	**OPEN**	Characterization	**OPEN**

Table 1: At column "$KB(G), G \in \mathcal{A}$" a brief description of $KB(G)$; at column "class $KB(\mathcal{A})$", class that is equal to (or a super-class of) $KB(\mathcal{A})$; at column "complexity", complexity (if known) of recognizing $KB(\mathcal{A})$.
(*) Note that to decide if G is the square of a graph of girth ≥ 5 is $\mathcal{NP}$-complete [2].

References

[1] E. P. Cruz, M. Groshaus, A. L. P. Guedes, and J. P. Puppo. Biclique graphs of interval bigraphs. *Discrete Applied Mathematics*, 281:134–143, 2020.

[2] Babak Farzad and Majid Karimi. Square-root finding problem in graphs, A complete dichotomy theorem. *CoRR*, abs/1210.7684, 2012.

[3] M. Groshaus, A. Guedes, and J. P. Puppo. The biclique graph of a subclass of split graphs. In *Latin American Workshop on cliques in graphs, La Plata, Argentina*, page 22, 2016.

[4] M. Groshaus, A. L. P. Guedes, and F. S. Kolberg. On the helly subclasses of interval bigraphs and circular arc bigraphs. In *Proceedings of the 14th Latin American Theoretical Informatics Symposium*, São Paulo, 2020.

[5] M. Groshaus, A. L. P. Guedes, and J. P. Puppo. Biclique graphs of split graphs, 2020. Submitted.

[6] Marina Groshaus and André L. P. Guedes. The biclique graph of K_3-free graphs are the square of some graph. In *Proc. VIII Latin American Workshop on Cliques in Graphs*, page 25, Rio de Janeiro, 2018.

[7] Marina Groshaus and André Luiz Pires Guedes. Biclique graphs of k_3-free graphs and bipartite graphs. *CoRR*, abs/2006.00040, 2020.

[8] Marina Groshaus and Jayme L. Szwarcfiter. Biclique graphs and biclique matrices. *Journal of Graph Theory*, 63(1):1–16, 2010.

[9] Marina E. Groshaus. *Bicliques, cliques, neighborhoods y la propiedad de Helly*. PhD thesis, Universidad de Buenos Aires, 2006.

[10] Guimei Liu, Kelvin Sim, and Jinyan Li. Efficient Mining of Large Maximal Bicliques. In *Proceedings of the 8th International Conference on Data Warehousing and Knowledge Discovery*, DaWaK'06, pages 437–448, Berlin, Heidelberg, 2006. Springer-Verlag.

Subclass Hierarchy on Circular Arc Bigraphs

A study of graph class containments

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Abstract

A bipartite graph is a circular arc bigraph if there exists a bijection between its vertices and a family of arcs on a circle such that vertices of opposing partite sets are neighbors precisely if their corresponding arcs intersect. The class is a relatively unexplored subject, with most results on it and its subclasses being quite recent. In our work, we provide a full exploration of the containment relations and intersections between seven subclasses of circular arc bigraphs.

Introduction

The class of ***Circular arc bigraphs*** is a bipartite variation of the class of circular arc graphs. A bipartite graph $G = (V, W, E)$ is a circular arc bigraph if there exists a bijection $b : V \cup W \to \mathbb{A}$ such that $\mathbb{A}$ is a family of arcs on a circle and two vertices $v \in V, w \in W$ are neighbors if and only if $b(v) \cap b(w) \neq \emptyset$. Unlike its non-bipartite counterpart, circular arc bigraphs are a mostly unexplored topic of research, with most results on it being relatively recent.

In 2013, Basu et al. [1] published matrix-based characterizations for the class of circular arc bigraphs, as well as for the subclasses of ***proper circular arc bigraphs*** and ***unit circular arc bigraphs***. In 2015, Das and Chakraborty [2] presented a vertex order based characterization of proper circular arc bigraphs and ***proper interval bigraphs***. More recently, in 2019, Safe [4] presented a forbidden structure characterization and efficient recognition algorithm for proper circular arc bigraphs.

In our work, we study seven different subclasses of circular arc bigraphs, and provide comprehensive results on their pairwise comparability and containment relations.

Classes studied

The classes studied in our work are the following, defined in detail in the Definitions section.

- ***Circular Convex Bipartite (CCB) graphs***, including its subclass of ***Doubly Circular Convex Bipartite (doubly-CCB) graphs***.
- ***Helly circular arc bigraphs***, including its subclasses of ***non-bichordal Helly circular arc bigraphs*** and ***Helly interval bigraphs***.
- ***Proper circular arc bigraphs*** including its subclass of ***proper interval bigraphs***.

Definitions

Assume a bipartite graph $G = (V, W, E)$. We use $N(v)$ and $N[v]$ to denote the open and closed neighborhoods of vertex v, respectively.

Proper family: no two elements in it are properly contained in one another.

***Biclique*:** a maximal subset of $V(G)$ that induces a bipartite-complete subgraph.

***Bichordal*:** bipartite graph that does not admit any induced cycles of length greater than 4.

***Non-bichordal*:** is not bichordal.

***Twins*:** two vertices $v, w \in V \cup W$ such that $N(v) = N(w)$.

Circular arc bigraph (CAB)*:** bipartite graph that admits a bijection $b : V \cup W \to \mathbb{A}$ where $\mathbb{A}$ is a family of arcs on a circle such that $v \in V, w \in W$ are neighbors if and only if $b(v) \cap b(w) \neq \emptyset$. Call such a bijection a ***bi-circular-arc model of G.

Interval bigraph*:** a bipartite graph that admits a bijection $b : V \cup W \to \mathbb{A}$ such that $\mathbb{A}$ is a family of intervals on the number line, and $v \in V, w \in W$ are neighbors if and only if $b(v) \cap b(w) \neq \emptyset$. Call such a bijection a ***bi-interval model of G.

Circular convex bipartite (CCB)*:** a bipartite graph that V can be circularly ordered such that for every $w \in W$, $N(w)$ is an interval in the order. Call such an order a ***CCB order of V. Graph G is ***doubly-CCB (D-CCB)*** if both V and W partite sets admit such an order.

***Proper circular arc bigraph (P)* (resp. *proper interval bigraph (PI)*):** a bipartite graph that it admits a bi-circular-arc (resp. bi-interval) model b such that $b(V)$ and $b(W)$ are proper families.

Helly circular arc bigraph (H)* (resp. *Helly interval bigraph (HI)*):** a bipartite graph that it admits a bi-circular-arc (resp. bi-interval) model b such that, for every biclique $K \subset V \cup W$, there exists a point p on the circle (on the number line) such that $p \in X$ for all $X \in b(K)$. Call such a model a ***Helly bi-circular-arc (resp. ***Helly bi-interval***) ***model*** of G.

***Non-bichordal Helly circular arc bigraph (NBH)*:** a non-bichordal Helly circular arc bigraph.

Findings

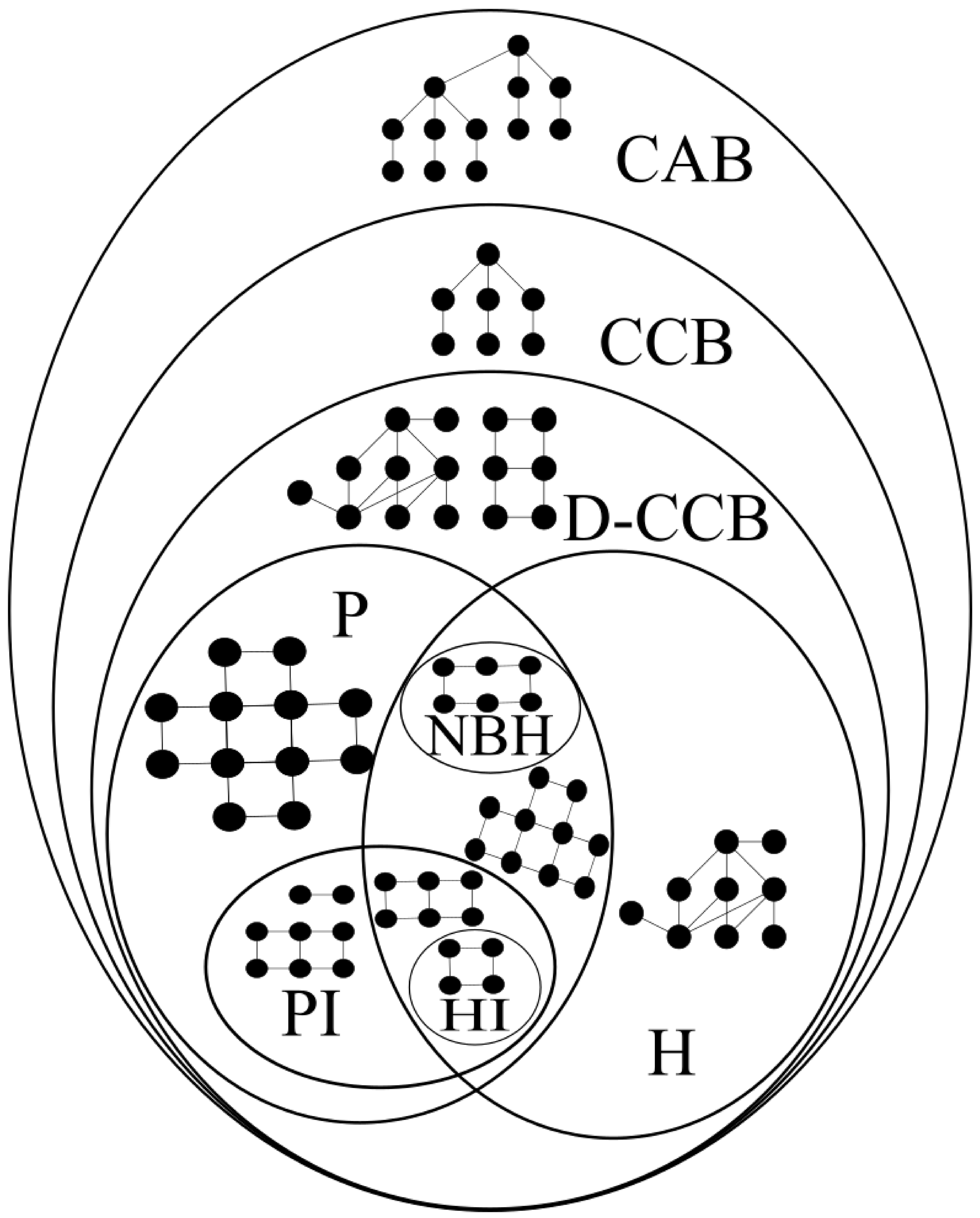

Figure 1: The Venn diagram of the classes studied, with an example graph in each region.

- **CCB ⊂ CAB**

  Proof idea. In a bi-circular-arc model b, attribute to the vertices of V pairwise disjoint arcs on the circle ordered according to the circular order of the vertices. For every $w \in W$, there is an arc that intersects every arc in $b(N(w))$ and no arcs in $b(V) - b(N(w))$. □

- **P ⊂ D-CCB**

  Proof idea. Let b be a bi-circular-arc model of G such that $b(V)$ and $b(W)$ are proper families. The clockwise order of the beginning points of the arcs in $b(V)$ (resp. $b(W)$) is a CCB order. □

- **H ⊂ D-CCB**

  Proof idea. Let b be a Helly bi-circular-arc model of G. For every $v \in V$ (every $w \in W$) there is a biclique K_v (K_w) that contains $N[v]$ ($N[w]$). For each of those, there is a point p_v (p_w) on the circle that every arc in $b(K_v)$ ($b(K_w)$) contains. The clockwise order of the points in $\{p_v | v \in V\}$ ($\{p_w | w \in W\}$) is a CCB order. □

- **NBH ⊂ P**

  Proof idea. In [3], we show that every twin-free **NBH** graph is an induced subgraph of a restrictive set of graphs. It is possible to show that every graph of that set is **P**. □

- **HI ⊂ PI**

  Proof idea. In [3], we show that every twin-free **HI** is an induced subgraph of a restrictive set of graphs. It is possible to show that every graph of that set is **PI**. □

- **H and P are uncomparable.**

  Proof idea. The Venn diagram has examples of a **P** graph that is not **H**, and vice-versa. □

Conclusion

We provided a comprehensive study of the relationship between seven different subclasses of circular arc bigraphs. We showed that doubly CCB graphs, a proper subclass of circular convex bipartite graphs, are a proper superclass of both Helly and proper circular arc bigraphs. We also showed that non-bichordal Helly circular arc bigraphs are a proper subclass of proper circular arc bigraphs, and that Helly interval bigraphs are a proper subclass of proper interval bigraphs. We also showed that proper and Helly circular arc bigraphs, which contain a non-empty intersection, are not comparable.

The results provide a full understanding of the containment hierarchies of the classes mentioned, allowing us to present a comprehensive Venn diagram of them.

Future Research

Future research includes looking into relationships between other subclasses of circular arc bigraphs, such as unit circular arc bigraphs (graphs that admit a bi-circular-arc model such that all arcs are of the same length), cross-proper circular arc bigraphs (graphs that admit a bi-circular-arc model where no two arcs corresponding to vertices of opposing partite sets are comparable), and normal circular arc bigraphs (graphs that admit a bi-circular-arc model where no union of two arcs equals the entire circle).

It also includes looking into the recognition problems of circular arc bigraphs, and any important subclasses of circular arc bigraphs for which no efficient recognition algorithm is known.

References

[1] Asim Basu, Sandip Das, Shamik Ghosh, and Malay Sen. Circular-arc bigraphs and its subclasses. *Journal of Graph Theory*, 73(4):361–376, 2013.

[2] Ashok Kumar Das and Ritapa Chakraborty. New characterizations of proper interval bigraphs and proper circular arc bigraphs. In Sumit Ganguly and Ramesh Krishnamurti, editors, *Algorithms and Discrete Applied Mathematics*, pages 117–125, Cham, 2015. Springer International Publishing.

[3] M. Groshaus, A. L. P. Guedes, and F. S. Kolberg. On the helly subclasses of interval bigraphs and circular arc bigraphs. In *Proceedings of the 14th Latin American Theoretical Informatics Symposium*, São Paulo - SP, Brazil, 2020.

[4] Martín D. Safe. Circularly compatible ones, d-circularity, and proper circular-arc bigraphs. *ArXiv*, abs/1906.00321, 2019.

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Efficient characterizations and algorithms of tree *t*-spanners

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Introduction

The t-admissibility problem has been widely studied specially because determining if a graph G is 3-admissible is still an open problem since it was proposed [2]. Although recognizing if a graph is 2-admissible is a polynomial time solvable problem, we realized that for some classes could be easier. Hence, in this work we present simple and efficient algorithms in order to characterize 2 and 3-admissible graphs for some graphs classes as cographs, split graphs, P_4-sparse and other superclasses.

Tree t-spanners

A tree *t*-spanner of a graph G is a spanning tree T of G in which the distance between adjacent vertices of G is at most *t* in T. In this case, we say that G is a t-admissible graph and the t-admissibility problem concerns in deciding if G is t-admissible. The minimum t for which G is t-admissible is the stretch index of G.

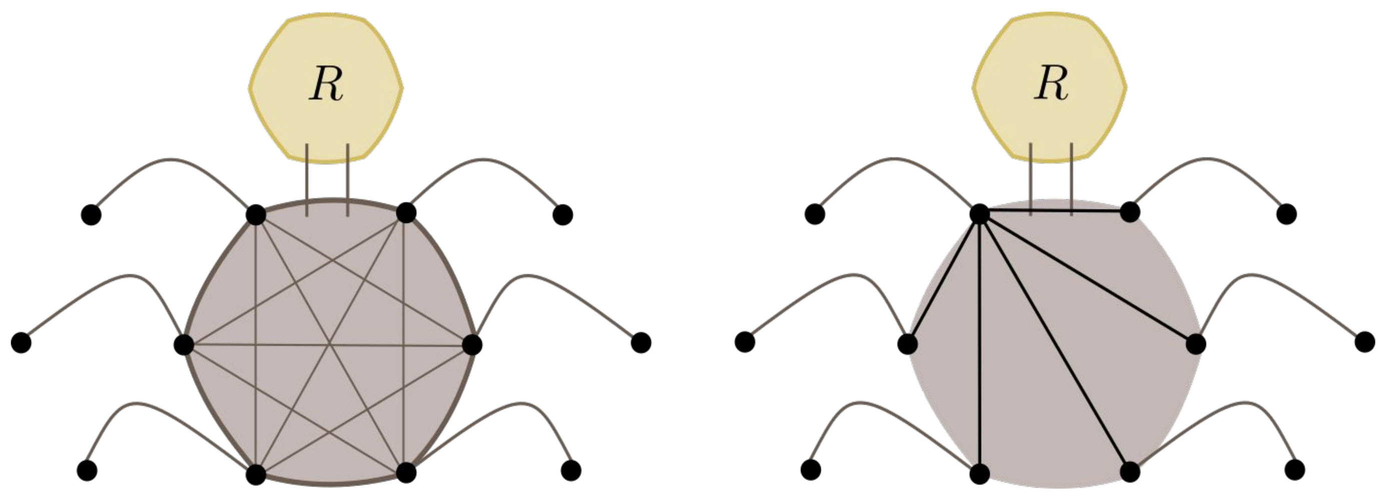

Figure 1: Thin spider graph and its tree 2-spanner T. Two parallel lines represent a join operation between the touched parts. Each vertex in the spider's body is connected to all other vertices in the spider's body and the vertices on the spider's head R. Thus there is a spanning star with respect to the body and R. Since spider's paws have degree one, we make them pendant in T, and then, the stretch index is equal 2.

Deciding whether G is 2-admissible can be solved in *O(n+m)* time, where *n* and *m* are the number of vertices and edges of G, respectively. t-admissibility is NP-complete for $t \geq 4$, and 3-admissibility remains an open problem.

Our goal is to provide simple and fast characterizations of tree *t*-spanners for graph classes in order to check 2- or 3-admissibility for them.

3-admissibility has been already efficiently solved for some graph classes, such as cographs, split graphs, cycle-power graphs and (2,1)-chordal graphs [1,3].

2-admissible P4-sparse graphs and (0,2)-graphs

For P_4-sparse graphs (graphs obtained from trivial graphs, by applying in any order union, join and spider operations), we have that, if G is not a thin spider (Figure 1) and has not a universal vertex, its stretch index is equal to 3.

Moreover, given a P_4-sparse graph G, G is 2-admissible if and only if either G has universal vertex; or G is a thin spider.

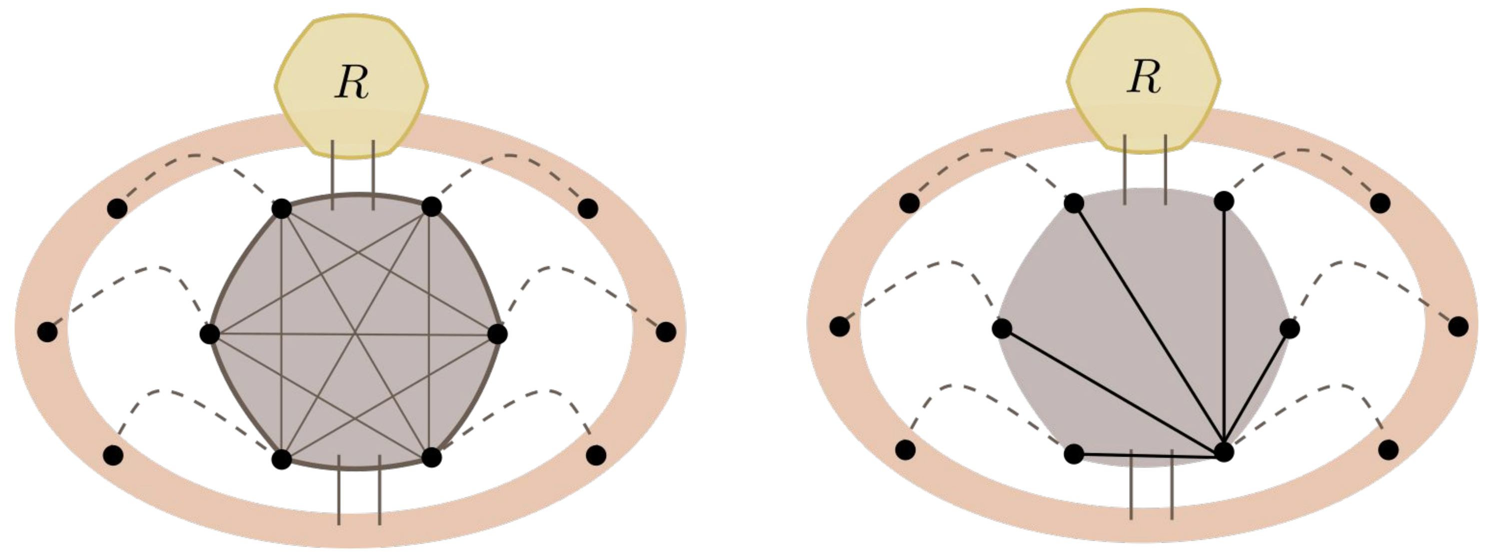

Figure 2: Thick spider graph and its tree 3-spanner. Two parallel lines represent a join operation between the touched parts. Dashed lines represent non-edges. Since each spider's paw is adjacent to all vertices of the body, except one, there is a spanning star with respect to the body and the head R with any vertex v of the spider's body as the center of the star. The paw that is not adjacent to v is placed in any of the leaves of the spider's body. And, thus, the stretch index of the graph is 3.

We present a linear time algorithm to decide 2-admissibility for P_4-sparse graphs. The algorithm consists in verifying the existence of a universal vertex and if the given graph is a thin spider. For this second part, we calculate its spider partition (S, K, R) and check the degrees of the vertices in order to differ the thin form the thick spider (Figure 2), which is not 2-admissible.

Considering (0,2)-graphs (graphs that can be partitioned into 0 independent set and 2 cliques) we also present a linear time algorithm to check the 2-admissibility.

Given a (0,2)-graph G, G is 2-admissible if and only if G has a universal vertex, a cut-vertex or between the parts of the (0,2)-partition is a strict 2-connected graph that has not an induced C_4.

Further work

In addition to the results presented above, we determined linear time algorithms to check 2-admissibility for P_4-tidy graphs, graphs that generalize P4-sparse graphs, as described above.

We also considered the *t*-admissibility problem for a superclass of (0,2)-graphs, the (k,l)-graphs. Specifically: split graphs (i.e. (1,1)-graphs) and (0,l)-graphs. We presented linear time algorithms to verify the existence of a tree 2-spanner.

As future work, we intend to extend this study to other graph classes and to deal with a recent study that is a variation of t-admissibility, called edge admissibility [4], concerning in obtaining a spanning tree of the line graph of G in which the distance between adjacent edges of G is at most *t*.

References

[1] F. COUTO; L. F. I. CUNHA; D. FERRAZ. Determining optimum tree t-spanners for split graphs and cographs. VIII Latin American Workshop on Cliques in Graphs, p.10, 2018.
[2] L. CAI ; D. G. CORNEIL. Tree spanners. SIAM J. Discrete Math., 8(3):359–387, 1995.
[3] F. COUTO; L. F. I. CUNHA. Hardness and efficiency on minimizing maximum distances for graphs with few P4's and (k,l)-graphs. Electronic Notes in Theoretical Computer Science, 346:355–367, 2019.
[4] F. COUTO; L. F. I. CUNHA.; D. POSNER. Edge tree spanner, CTW 2020. 1-12, 2020.

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Hamiltonicity of Token Graphs of Some Fan Graphs

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Abstract

In this poster we present some recent results about the Hamiltonicity of the 2-token graph $F_2(G)$ and the 2-multiset graph $M_2(G)$ of some fan graphs G. In particular, we exhibit an infinite family of graphs for which $F_2(G)$ and $M_2(G)$ are Hamiltonian.

Introduction

As far as we know, token graphs have been defined, independently, at least four times since 1988. Since then, several combinatorial parameters of token graphs have been studied, such as connectivity, regularity, planarity, Hamiltonicity, Eulerianicity, and, chromatic, clique, independence and packing numbers; as well as their automorphism group and spectrum. Also, several connections between token graphs and other research areas have been discovered, such as Quantum Mechanics and Coding Theory. For example, token graphs model the following system in Quantum Mechanics: consider a cluster of n interacting qubits (two-level atoms) represented by a graph G (where the qubits are interacting via an (excitation)-exchange Hamiltonian), in which, at each moment, exactly k qubits are in the *excited* state and the remaining in the *ground* state; this system corresponds to the k-token graph of G. In Coding Theory, the packing number of the k-token graph of P_n corresponds to the largest code of length n and constant weight k that can correct a single adjacent transposition; also the k-token graph of the Complete graph K_n is isomorphic to the Johnson graph $J(n,k)$, which have several applications in Coding Theory. Besides, token graphs have been used to study the Isomorphism Problem of Graphs.

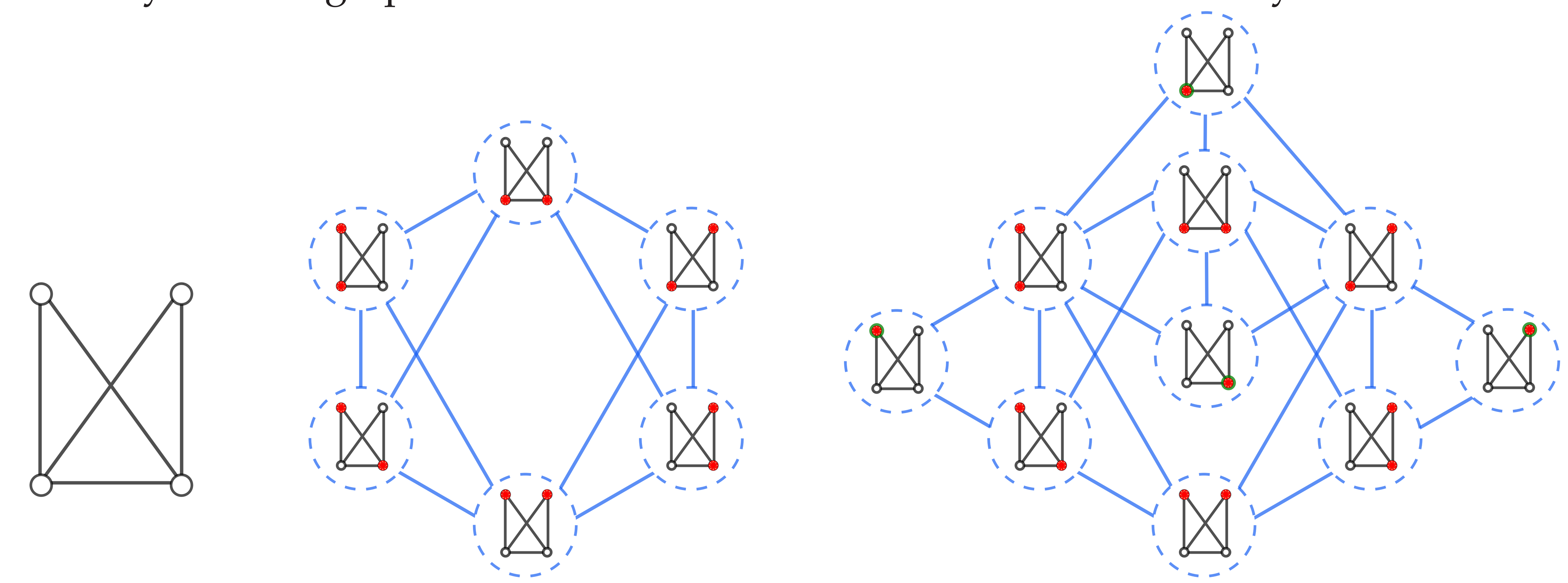
Motivation

Besides the possible applications of token graphs, one of our motivations to study the Hamiltonicity of token graphs was to extend our result of 2018 [5]:

Theorem 1 *If $n \geq 3$, and $1 \leq k \leq n-1$, then the k-token graph of the fan graph $F_{1,n-1}$ is Hamiltonian.*

Definitions

For two disjoint graphs G_1 and G_2, the *join graph* $G = G_1 + G_2$ of graphs G_1 and G_2 is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and its edge set is $E(G_1) \cup E(G_2) \cup \{uv : u \in G_1 \text{ and } v \in G_2\}$, a simple example is the *fan graph* $F_{m,n} = E_m + P_n$, where E_m denotes the graph of m isolated vertices and P_n denotes the path graph of n vertices.
Let G be a simple graph of order n. The *k-token graph* $F_k(G)$ of G is the graph whose vertices are the k-subsets of $V(G)$, where two of such vertices are adjacent if their symmetric difference is a pair of adjacent vertices in G. The *k-multiset graph* $M_k(G)$ of G is the graph whose vertices are the k-multisubsets of $V(G)$, and two of such vertices are adjacent if their symmetric difference (as multisets) is a pair of adjacent vertices in G. See an example of these constructions in the figure below. The 2-token graph is usually called the *double vertex graph* and the 2-multiset graph is called the *complete double vertex graph*.
A *Hamiltonian path* (resp. a *Hamiltonian cycle*) of a graph G is a path (resp. cycle) containing each vertex of G exactly once. A graph G is *Hamiltonian* if it contains a Hamiltonian cycle.



Fan graph $F_{2,2}$ 2-token graph $F_2(F_{2,2})$ 2-multiset graph $M_2(F_{2,2})$

Previous results

It is well known that the Hamiltonicity of G does not imply the Hamiltonicity of $F_k(G)$. For example it is know that if $n = 4$ or $n \geq 6$, then $F_2(C_n)$ is not Hamiltonian. On the other hand, there exist non-Hamiltonian graphs for which its double vertex graph is Hamiltonian, for example $F_2(K_{1,3})$ is Hamiltonian. Next, we list the known results about the Hamiltonicity of $F_k(G)$ or the existence of a Hamiltonian path in $F_k(G)$, when k may be greater than two.

- If $n \geq 3$ and $1 \leq k \leq n-1$, then $F_k(K_n)$ is Hamiltonian, see for example [3].
- If $m \geq 2$, then $F_k(K_{m,m})$ has a Hamiltonian path if and only if k is odd [4].
- If G is a graph containing a Hamiltonian path and n is even and k is odd, then $F_k(G)$ has a Hamiltonian path [4].
- If $n \geq 3$ and $1 \leq k \leq n-1$, then $F_k(F_{1,n-1})$ is Hamiltonian [5].

In addition to these results, the following are some known results for the double vertex graph ($k = 2$).

- $F_2(C_n)$ is non-Hamiltonian [2].
- If G is a cycle with an odd chord, then $F_2(G)$ is Hamiltonian [2].
- $F_2(K_{m,n})$ is Hamiltonian if and only if $(m-n)^2 = m+n$ [2].

More results about the Hamiltonicity of double vertex graphs can be found in the survey of Alavi et. al. [1].

Results

These results were obtained by Luis Adame and the authors of this poster.

Theorem 2 *Let $m \geq 1$ and $n \geq 2$. Then, $F_2(F_{m,n})$ is Hamiltonian if and only if $m \leq 2n$, and $M_2(F_{m,n})$ is Hamiltonian if and only if $m \leq 2(n-1)$.*

This theorem implies the following result.

Corollary 3 *Let G_1 and G_2 be two graphs of order $m \geq 1$ and $n \geq 2$, respectively, such that G_2 has a Hamiltonian path. Let $G = G_1 + G_2$. If $m \leq 2n$ then $F_2(G)$ is Hamiltonian, and if $m \leq 2(n-1)$ then $M_2(G)$ is Hamiltonian.*

Open Questions

1. To study the Hamiltonicity of $F_k(F_{m,n})$ and $M_k(F_{m,n})$, for $k > 2$.
2. Given two graphs G and H, to study the Hamiltonicity of $F_k(G \square H)$ and $M_k(G \square H)$.
3. To find other families of non-Hamiltonian graphs for which their k-token graph and k-multiset graph are Hamiltonian.
4. What is the smallest Hamiltonian graph G for which $F_k(G)$ and $M_k(G)$ are Hamiltonian?

References

[1] Y. Alavi, D. R. Lick and J. Liu. Survey of double vertex graphs, *Graphs Combin.*, **18**(4) (2002), 709–715.
[2] Y. Alavi, D. R. Lick and J. Liu, Hamiltonian cycles in double vertex graphs of bipartite graphs, *Congr. Numerantium*, **93** (1993), 65–72.
[3] B. Alspach, Johnson graphs are Hamilton-connected. Ars Mathematica Contemporanea, 6(1), 21-23 (2012).
[4] R. Fabila-Monroy, D. Flores-Peñaloza, C. Huemer, F. Hurtado, J. Urrutia and D. R. Wood, Token graphs, *Graph Combinat.* **28**(3) (2012), 365–380.
[5] L. M. Rivera and A. L. Trujillo-Negrete, Hamiltonicity of token graphs of fan graphs, *Art Discr. Appl. Math.*, **1** #P07 (2018).

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