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Total coloring of some unitary Cayley graphs

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Unitary Cayley graphs

For a positive integer \( n \), the unitary Cayley graph \( X_n = \text{Cay}(\mathbb{Z}_n, \mathbb{U}_n) \) is defined by the additive group of the ring \( \mathbb{Z}_n \) of integers modulo \( n \) and the multiplicative group \( \mathbb{U}_n \) of its units, where \( \mathbb{U}_n = \{ a \in \mathbb{Z}_n : \gcd(a, n) = 1 \} \). The vertex set of \( X_n \) is the set \( V(X_n) = \mathbb{Z}_n = \{ 0, 1, \ldots, n-1 \} \) and its edge set is \( E(X_n) = \{ ab : a, b \in \mathbb{Z}_n \text{ and } gcd(a-b, n) = 1 \} \). The unitary Cayley graphs \( X_n \) are regular of degree \( |\mathbb{U}_n| = \phi(n) \), where \( \phi(n) \) is the Euler function.

Total coloring

A \( k \)-total coloring of \( G \) is an assignment of \( k \) colors to the edges and vertices of \( G \), such that no adjacent elements (vertices and edges) receive the same color. The total chromatic number of \( G \), denoted by \( \chi_T(G) \), is the least \( k \) for which \( G \) has a \( k \)-total coloring. Let \( \Delta(G) \) be the maximum degree of \( G \). Clearly, \( \chi_T(G) \geq \Delta(G) + 1 \) and the Total Coloring Conjecture (TCC) [1, 6] states that \( \chi_T(G) \leq \Delta(G) + 2 \). This conjecture has been verified for some classes but the general statement has remained open for more than fifty years and has not been settled even for regular graphs. If \( \chi_T(G) = \Delta(G) + 1 \), then \( G \) is said to be Type 1, and if \( \chi_T(G) = \Delta(G) + 2 \) then \( G \) is said to be Type 2. The problem of deciding if a graph is Type 1 has been shown NP-complete [5]. For more information, we refer to [3], which is the first PhD thesis on total coloring developed in Brazil.

Total coloring of unitary Cayley graphs

Prajnanaswaroopa et al. [4] established the TCC for all unitary Cayley graphs. Some unitary Cayley graphs are already known to be Type 1 or Type 2. If \( n = p^r \) is a prime power, then \( X_p \) is a complete \( p \)-partite graph and the total chromatic number is well known: if \( p \) is odd, then \( X_p \) is Type 1, and if \( p \) is even, then \( X_p \) is Type 2 [3]. We determine the total chromatic number of all members of two families of unitary Cayley graphs \( X_{np} \), when \( n = 6s \), for a positive integer \( s \), and when \( n = 3p \), for prime \( p \geq 5 \).

Boggess et al. [2] proved that for \( n \geq 3 \), graph \( X_n \) can be decomposed into \( \frac{\Delta - 1}{2} \) edge-disjoint Hamiltonian cycles, denoted by \( H_\Delta^n \), with \( j \in \mathbb{U}_n \); and this result is used to prove the following theorems. Consider directed edges \( \langle i, i+j \ mod \ n \rangle : 0 \leq i \leq n-1 \) to indicate the direction used to construct the cycles \( H_\Delta^n \), as \( H_3^{12} \) and \( H_{2p}^{12} \) are the same cycle.

**Theorem 1.** For positive integer \( s \), the graph \( X_{6s} \) is Type 1.

**Proof.** Graph \( X_{6s} \) is bipartite with parts \( A = \{ 2i : 0 \leq i \leq \frac{6s-2}{3} \} \) and \( B = \{ 2i+1 : 0 \leq i \leq \frac{6s-2}{3} \} \). Consider the Hamiltonian cycle \( H_{2p}^{12} \), since it has 6s vertices, it is well known that admits a 3-total coloring \( T \) such that vertices \( i \), with \( i \equiv 0 \ mod \ 3 \) (resp. \( i \equiv 1 \ mod \ 3 \) and \( i \equiv 2 \ mod \ 3 \)) receive the same color. As \( 3 \notin \mathbb{U}_6 \), the adjacent vertices in \( X_{6s} \) do not have the same color assigned by \( T \). Now, remove from \( X_{6s} \) all the edges in \( H_3^n \). Clearly, the resulting bipartite graph is \( (\Delta(X_{6s}) - 2) \)-regular and, by Hall’s theorem, it can be edge colored with \( \Delta(X_{6s}) - 2 \) colors. Therefore, \( X_{6s} \) is Type 1. The following figure presents a 5-total coloring of \( X_{12} \).

**Theorem 2.** For prime \( p \geq 5 \), the graph \( X_{3p} \) is Type 1.

**Idea of the proof.** Graph \( X_{3p} \) is a 3-partite graph with parts \( A = \{ 3i : 0 \leq i \leq p - 1 \} \), \( B = \{ 3i+1 : 0 \leq i \leq p - 1 \} \) and \( C = \{ 3i+2 : 0 \leq i \leq p - 1 \} \). By Vizing’s theorem, each Hamiltonian cycle \( H_{3p}^{12} \) admits a 3-edge coloring. For \( p > 1 \), assign 3 colors to the edges of each \( H_{3p}^{12} \) such that a special color \( c_0 \) is used in all cycles in a particular directed edge \( (a, a+j \mod 3p) \), and the endpoints \( \{ a, a+j \mod 3p \} \) receive 2 different colors already used in the respective cycle. For \( j = 1 \in U_{3p} \), assign 3 colors to the edges of \( H_{3p}^{12} \) so that the special color \( c_0 \) is assigned to exactly 3 directed edges: \{1, 2\}, \{4, 5\}, \{7, 8\}; and the endpoints \{1, 4\} \in \mathcal{B} \text{ and } \{2, 5, 8\} \in \mathcal{C} \text{ receive the 2 colors already used in the respective cycle, one color to each part. The remaining vertices not colored in } X_{3p} \text{ are in part } A \text{, and we assign color } c_0 \text{ to these vertices.}

Notice that the assignment of colors does not have conflict. We used 2 colors for the elements of each of the \( p - 1 \) Hamiltonian cycles and used color \( c_0 \) in all cycles. Thus, we obtain a \( (2p - 1) + 1 = \Delta(X_{3p}) + 1 \)-total coloring. The figure below presents the four edge-disjoint Hamiltonian cycles \( H_3^{12}, H_2^{12}, H_3^{12}, H_2^{12} \) and \( H_3^{12} \) of \( X_{15} \) with a 9-total coloring such that the color \( c_0 \) is represented by purple color.

References


Let $G$ be a B$_1$-EPG graph. Recognition of $G$ is simultaneously in the VPT and EPT graph classes. In addition, we describe a set of graphs that defines Helly-B$_1$-EPG families. In particular, this work presents some features of non-trivial families of graphs properly contained in Helly-B$_1$ EPG, namely Bipartite, Block, Cactus and Line of Bipartite graphs.

### Objective

In this work we will mainly explore the EPG graphs, in particular B$_1$-EPG graphs. However, other classes of intersection graphs will be studied such as EPT and VPT graph classes.

### Definitions and Technical Results

- A graph is a B$_k$-EPG graph if it admits an EPG representation in which each path has at most $k$ bends;
- When $k=1$ we say that this is a single bend EPG representation or simply a B$_1$-EPG representation;
- In a B$_1$-EPG representation, a clique $K$ can be edge-clique or claw-clique [3].

![Figure 1: Representation of a clique as edge-clique and as claw-clique.](image)

- A collection of sets satisfies the Helly property when every pairwise intersecting sub-collection has at least one common element;
- When this property is satisfied by the set of paths used in a representation, we get a Helly representation;
- Helly-B$_1$-EPG graphs were studied in [2];
- EPG, EPT and VPT representations arise in circuit layout problems and layout optimization [4];
- VPT and EPT graphs are the vertex-intersection and edge-intersection graphs of paths on trees, respectively;
- VPT and EPT graphs are incomparable families of graphs.

### Subclasses of Helly-B$_1$-EPG Graphs

**Theorem 1:** Let $G$ be a B$_1$-EPG graph. If $G$ is $\{S_1, S_3, S_3', C_4\}$-free then $G$ is a Helly-B$_1$-EPG graph.

![Figure 2: Reconstruction of intersection model.](image)

**Theorem 2:** If the graph $G$ is B$_1$-EPG and diamond-free then $G$ is Helly-B$_1$-EPG.

**Corollary:** Bipartite, Block, Cactus and Line of Bipartite graphs are Helly-B$_1$-EPG.

### Relationship among Chordal B$_1$-EPG, VPT and EPT graphs

**Theorem 3:** Chordal B$_1$-EPG $\subseteq$ VPT.

**Theorem 4:** Chordal B$_1$-EPG $\subseteq$ EPT.

![Figure 4: Graph $S_4$ and one of its possible VPT and EPT representations.](image)

### References


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Alcón, Liliana - CONICET/UNLP - liliana@mate.unlp.edu.ar; Mazzoleni, Maria Pia - UNLP - pia@mate.unlp.edu.ar; Santos, Tanilson Dias - UFT/UFRJ - tanilson.dias@uft.edu.br
**INTRODUCTION**

This work presents a hybrid exact-heuristic algorithmic approach, based on an arc-time indexed mixed-integer programming formulation and a generalized evolutionary based on a strong local search, in order to better solve the problem $P|\sum \alpha_j E_j + \beta_j T_j| \text{WET}$ (WET). The selected arcs from local optimal solutions generated by a Genetic Algorithm based on a strong Local Search (GLS), are given as input to an IP Arc-time indexed formulation, which is solved to produce better solutions at CPLEX. The proposed Hybrid Matheuristic method is capable to produce better results when compared with the previous best results in the literature.

**OBJECTIVE**

The objective of this work is to develop an exact-heuristic method to solve large instances of the identical parallel machine Weighted Just-In-Time Scheduling Problem.

**JUST-IN-TIME SCHEDULING PROBLEM**

Considering the classical NP-hard parallel-machine weighted earliness-tardiness scheduling problem, $P|\sum \alpha_j E_j + \beta_j T_j| \text{WET}$, in 3-field notation [1], where $j = \{1, \ldots, n\}$ is the set of independent jobs to be processed without preemption, in $m$ identical parallel machines, where each one can process at least one job on a given time. Every job $j$ has a positive processing time $p_j$, a due date $d_j$ and a positive earliness ($\alpha_j$) and tardiness ($\beta_j$) weights. The earliness of a job is defined as $E_j = \max(0; d_j - C_j)$ and the tardiness of a job is defined as $T_j = \max(0; C_j - d_j)$, where $C_j$ is the completion time of the job [2]. Figure 1 (a) presents an example of 8 jobs for the problem followed by a solution representation for single machine scheduling in Figure 1 (b) and its corresponding representation for identical parallel machines in Figure 1 (c), considering three identical parallel machines.

**THE HYBRID MATHEURISTIC**

The Hybrid Matheuristic (MathGLS-IP) is based on two steps:

- **STEP 1: Heuristic approach (GLS)**

The best local optimal solution generated by the GLS (Figure 2) is kept in a Hash Table on every generation, which will be used as a selected set of arcs to the IP Arc-time formulation. A solution representation of the Arc-time is presented in Figure 3.

- **STEP 2: Exact approach (Solving the Arc-time)**

When GLS procedure finishes, the selected arcs are kept in the Hash Table are used to build the Arc-time, and then, solve it in CPLEX to get better convergence or improve the solution for a given instance of the problem. The Arc-time indexed formulation, proposed by Pessoa et al. [3], is presented below. The MathGLS-IP method eliminates the Constraints (4), in order to decrease the number of binary variables of idle time at the end of a scheduling.

**COMPUTATIONAL EXPERIMENTS**

In Table 1 we present a resume of the computational experiments, compared with the literature. MathGLS-IP solves large instances up to 500 jobs and 2, 4 and 10 identical parallel machines. Our method also presents results for 200 instances, not yet known in the literature, and improved 4. Detailed results can be observed at Amorim [4].

<table>
<thead>
<tr>
<th>Instance group</th>
<th>MathGLS-IP</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>0.106</td>
<td>0.157</td>
</tr>
<tr>
<td>BKS</td>
<td>708</td>
<td>61,243</td>
</tr>
<tr>
<td>BRAKOV</td>
<td>0.010</td>
<td>40.31</td>
</tr>
<tr>
<td>Number of solutions equal to BKS</td>
<td>0.049</td>
<td>50.024</td>
</tr>
</tbody>
</table>

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ACKNOWLEDGMENT

A Matheuristic Approach for the Weighted Just-in-Time Scheduling Problem

AMORIM, Rainer X. de - ICET/UFAM - raineramorim@ufam.edu.br, DE FREITAS, Rosiane - ICOMP/UFAM - rosiane@icomp.ufam.edu.br, PESSOA, Artur - artur@producao.uff.br, UCHOA, Eduardo - uchoa@producao.uff.br
Layout de Redes de Sensores Sem Fio com Múltiplas Origens e Destinos: Uma Abordagem Combinatória

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Introdução

Há diversas aplicações para as Redes de Sensores sem Fio (RSSF): monitoramento de sinais ambientais[1], aplicações militares[2], entre outras. Neste trabalho, investiga-se o problema de Planejamento de Redes de Sensores Sem Fio (PRSSF-MOD), onde a rede é formada por múltiplas origens (sensores) e múltiplos destinos (sorvedouros). A topologia da rede é representada através de um grafo e na resolução do problema proposto, iremos definir um modelo de Programação Linear Inteira (PLI) e um grafo auxiliar que será utilizado junto ao modelo.

Objetivo

O objetivo deste trabalho é minimizar o número de sensores da topologia da rede em uma dada região de interesse, de modo a atender as conexões entre múltiplas origens e destinos.

Definição do problema

Dado um conjunto $S$ de sensores, onde para cada $s \in S$ é associado um conjunto $\{s_i\}_{i=1..k}$, um raio de comunicação $r$, um custo de alocação $c$ e um conjunto $P$ de origens e destinos $p = \{op, dp\}$. Utilizando essas informações pretende-se construir uma topologia $T \subseteq S$ que conecte todos os pares de origens destinos $p = \{op, dp\}$, de forma direta ou por múltiplos saltos entre sensores intermediários, de modo a minimizar o custo de instalação da rede.

Modelo PLI e grafo auxiliar

Para resolver o problema foram definidos um modelo PLI[3] (Figura 1) e um grafo auxiliar $G = (V, E)$ (Figura 2), onde $V$ é definido pelas possíveis posições de $S$ e vértices artificiais $A$ que representam origens e destinos e $E$ compreende as arestas definidas pela intersecção entre os raios de comunicação dos sensores em diferentes grupos. Para finalizar a aplicação são adicionadas arestas entre os nós artificiais de origem e destino e seus respectivos grupos de sensores.

Restrição (1) garante a existência de um caminho entre origens e destinos, Restrição (2) garante que apenas uma posição dentre as candidatas será escolhida, (3) e (4) representam que uma aresta só pode ser usada se existe um sensor naquela posição e a Restrição (5) define que todo nó artificial está na solução.

Experimentos computacionais

Os experimentos computacionais se baseiam na instância real do intel lab data[4], que possui 54 sensores de monitoramento ambiental. Os grupos de possíveis posições e as origens e destinos de cada experimento estão apresentadas abaixo.

Conclusões

Este trabalho apresentou uma versão modificada do PRSSF que considera múltiplas origens e destinos. O modelo foi avaliado em uma instância real e obteve uma redução na quantidade de sensores de 25% utilizados na topologia.

Referências

The random walk on the Tower of Hanoi

The Tower of Hanoi puzzle is a single-player game where at each turn the player moves a disk to a tower that is different from the one it previously was. The game does not allow a disk above smaller disks and its aim is to move all disks from a tower to another one (see Figure 1).

The Hanoi graph \( \mathcal{H}_m \) is the graph whose vertices represent the possible configurations of the tower of Hanoi puzzle with \( m \) towers and \( m \) disks. Its edges represent the moves between these configurations. Thus, \( \mathcal{H}_2 \) is isomorphic to a triangle and for each \( m \geq 2 \) we can construct \( \mathcal{H}_m \) in the following way which is illustrated in Figure 2: we consider three isomorphic copies of \( \mathcal{H}_{m-1} \) and we label them as \( \mathcal{H}_{m-1}^1 \), \( \mathcal{H}_{m-1}^2 \), \( \mathcal{H}_{m-1}^3 \), \( i \in \{1,2,3\} \). For each \( i \in \{1,2,3\} \) let \( v_i^{m-1} \) and \( v_i^{m} \) be the vertices on the top, on the left, and on the right of the basis of the biggest triangle in \( \mathcal{H}_{m-1} \). The graph \( \mathcal{H}_m \) is the graph with vertex set \( V_m = \{v_1^{m-1}, v_2^{m-1}, v_3^{m-1} \} \) and edge set \( E_m = \{v_i^{m-1}v_j^{m} \} \cup E_{m-1} \), where \( E_m \) is defined as \( E_m = \{(v_1^{m-1}v_2^{m}), (v_1^{m-1}v_3^{m}), (v_2^{m-1}v_3^{m}) \} \).

The simple random walk on \( \mathcal{H}_m \) is the process \( \{X_t, t \geq 0 \} \) described as follows: an \( m \)-disk to a tower that is different from the one it previously was. The game does not allow a disk above smaller disks. Not only the above function is decreasing, but it also takes values in the interval \([0,1]\).

Algebraic connectivity and Poincaré inequalities

The spectral gap \( \gamma_m \) of the simple random walk on \( \mathcal{H}_m \) (also known as the algebraic connectivity of the graph \( \mathcal{H}_m \)) is defined as the symmetric of the second largest eigenvalue \( \lambda_2 \) of the operator \( \Delta_m \). It also presents a variational formula [2]. Indeed, let \( \mathcal{E}_m \) be the Dirichlet form of the simple random walk on \( \mathcal{H}_m \), which is given by

\[
\mathcal{E}_m(f,f) = \frac{1}{2} \sum_{i,j \in \mathcal{H}_m} [f(i) - f(j)]^2 U_{ij}(x),
\]

Let \( \text{Var}(f, U_m) \) be the variance of a function \( f: V_m \to \mathbb{R} \) for the simple random walk on \( \mathcal{H}_m \), which is given by

\[
\text{Var}(f, U_m) = \frac{1}{2} \sum_{i,j \in \mathcal{H}_m} [f(i) - f(j)]^2 U_{ij}(x).
\]

The spectral gap \( \gamma_m \) of the simple random walk on \( \mathcal{H}_m \) can be defined as

\[
\gamma_m = \min \left\{ \frac{\mathcal{E}_m(f,f)}{\text{Var}(f, U_m)} : |f| \leq 1, \text{Var}(f, U_m) \neq 0 \right\}.
\]

Namely, the relaxation time \( \tau_m = \frac{1}{\gamma_m} \) of the simple random walk on \( \mathcal{H}_m \) is the smallest constant that satisfies the Poincaré inequality

\[
\text{Var}(f, U_m) \leq \gamma_m \mathcal{E}_m(f,f) \quad \text{for every function } f.
\]

The spectral gap is strongly related to mixing because \( \gamma_m \leq \sqrt{\mathcal{E}_m(f,f)} \) (see [5], for instance).

Logarithmic-Sobolev inequalities

The log-Sobolev constant \( \alpha_m \) of the simple random walk on \( \mathcal{H}_m \) presents a similar variational definition. Here the variance is replaced by the entropy-like quantity \( \mathcal{L}_m \) given by

\[
\mathcal{L}_m(f) = \sum_{x \in \mathcal{H}_m} (f(x))^2 \log \left( \frac{(f(x))}{\mathcal{E}_m(f,f)} \right) U_{mx}(x).
\]

Namely

\[
\gamma_m = \min \left\{ \frac{\mathcal{E}_m(f,f)}{\mathcal{L}_m(f,f)} : |f| \leq 1, \mathcal{L}_m(f,f) \neq 0 \right\}.
\]

that is, \( 1/\gamma_m \) is the smallest constant that satisfies the logarithmic Sobolev inequality

\[
\mathcal{L}_m(f) \leq \gamma_m \mathcal{E}_m(f,f) \quad \text{for every function } f.
\]

References


On total coloring of circulant graphs

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Introduction

A k-total coloring of a graph G is an assignment of k colors to the elements of G such that adjacent elements have different colors. The total chromatic number \( \chi''(G) \) is the smallest integer k for which G has a k-total coloring. Clearly, \( \chi''(G) \geq \Delta+1 \), and the Total Coloring Conjecture (TCC) states that for any simple graph G, \( \chi''(G) \leq \Delta+2 \), where \( \Delta \) is the maximum degree of G [2, 8]. Graphs with \( \chi''(G) = \Delta(G)+1 \) are called Type 1, and graphs with \( \chi''(G) = \Delta(G)+2 \) are called Type 2. A circulant graph \( C_n(d_1, d_2, \ldots, d_l) \) with \( 1 \leq d_1 < \cdots < d_l \leq n \) has vertex set \( V = \{v_1, v_2, \ldots, v_n \} \) and edge set \( E = \{\{v_i, v_{i+j}\} : 0 < j < n \} \) where the indices of the vertices are considered modulo n. An edge of E is called an edge of length \( d_i \). In this work, we determine the Type of an infinite family of 4-regular circulant graphs, that is, \( C_n(a, b) \). When a divides n (or b divide n), we will have a Prism graph \( G(\frac{n}{2}, 1) \) as subgraph of \( C_n(a, b) \). A Prism graph \( G(n, 1) \) is defined by \( V(G(n, 1)) = \{w_i, v_i : 0 \leq i < n \} \) and \( E(G(n, 1)) = \{u_{i, i+1}, v_{i, v_{i+1}}, w_i, v_i : 0 \leq i < n \} \). See some examples of \( C_n(a, b) \) with \( G(\frac{n}{2}, 1) \) as a subgraph in Figure 1.

General results

In the table below, we present some results already known about the total coloring of circulant graphs.

<table>
<thead>
<tr>
<th>Circulant graph</th>
<th>Type 1</th>
<th>Type 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1(1) )</td>
<td>( n \equiv 0 \mod 3 )</td>
<td>otherwise</td>
</tr>
<tr>
<td>( C_{1, 2, \ldots, (\frac{\mu}{2})} )</td>
<td>( n \equiv 0 \mod 3 )</td>
<td>otherwise</td>
</tr>
<tr>
<td>( C_{\mu}(d, n) )</td>
<td>( l = \gcd(d, n) ) with ( d = \lambda m ). ( n ) is even</td>
<td>otherwise</td>
</tr>
<tr>
<td>( C_{\mu}(1, 2) )</td>
<td>( k \equiv 2 \mod 5 ) or ( k \equiv 3 \mod 5 )</td>
<td>otherwise</td>
</tr>
<tr>
<td>( C_{\mu}(1, 3) )</td>
<td>( k \equiv 1 \mod 3 ) or ( k \equiv 2 \mod 3 )</td>
<td>otherwise</td>
</tr>
</tbody>
</table>

Tabela 1: State of the art

Our results

It is known that the Prism graphs \( G(n, 1) \) are Type 1, except \( G(5, 1) \) [7, 4]. The 4-total coloring for this family will be useful in the proof of the following theorem about 4-regular circulant graphs in which \( G(n, 1) \) is a subgraph.

Theorem 1. Let \( C_{2k}(2, 3) \) be a 4-regular circulant graph. The graph \( C_{2k}(2, 3) \) is Type 1 for \( n = (8\mu + 6)k \), with \( k \geq 1 \) and non-negative integers \( \mu \) and \( \lambda \).

A semigraph is a triple \( B = (V, E, S) \), where \( V \) is the set of vertices of \( B \), \( E \) is a set of edges having two distinct endpoints in \( V \), and \( S \) is a set of semiedges having one endpoint in \( V \). In this work we consider 4-regular semigraphs. Notice that a k-total coloring of a semigraph \( B \) is an assignment of k colors to the edges, semiedges and vertices of \( B \) such that adjacent elements have different colors.

Sketch of the proof. The result was proved in [1] when \( C_n(2k, 3) \) is connected, using the Figure 2(a). Hence, suppose that \( C_{10(2k, 3)} \) is disconnected, that is \( k = 3 \alpha \). In this case, note that \( C_{10(2k, 3)}(3, 6n) \) isomorphic to three copies of \( C_{10(2k, 3)}(1, 2n) \). To construct the colorings of these graphs, we consider two cases: \( \mu = 0 \) and \( \mu \neq 0 \). When \( \mu = 0 \), we construct the desired coloring by making the junction of \( \lambda \) copies of the semigraph \( B(1, 2) \) (Figure 2(c)) vertically and horizontally, recursively. When \( \mu \neq 0 \), we make the junction of \( \mu \) copies of the semigraph \( B(8, 2) \) with \( \lambda \) copies of \( B(8, 2) \) (Figure 2(b)) vertically and horizontally, (same for the case when \( C_n(2k, 3) \) is connected). However the process of joining its semiedges to construct the desired graph is different. See an example in Figure 3.

Figura 1: Examples of \( C_n(a, b) \) with \( G(\frac{n}{2}, 1) \) as a subgraph.

Figura 2: Semigraph \( B(n, a) \).

Figura 3: The graph \( C_{2k}(3, 12) \) with a total coloring with 5 colors.

Conclusion

The total chromatic number of several circulant graphs has been determined, including the total chromatic number of the cubic circulant graphs \( C_n(2k, 3) \). As a future work, we would like to determine the total chromatic number of all 4-regular circulant graphs \( C_n(a, b) \).

References


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Figure 3: The graph \( C_{2k}(3, 12) \) with a total coloring with 5 colors.
EQUITABLE TOTAL COLORING OF BLOWUP SNARKS

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INTRODUCTION

Let G be a simple graph. A k-total-coloring of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The total chromatic number of G, denoted by χ"(G) is the least k for which G has a k-total-coloring. Evidently, χ"(G) ≥ Δ(G) + 1, where Δ(G) is the maximum degree of G. The Total Coloring Conjecture [1] affirms that χ"(G) ≤ Δ(G) + 2. This conjecture has been proved for cubic graphs [2], so the total chromatic number of a cubic graph is 4 or 5 (graphs with χ"(G) ≥ Δ(G) + 1 are said to be Type 1 and graphs with χ"(G) ≤ Δ(G) + 2 are said to be Type 2). Deciding whether a graph is Type 1 has been shown NP-complete [3].

A k-total-coloring is equitable if the cardinalities of any two color classes differ by at most one. The least k for which G has an equitable k-total-coloring is the equitable total chromatic number of G and its denoted by ϕ(G).

The search for connected, bridgeless, 3-regular graphs with chromatic index equals 4, was motivated by the Four Color Problem. Due the difficult to find them, they were named Snarks after Lewis Carrol poem "The hunting of the Snark", by M. Gardner [4]. Snarks were fictional animal species described by Carrol as unimaginable creatures.

The girth of G is the length of the shortest cycle contained in G. One condition often imposed on snarks is that they must have girth at least 5, to avoid graphs that can be reduced to a smaller graph by replacing a subgraph for structures that do not affect the edge colorability.

In this study we investigate the k-total-coloring of blowup infinite family of snarks, recently defined by Hägglund [5], with girth at least 5.

RESULTS

Let B be the cubic semigraph with 6 semi-edges, illustrated in Figure 2. Blowup graphs are constructed by connecting copies of B as in the examples of Figures 3.

An n-Blowup is a graph build with n copies of B.

Theorem: All n-Blowups with n≥5 have equitable total chromatic number equals 4.

The sketch of the proof is by construction and two different equitable 4-total-colorings were necessary to obtain the result. We represent 1 for blue, 2 for green, 3 for red and 4 for yellow. First coloring is showed in Figure 3(a). It's composed by two copies of B colored with 4 colors. More specifically, in this figure, ϕ(1) = ϕ(2) = ϕ(3) = ϕ(4) = 15 (semi-edges counts 0.5).

When n≡0 (mod 2) we repeat this coloring 2 times. Evidently, ϕ(1) = ϕ(2) = ϕ(3) = ϕ(4) = 2·15, Figure 3(b) shows 6-Blowup colored following this rule.

The second coloring is showed in Figure 3(c) and its composed by 3 copies of B colored with 4 colors. In this coloring, ϕ(1) = ϕ(2) = 22 and ϕ(3) = ϕ(4) = 23.

When n≡1 (mod 2) we use this coloring once and for the remaining n-3 copies of B we repeat 4 times the coloring showed in (a).

Thus, ϕ(1) = ϕ(2) = 22 + 4·3 = 22 + 12 = 34 and ϕ(3) = ϕ(4) = 23 + 4·3 = 23 + 12 = 35. Evidently, ϕ(1), ϕ(2), ϕ(3) and ϕ(4) differ at most one. Figure 3(d) shows 5-Blowup colored following this rule.

ACKNOWLEDGMENT
Let $G$ be a graph and $W \subseteq V(G)$ be a non-empty set, called terminal set. A strict connection tree of $G$ for $W$ is a tree subgraph of $G$ whose leaf set is equal to $W$. A non-terminal vertex of a strict connection tree $T$ is called linker if its degree in $T$ is exactly 2, and it is called router if its degree in $T$ is at least 3. We remark that the vertex set of every connection tree can be partitioned into terminal vertices, linkers, and routers. For each connection tree $T$, we let $L(T)$ denote the linker set of $T$ and $R(T)$ denote the router set of $T$. Figure 1 illustrates a graph $G$, a terminal set $W$ and a strict connection tree of $G$ for $W$.

![Figure 1](image1.png)

Figure 1: (a) Graph $G$ and terminal set $W$ (blue squared vertices). (b) Strict terminal connection tree $T$ of $G$ for $W$, such that $|L(T)| = 3$ and $|R(T)| = 3$.

Motivated by applications in information security, network routing and telecommunication, Dourado et al. [1] introduced the strict terminal connection problem, which is formally defined below.

**Strict Terminal Connection (S-TCP)**

**Input:** A graph $G$, a non-empty terminal set $W \subseteq V(G)$ and two non-negative integers $\ell$ and $r$.

**Question:** Does there exist a strict connection tree of $G$ for $W$, such that $|L(T)| \leq \ell$ and $|R(T)| \leq r$?

Table 1 summarises the complexity of S-TCP with respect to the parameters $\ell$, $r$, $\Delta(G)$, and the classes of split graphs and cographs. In addition to these results, it is known that S-TCP is NP-complete even if $\Delta(G) = 4$ and $r \geq 0$ is fixed, or $\Delta(G) = 3$ and $\ell$ is arbitrarily large [3]. On the other hand, if $\Delta(G) = 3$, the problem can be solved in time $O(n^3)$ [3].

<table>
<thead>
<tr>
<th>Graph class</th>
<th>$\ell$</th>
<th>$r$</th>
<th>$\ell + r + \Delta(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Split NPC [3]</td>
<td>$P$</td>
<td>$r \in {0, 1}$ [3]</td>
<td>$P$ [1, 3] $\Delta(G)$</td>
</tr>
</tbody>
</table>

Table 1: Computational complexity of S-TCP. (Adapted from [3]).

**Construction.** Let $I = (G, k)$ be an instance of vertex cover and $\ell \geq 0$ be a constant. Assume that $V(G) = \{v_1, \ldots, v_n\}$ for some positive integer $n \geq 2$. Moreover, assume that $G$ has at least one edge, i.e. $m = |E(G)| \geq 1$. We let $f(I, \ell) = (H, W, \ell = c, r)$ be the instance of S-TCP defined as follows.

- For each $v_i \in V(G)$, create the gadget $H_i$, as illustrated in Figure 2.

![Figure 2](image2.png)

Figure 2: Gadget $H_i$.

Subdivide the edge $w_{ik} \in H_i$ into $\ell$ new vertices $u_1, u_2, \ldots, u_\ell$, creating the induced path $(w_{ik}, u_1, u_2, \ldots, u_\ell, v_i)$.

- For each pair $v_i, v_j \in V(G)$, with $i \neq j$, add the edges $x_i y_j$ and $x_j y_i$, making the subgraph of $H$ induced by $X \cup Y$ a complete bipartite graph with bipartition $(X, \cup Y)$, where $X = \{x_i \mid v_i \in V(G)\}$, $Y = \{y_i \mid v_i \in V(G)\}$ and $Z = \{z_i \mid v_i \in V(G)\}$.

Finally, define $W = W_1 \cup W_2 \cup W_3$ and $r = k + 4n + 4m$, where $W_1 = \{w_{ij}^k \mid v_i \in V(G)\}$,

$W_2 = \{w_{ij}^1, w_{ij}^2, w_{ij}^3, w_{ij}^4, w_{ij}^5 \mid v_i \in V(G)\}$, and

$W_3 = \{w_{ij}^j \mid v_i \in V(G)\}$.

**Theorem.** Let $I = (G, k)$ be an instance of VERTEX-COVER, such that $G$ has at least one edge, and let $c \geq 0$ be a constant. The graph $H$ of $f(I, \ell)$ is chordal bipartite. Moreover, $I$ is a yes-instance of VERTEX-COVER if and only if $f(I, \ell)$ is a yes-instance of S-TCP.

**Concluding remarks**

We conclude this work by posing some open questions.

- Is S-TCP parameterized by $r \geq 2$ in XP?
- Is S-TCP parameterized by $r \geq 2$ in FPT when restricted to chordal bipartite graphs? Is not, is it in XP?
- Is S-TCP parameterized by $\ell$ in FPT when restricted to graphs of maximum degree 3?
- In addition to cographs, on which graph classes is S-TCP in P?

**References**


INTRODUCTION
Tuza [1] contributed to the area of graph labeling presenting many results in his seminal paper and proposing new labeling games. We investigate the Range-Relaxed Graceful game (RRG game) and present a lower bound for the number of available labels for which Alice has a winning strategy in the RRG game on a simple graph G, on a cycle and on a path graph.

RANGE RELAXED GRACEFUL LABELING
Given a graph G and the set of consecutive integer labels L = {0, ..., k}, k ≥ |E(G)|, a labeling f : V(G) → L is said to be a Range-Relaxed Graceful Labeling if: (i) f is injective; (ii) each edge uv ∈ E(G) is assigned the (induced) label g(uv) = |f(u) − f(v)|, then all induced edge labels are distinct.

RRG GAME
Two players, called Alice and Bob, alternately assign a previously unused label f(v) ∈ L = {0, ..., k}, k ≥ |E(G)| to an unlabeled vertex v of a given graph G. If both ends of uv ∈ E(G) are already labeled, then the label of the edge is defined as |f(u) − f(v)|. A move is said legal if, after it, all edge labels are distinct. Alice’s goal is to end up with a vertex labeling of the whole G where all of its edges have distinct labels and Bob’s goal is to prevent it from happening.

OBJECTIVE
To investigate the Range-Relaxed Graceful game, present a lower bound on the number of consecutive nonnegative integer labels necessary for Alice to win the RRG game on a simple graph G and contribute to the study of the question posed by Tuza [1]:

TUZA’S QUESTION
Given a simple graph G and a set of consecutive nonnegative integer labels f(v) ∈ L = {0, ..., k}, for which values of k can Alice win the range-relaxed graceful game?

RESULTS

THEOREM 1
Let G be a simple graph on n vertices and maximum degree Δ. Alice wins the RRG game on G for any set of integer labels L = {0, ..., k}, with

\[ k \geq (2\Delta^2 + 1)(n - 1) + (2\Delta + 1) \left( \frac{n - 1}{2} \right). \]

SKETCH OF THE PROOF
For each vertex v ∈ V(G), we define a set of available labels L_v. When the game starts, L_v = L for every v ∈ V(G). At each iteration, a player assigns a label to an unlabeled vertex u from its set L_u, and, then, the set of available labels of each remaining vertex is updated. Only vertex labels that can not generate repeated edge labels in future iterations can last at each set. We consider four cases that can give rise to repeated edge labels and, for each one, we count how many labels are deleted, throughout the game, from each set of available labels. From our analysis, we conclude that at most \((2\Delta^2 + 1)(n - 1) + (2\Delta + 1) \left( \frac{n - 1}{2} \right)\) labels are deleted from each set of available labels. Since |L| is greater than this value, there is always an available label at each set that can be assigned to a vertex.

EXAMPLE
Consider K_5 and the set L = {0, 1, 2, 3, ..., 66}. Suppose that Alice starts the game by assigning label 1 to a vertex v_1. Below, we present the first three iterations, where the players play at v_1, v_2, v_3 consecutively, and we show the last iteration.

A similar proof is obtained for the following result.

THEOREM 2
Given any integer n ≥ 4, Alice wins the RRG game on the path P_n and on the cycle C_n for any set of integer labels L = {0, ..., k}, with k ≥ 9n − 17.

REFERENCES
The sandpile group of outerplanar graphs

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Introduction

We compute the sandpile groups of outerplanar planar graphs. The method can be used to determine the algebraic structure of the sandpile groups of other planar graph families.

Sandpile groups

The sandpile group was originated in statistical physics. It was the first model of a dynamical system exhibiting self-organized criticality. The dynamics of the sandpiles are developed over a graph \( G \) in the following way. Consider a graph \( G \) with a special vertex \( q \), called sink. A configuration \( c \) is a vector whose entries are associated with the number of grains of sand at each vertex of \( G \). The sink vertex collects the sand quiting the system. A vertex is stable if the number of sand grains on it is lower than its degree, that is, the number of edges incident to the vertex. Otherwise, the vertex is unstable. A configuration is stable if all the non-sink vertices of \( G \) are stable. A toppling of an unstable configuration consists of selecting an unstable vertex \( v \) and moving \( \deg(v) \) grains from \( v \) to its neighbors, such that each neighbor \( u \) receives \( m(u,v) \) grains, where \( m(u,v) \) denotes the number of edges between \( u \) and \( v \). In Figure 1, we show a sequence of topplings.

Over connected graphs with a sink, we will always obtain a stable and unique configuration after a finite sequence of topplings. The stable configuration obtained from the configuration \( c \) will be denoted by \( s(c) \). The sum of two configurations \( c \) and \( d \) is performed entry-by-entry. Let \( c \oplus d := s(c + d) \). A configuration \( c \) is recurrent if there exists a non-zero configuration \( d \) such that \( c = c \oplus d \). Recurrent configurations play a central role in the dynamics of the Abelian sandpile model since recurrent configurations together with the \( \ominus \) operation form an Abelian group known as sandpile group and denoted \( K(G) \). An introduction to the topic can be found in [1].

For example, the recurrent configurations for the cycle with 5 vertices and sink vertex \( q \) are \( (0,1,1,1,0,1), (1,0,1,1,0,1), (1,1,0,1,0,1) \) and \( (1,1,1,1,1) \). Could the reader verify that these configurations form an Abelian group with the \( \ominus \) operation? Which configuration is the identity?

Smith normal form and graphs

Let \( GL_n(\mathbb{Z}) \) denote the group of \( n \times n \) invertible matrices with entries in the integers whose inverses also have entries in the integers. Two matrices \( M \) and \( N \) are equivalent if there exist two matrices \( P, Q \in GL_n(\mathbb{Z}) \) such that \( M = QNP \). The Smith normal form of the matrix \( M \) is the unique diagonal matrix \( diag(d_1, ..., d_r, 0, ..., 0) \) equivalent to \( M \) such that \( r \) is the rank of \( M \) and \( d_i | d_j \) for \( i < j \). The integers \( d_1, ..., d_r \) are called invariant factors.

Let \( G \) be a planar graph with \( s \) interior faces \( F_1, ..., F_s \), let \( c(F_i) \) denote the number of non-edges in the cycle bounding \( F_i \). We define the cycle-intersection matrix, \( C(G) = (c_{ij}) \) to be a symmetric \( s \times s \) matrix, where \( c_{ii} = c(F_i) \), and \( c_{ij} \) is the negative of the number of common edges in the cycles bounding \( F_i \) and \( F_j \), if \( i \neq j \).

Lemma [2]. Let \( d_1, ..., d_r \) be the invariant factors of \( C(G) \), where \( G \) is a planar graph. Then \( K(G) \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r} \).

Sandpile groups of outerplanar graphs

We call a graph outerplanar if it has a planar embedding with the outer face containing all the vertices. The weak dual graph \( G^* \) is constructed the same way as the dual graph but without placing the vertex associated with the outer face. A graph \( G \) is biconnected outerplanar if and only if its weak dual is a tree. Note that \( C(G) + A(G) = diag(c(F_1), ..., c(F_s)) \), where \( A(G) \) is the adjacency matrix of \( G \). A 2-matching \( M \) is a set of edges of a graph \( G \) such that each vertex of \( G \) is incident with at most 2 edges of \( M \). Denote by \( G^o \), the graph \( G \) where each vertex has a loop added. Given a 2-matching \( M \) of \( G^o \), let \( \Omega(M) \) denote the set of loops in \( M \). A 2-matching \( M \) of \( G^o \) is minimal if there is no 2-matching \( M' \) of \( G^o \) such that \( \Omega(M') \) is not contained in \( \Omega(M) \) and \( |M'| \leq |M| \). The set of minimal 2-matchings of a tree with loops \( T^o \) with \( k \) edges will be denoted by \( 2M_k(T^o) \). Let \( d(M) \) denote the determinant of the submatrix of \( C(G) = diag(c(F_1), ..., c(F_s)) - A(T) \) created by taking the rows and columns associated with the loops of \( M \) of \( T^o \).

Theorem [2]. Let \( G \) be a planar biconnected graph whose weak dual is the tree \( T \) with \( n \) vertices. Let \( \Delta_k = gcd((d(M) : M \in 2M_k(T^o))) \). Then the spanning-tree number \( t(G) \) coincides with \( \Delta_n \) and \( K(G) \cong \mathbb{Z}_{\Delta_1} \oplus \mathbb{Z}_{\Delta_2} \oplus \cdots \oplus \mathbb{Z}_{\Delta_n} \).

References

Properties of fullerene graphs with icosahedral symmetry

Thiago M. D. Silva, Diego Nicodemos and Simone Dantas

A graph $G$ is planar if it has an immersion in the plane so that its edges intersect only at their endpoints. The diameter of a graph is the maximum distance between any pair of vertices of $G$. As an example, Figure 2 displays the Fullerene graph $C_{60}$, it is planar (no two edges intersect each other); it is 3-regular (all vertices have degree 3); and it is 3-connected (it remains connected if we remove one or two edges). Fullerene graphs are 3-regular and 3-connected planar graphs with only pentagonal and hexagonal faces. Figure 2 shows the Fullerene graph $C_{60}$.

The Fullerene graph with icosahedral symmetry have exactly 12 pentagonal faces. All other faces are hexagons. Moreover, their pentagonal faces shape the planning of an icosahedron. They are described by $G_{i,j}$, $i, j \in \mathbb{N}^*$, $j \geq i$, where $i$ and $j$ determine the distance between the vertices, with $i$ as the number of hexagons in direction $\mathbb{F}$ and $j$ as the number of hexagons in direction $\mathbb{F}'$. Figure 3 displays the planning of the graph Fullerene graph with icosahedral symmetry $G_{1,4}$. The proofs of both theorems are based on vectorial operations of the vector $\mathbb{F}$ and $\mathbb{F}'$ and the hexagonal lattice’s symmetry characteristics. Figure 4 displays the results of Theorems 1 and 2 for the graph $G_{1,4}$. As a visual proof of Theorem 1, note that the blue triangle corresponds to the graph $G_{0,3}$ entirely included in $G_{1,4}$. Similarly for Theorem 2, the red triangle corresponds to the $G_{1,4}$, which wholly contains the graph $G_{1,4}$.
A relationship between D-eigenvalues and diameter.

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Objective

Our goal is to provide examples of connected graphs having diameter d and less than d + 1 D-eigenvalues. This answers a question stated by Atik and Panigrahi in [4, Problem 4.3].

Introduction

It is known, by [1], that if G is a graph of diameter d then the adjacency matrix of G has at least d + 1 distinct eigenvalues. We can see in [2] that distance-regular graphs actually attain this minimum, that is, they have exactly d + 1 distinct adjacency eigenvalues.

A simple connected graph G is called distance-regular if it is regular, and if for any two vertices x, y ∈ V(G) at distance i, there are constant number of neighbors c_i and b_i of y at distance i − 1 and i + 1 from x, respectively.

![Graph](image1)

Figure 1: C_5 and Petersen graph are examples of distance regular graphs. More generally, C_n is a distance regular graph.

It seems reasonable to ask whether these results can be extended to the eigenvalues associated with the distance matrix (D-eigenvalues) of a simple connected graph G. Indeed, Lin et al. [5] ask if, for a graph G with diameter d, its distance matrix has at least d + 1 distinct eigenvalues. Atik and Panigrahi give a negative answer to this problem in [4]. Moreover, they prove that a distance-regular graph with diameter d has at most d + 1 distinct D-eigenvalues and leave the following question: “Are there connected graphs other than distance regular graphs with diameter d and having less than d + 1 distinct D-eigenvalues?”

In what follows, we answer this question positively by given two examples of connected graphs with diameter d having less than d + 1 distinct D-eigenvalues.

Examples

In our example we consider two bipartite graphs G_1 and G_2 described in figures 2 and 3. We have that |V(G_1)| = 20 and |V(G_2)| = 70, and that diam(G_1) = 5 and diam(G_2) = 7. However, both graphs have exactly four distinct D-eigenvalues. These graphs and their respectively D-spectrum are shown as follows.

![Graph](image2)

Figure 2: The graph G_1.

![Graph](image3)

Figure 3: The graph G_2.

\[
spect(G_1) = \begin{bmatrix} 50 & 0 & -2 & -12 \\ 1 & 14 & 1 & 4 \end{bmatrix} \]

\[
spect(G_2) = \begin{bmatrix} 245 & 0 & -5 & -40 \\ 1 & 62 & 1 & 6 \end{bmatrix} \]

Conclusions

About the problem proposed by Atik and Panigrahi in [4], it can be said that there are other connected graphs with diameter d, in addition to distance regular graphs, having less than d + 1 distinct D-eigenvalues. More specifically, the graphs presented in this work have exactly 4 distinct D-eigenvalues. For future works, we are interested in characterize a class of distance-biregular graphs with this property.

References


Acknowledgment
Conflict Free Closed Neighborhood Coloring Game

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Introduction

In Cellular Networks, communication between bases and mobile devices is established via radio frequencies. Interference occurs if one particular device communicates with two different bases that have the same frequency. So, every device must contact a base with an unique frequency and, since having a lot of different frequencies is expensive, it’s important trying to minimize their quantity, in a way that there exists no interference.


Inspired by this problem, and by the well known coloring game, we introduce a game theoretical approach to CFCN coloring, and determine the minimum number of colors necessary for Alice to have a winning strategy in the case of Complete Graphs.

CFCN Coloring

A CFCN coloring of a graph $G$ is an assignment of colors to the vertices of $G$ such that each vertex $v$ in $G$ has an uniquely colored vertex in its closed neighborhood $N[v]$ (the set of all vertices adjacent to $v$ including itself). A CFCN $k$-coloring of a graph $G$ is a CFCN coloring with at most $k$ colors. We say that $N[v]$ is fully colored if each vertex of $N[v]$ has a color assigned to it. A graph together with a CFCN $k$-coloring is said to be CFCN $k$-colored.

CFCN Coloring Game

Given a graph $G$ and $k > 1$ colors, two players, Alice and Bob, take turns coloring vertices of $G$ such that at each turn for every $v$ with a fully colored $N[v]$, the induced subgraph $G[N[v]]$ is CNF $k$-colored. The goal of Alice is to obtain a CFCN $k$-coloring of $G$ while Bob does his best to prevent it. Alice wins if at the end $G$ has a CFCN $k$-coloring; otherwise Bob wins.

We refer to the next figure for a CFCN 2-coloring game on $K_5$, where white vertices are uncolored ones. The game ends on the $4$th turn because, no matter which color Alice chooses for the $5$th turn, it creates a fully colored neighborhood that is not CFCN 2-colored.

The figure below shows a CFCN 3-coloring game on $K_6$, where white vertices are uncolored ones. The game ends on the $6$th turn because the Graph is CFCN 3-colored.

Results

Theorem: Alice wins CFCN $k$-coloring game on a complete graph $G$ on $n$ vertices if and only if $k > \lceil \frac{n}{2} \rceil$.

Sketch of the proof: Let $k > 1$ be the number of available colors. Without loss of generality, Alice starts playing in any vertex with color $1$.

We claim that Alice always wins if $n \leq 4$ and $k = 2$ (winning for any $k$).

Indeed if $n = 1$, Alice colors the vertex with $1$. If $n = 2$, Alice colors a vertex with $1$ and then Bob is forced to color the other vertex with $2$. If $n = 3$, on the $1$st turn she colors a vertex with $1$ and on the $3$rd with $2$. If $n = 4$, Alice guarantees that by the $3$rd turn, without loss of generality, there are two vertices colored with $1$ and one vertex colored with $2$, thus Bob has to finish the coloring with $1$ or another color different from $2$.

If $n > 4$, the proof is based on the following strategy.

Assume that $k \leq \lceil \frac{n}{2} \rceil$, Bob colors a vertex with $2$. If Alice colors the next vertex with $1$ (resp. $2$), Bob colors a vertex with $2$ (resp. $1$). On the following turns, independently of the colors chosen by Alice, Bob chooses the other colors twice and the game ends. If Alice colors the next vertex with a color $c$ not in $\{1, 2\}$, then Bob colors the next ones with $1, 2, c$, and then chooses the remaining colors twice. In any case Bob wins the game.

Now assume $k > \lceil \frac{n}{2} \rceil$. If the number of vertices is even then Alice always plays $1$. If the number of vertices is odd then Alice does the same strategy until her last turn, in which she chooses $1$ or one of the remaining colors. In both cases, Alice wins because the graph doesn’t have enough vertices for Bob to guarantee that each color is used twice.

References


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TEACHING NEWTON’S BINOMIAL WITH GENETICS

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Table 1: Cross between two heterozygotes

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<tr>
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Application

In genetics, phenotype refers to characteristics of the individual that can be visible or detectable, and polygenes are groups of genes that produce repeated variations. Polygenic inheritance refers to a single inherited phenotypic trait that is controlled by two or more different genes. The interaction that occurs between genes (polygenes) that convey the inherited characteristics happens in such a way that each one of them is responsible for a portion of the resulting phenotype. The pattern of inheritance distribution, in this case, follows the pattern of Newton’s Binomial, \((p + q)^n\), where \(n\) is the number of polygenes, \(p\) represents the dominant genes (B and G) and \(q\) represents the recessive (b). In our study, we develop Newton’s binomial for the eye color problem [3].

The eye color results from at least two genes. The first, OCA2 (oculocutaneous albinism II), comes in two forms: B (brown) and b (blue). The second gene, called GEY (green eye color), comes also in two forms: G (green) and b (blue). The first thing to notice is that the gene B is dominant over both G and b. And, as well, G is dominant over b (recessive). In other words, a person heterozygote BbGb, despite having the gene G, she has brown eyes. Thus, we could calculate the probability of their progenies being born with brown, green or blue eyes shown in Table 1 [4]. Other genes produce spots, rays, rings and pigment diffusion patterns.

References


Acknowledgement

The teaching of Combinatorial Analysis is still done in a very mechanical way by some teachers who, for the most part, memorize formulas without real content domain. This practice is repeated superficially, thus not stimulating combinatorial reasoning [1]. The vast majority of books and websites present this content only through formulas, without showing their relationship to applicability, making it difficult for students to learn. Thus, we present an application of Newton’s Binomial, as a way of intuitively teaching such content. Since the binomial is used in many areas, we choose an interdisciplinary study with Biology, more specifically, in Genetics. In this work, we show how the binomial is presented in Genetics and why it is so important to understand certain characteristics inherited from our ancestors, such as the color of the eyes. We use concepts of Polygenic (or Quantitative) Inheritance [2].

The aim of this work is to present a new way of teaching Newton’s Binomial through an interesting application related to Genetics, without the early use of formulas. In addition, we show the relationship between the binomial and the combinatorial analysis: how is the combination present in terms of the binomial and what do they represent in its expansion?

The methodology consisted of studying applications in genetics that involve Newton’s Binomial; choosing an application and developing playful material for teaching the content which included simulations and short films.
The strong pseudoachromatic number of split graphs

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Introduction

Given a graph \( G \) and a set of colors \( C \), a vertex coloring \( \alpha: V(G) \to C \) is an assignment of colors from \( C \) to the vertices of \( G \). If there are no adjacent vertices with the same color, \( \alpha \) is proper. Let \( \beta \) be a not necessarily proper vertex coloring of \( G \) such that for every two distinct colors, there are adjacent vertices in \( G \) assigned these colors. If \( \beta \) is proper, then it is an achromatic (or complete) coloring of \( G \). If \( \beta \) is nonproper, then it is a pseudoachromatic (or nonproper complete) coloring of \( G \). If \( \beta \) is a pseudoachromatic coloring of \( G \) and for every color \( i \), there is an edge of \( G \) whose both vertices are colored \( i \), then \( \beta \) is a strong pseudoachromatic (or strong nonproper complete) coloring of \( G \). (See Figure 1.) The maximum number of colors of a strong pseudoachromatic coloring is its strong pseudoachromatic number (or strong achromatic number), \( \psi^*(G) \).

![Figure 1: A strong pseudoachromatic coloring for \( P_5, P_3, C_5 \) and \( K_{3,3} \).](image)

Previous results

Although there are many studies of the achromatic coloring (see Chartrand and Zhang [1, p. 329]), the only published paper on strong pseudoachromatic coloring is by Liu, Li, and Liu [2]. They present bounds for the strong pseudoachromatic number in the general case and determine the strong pseudoachromatic number of complete graphs, paths, cycles, complete multipartite graphs, complete biequipartite graphs from which a perfect matching is deleted, wheels, fans, and some line graphs.

Motivation

Let \( G \) be a graph and \( \beta: V(G) \to C \) be a pseudoachromatic coloring of \( G \). By the definition of pseudoachromatic coloring, for each color \( i \in C \), there must be an edge whose both vertices are colored \( i \). So, \( |C| \) is at most the size of a maximum matching of \( G \), denoted by \( \alpha^*(G) \). Consequently, \( \psi^*(G) \leq \alpha^*(G) \). By the previous results [2], this upper bound is tight, since \( \psi^*(G) = \alpha^*(G) \) when \( G \) is a complete graph or a complete multipartite graph. (See Figure 2.)

![Figure 2: A maximum strong pseudoachromatic coloring and a maximum matching (in red) of \( K_5 \) and \( K_{2,2,3} \).](image)

Sketch of proof. Since \( \psi^*(G) \leq \alpha^*(G) \) for any graph \( G \), it is sufficient to exhibit a strong pseudoachromatic coloring with \( \alpha^*(G) \) colors. Let \( Q \) be a maximum clique in \( G \). Consider a maximum matching \( M_B \) in \( B_G \) and a maximum matching \( M_Q \) in \( G \setminus V(B_G) \). For each edge in \( M_B \cup M_Q \), assign a new color to its vertices (the same color for both vertices). Assign a color previously used to the remaining vertices. Since, for each color \( i \), there is a vertex in \( Q \) colored \( i \) and an edge of \( M_B \cup M_Q \) whose vertices are colored \( i \), we have a strong pseudoachromatic coloring. \( \Box \)

Our contribution

\textbf{Theorem 2} If \( G \) is a split graph, then
\[ \alpha'(G) = \alpha'(B_G) + \left( \omega(G) - \alpha'(B_G) \right) \frac{1}{2} \]

\textbf{Theorem 3} If \( G \) is a split graph, then \( \psi^*(G) = \alpha^*(G) \).

Historical context

Chartrand and Zhang [1, p. 329] presented the strong pseudoachromatic coloring (they use the term "nonproper complete coloring") in the Study Project 6 [1, p. 442]. They ask for bounds to the pseudoachromatic number in terms of the number of edges and suggest investigating the strong pseudoachromatic number of paths and graphs in general.

A graph \( G \) is a split graph iff \( V(G) \) can be partitioned into a maximum clique \( Q \) and a stable set \( S \). Figure 3 exhibits a split graph. The size of \( Q \) is denoted \( \omega(G) \). The bipartite subgraph of \( G \) obtained by removing the edges between vertices of \( Q \) is denoted \( B_G \).

![Figure 3: A maximum strong pseudoachromatic coloring of a split graph.](image)

References

Enumeration of cospectral and coinvariant graphs

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Introduction

Starting from the eigenvalues of a matrix associated to a graph, spectral graph theory seeks to deduce combinatorial properties of the graph. For this, we associate a graph $G$ to a matrix $M$ and analyze the eigenvalues of $M$. Motivated by the graph isomorphism problem, it is of interest to study, for a graph $G$, what fraction of all graphs is uniquely determined by the $M$-spectrum of $G$. We propose representing a graph using the Smith Normal Form (SNF) of certain distance matrices. We provide numerical evidence that this algebraic representation may do a better job in distinguishing graphs.

Spectrum and invariant factors

The eigenvalues of a matrix $M(G)$ associated with a graph $G$ are called the $M$-spectrum of $G$, which is the multiset that allows multiple instances for each of its eigenvalues. $M$-cospectral graphs are graphs that share the same $M$-spectrum.

The Smith Normal Form of an integer matrix $M$, denoted by $SNF(M)$, is the unique diagonal matrix such that $SNF(M) = \text{diag}(d_1, \ldots, d_r, 0, \ldots, 0) = PMQ$ for invertible matrices $P, Q \in \text{GL}(n, \mathbb{Z})$ such that $r$ is the rank of $M$ and $d_i | d_j$ for $i < j$. The invariant factors of $M$ are the integers in the diagonal of $SNF(M)$. We say that graphs $G$ and $H$ are $M$-coinvariant, if the SNFs of integer matrices $M(G)$ and $M(H)$ are the same.

 Enumeration

We focus on the following matrices for connected graphs: the adjacency matrix $A$, the Laplacian matrix $L$, the distance matrix $D$, the signless Laplacian matrix $Q$, the distance Laplacian matrix $D^L$ and the distance signless Laplacian matrix $D^Q$.

Extensive research has been devoted to understand cospectral graphs, but much less has been dedicated to understand coinvariant graphs and its potential to characterize graphs. The reason for this could be that for matrices $A, L, Q$ and $D$, there is a large proportion of connected graphs having a coinvariant graph, as Figure 1.1 shows.

Figure 1.2 displays the number of cospectral and coinvariant graphs for matrices $D^L$ and $D^Q$. We also include the spectral graphs for matrix $Q$, since according to Figure 1.1, this would be the best invariant for distinguishing graphs using only the spectrum. According to our results, the SNF of $D^Q$ performs better than the spectrum for distinguishing graphs for the considered matrices. Details can be found in [1].

Coinvariant trees

Aouchiche and Hansen reported in [2] enumeration results on cospectral trees with at most 20 vertices with respect to $D$, $D^L$ and $D^Q$ matrices. For $D$, they found that among the 123,867 trees on 18 vertices, there are two pairs of $D$-cospectral trees. Among the 317,955 trees on 19 vertices, there are six pairs of $D$-cospectral trees. There are 14 pairs of $D$-cospectral trees over all the 823,065 trees on 20 vertices. Surprisingly, after the enumeration of all 1,346,023 trees on at most 20 vertices, they found no $D^L$-cospectral trees and no $D^Q$-cospectral trees. This fact led Aouchiche and Hansen to conjecture that every tree determined by its distance Laplacian spectrum, and by its distance signless Laplacian spectrum.

Analogously, for the SNF of $D$, $D^L$ and $D^Q$ of trees, one can obtain some similar insights. Hou and Woo obtained in [3] that the SNF of the distance matrix for any tree with $n + 1$ vertices equals $I_n \oplus I_{n-2} \oplus (2n)$. From which follows that all trees with $n$ vertices are $D$-coinvariant graphs. On the other hand, after enumerating coinvariant trees with at most 20 vertices with respect to $D^L$ and $D^Q$, we found no $D^L$-coinvariant trees and no $D^Q$-coinvariant trees among all trees with up to 20 vertices. This fact led us to conjecture that all trees are determined by the SNF of $D^L$, and, analogously, by the SNF of $D^Q$.

References

Intersection models for 2-thin and proper 2-thin graphs

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Thinness and proper thinness

A graph $G = (V, E)$ is $k$-thin if there exist an ordering and a $k$-partition of $V$ s.t., for $u < v < w$, if $u, v$ belong to the same class and $uv \in E$, then $vw \in E$. The minimum such $k$ is called the thinness of $G$ and denoted $\text{thin}(G)$ [1].

Interval graphs are exactly the 1-thin graphs, and 2-thin graphs include convex bipartite graphs. Complements of induced matchings have unbounded thinness.

2-diagonal box intersection models

A set of boxes drawn with sides parallel to the Cartesian axes of the plane is 2-diagonal if their upper-right corners are pairwise distinct and lie in two diagonals $y = x + d_1$, $y = x + d_2$, either in the 2nd or in the 4th quadrant, and weakly 2-diagonal if there is no quadrant restriction.

A 2-diagonal and a weakly 2-diagonal model.

Blocking models

A model is blocking if for two non-intersecting boxes $b_1, b_2$ in the upper and lower diagonal, resp., either the vertical prolongation of $b_1$ intersects $b_2$ or the horizontal prolongation of $b_2$ intersects $b_1$.

An L-graph is a graph $G$ is proper 2-thin if and only if it has a blocking 2-diagonal model.

Theorem. A graph is 2-thin if and only if it has a blocking 2-diagonal model.

Blocking 2-diagonal model and not.

Theorem. The following statements are equivalent:
1. $G$ is a proper 2-thin graph.
2. $G$ has a bi-semi-proper blocking 2-diagonal model.
3. $G$ has a bi-semi-proper weakly 2-diagonal model.

Theorem. The blocking property is necessary since there are graphs with thinness 3 and a 2-diagonal model.

Characterizations

The main results of this work are the following characterizations of 2-thin and proper 2-thin graphs as intersection graphs of boxes in the $k$-dimensional Euclidean space.

2-thin graphs as VPG graphs

A graph is $B_2$-VPG if it is the vertex intersection graph of paths with at most $k$ bends in a grid. An $L$-graph is a $B_2$-VPG graph admitting a representation with all the paths having the same of the four possible shapes $L, I, \Gamma, \Upsilon$.

- $B_2$-VPG graphs have unbounded thinness.
- 2-thin graphs are $L$-graphs (thus $B_2$-VPG).
- The wheel $W_4$ is 2-thin and not $B_2$-VPG.
- 3-thin graphs are $B_3$-VPG.

Bonus track: new upperbound

The pathwidth (resp. bandwidth) of a graph $G$ can be defined as one less than the maximum clique size of an interval (resp. proper interval) supergraph of $G$, chosen to minimize its maximum clique size [3]. It was proved in [1] that $\text{thin}(G) \leq \text{pw}(G) + 1$.

We prove that, if $|E(G)| \geq 1$, then $\text{pthin}(G) \leq \text{bw}(G)$.

References

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On the thinness and proper thinness of a graph.
Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques.

1 Due to lack of space, some standard graph classes, graph parameters, and small graphs are not defined here. The definitions of those concepts can be found in http://graphclasses.org
A Near-tight Bound for the Rainbow Connection Number of Snake Graphs

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Introduction

The rainbow connection number of a connected graph $G$, denoted $rc(G)$, is the least $k$ for which $G$ admits a (not necessarily proper) $k$-edge-coloring such that between any pair of vertices there is a path whose edge colors are all distinct. This parameter has important applications [3].

Remark ([1]) If $G$ is a connected and not trivial graph with $n$ vertices, then $diam(G) \leq rc(G) \leq |E(G)|$.

We present a near-tight bound for the rainbow connection number of snake graphs, a class commonly studied in labeling problems [2, 5].

Let $\ell \geq 3$, $k \geq 1$, $n \geq 2$. An $\ell$-gon $k$-multiple snake graph over $n$ vertices, denoted $S(\ell, k, n)$, is obtained from $P_n$: $v_0 v_1 \ldots v_{n-1}$ by adding $k$ multiple edges between $v_i$ and $v_{i+1}$ for $0 \leq i \leq n - 2$ and making $\ell - 2$ successive subdivisions at each edge added. See Fig. 1.

![Figure 1. $S(7, 4, 6)$](image)

The rainbow connection number is already known [4] for $G = S(3, k, n)$ with $k \in \{1, 2, 3\}$. In this case,

$$rc(G) = \begin{cases} \frac{diam(G)}{2} + 1, & \text{if } n = k = 3; \\ \frac{diam(G)}{2}, & \text{otherwise}. \end{cases}$$

Result

Lemma

Let $G = S(\ell, k, n)$.

$$diam(G) = \begin{cases} \ell / 2, & \text{if } n = 2 \text{ and } k = 1; \\ \ell - 1, & \text{if } n = 2 \text{ and } k > 1; \\ 2 \lfloor \ell / 2 \rfloor + n - 3, & \text{if } n > 2. \end{cases}$$

Theorem

Let $G = S(\ell, k, n)$.

$$rc(G) \leq \begin{cases} diam(G) + 1, & \text{if } \ell \text{ is even or } n = 2; \\ diam(G) + 2, & \text{if } \ell \text{ is odd.} \end{cases}$$

This bound is near-tight, since we know snake graphs which have $rc(G) = diam(G) + 1$.

Proof (sketch). Fig. 2 shows a rainbow coloring of the block $B_{i,i+1}$, for $0 \leq i \leq n - 2$.

![Figure 2. $B_{1,1+1}$](image)

References


Reduced indifference graphs are type 1
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Introduction

In the year that Celina Figueiredo, João Meidanis and Célia de Mello celebrate another decade of life, we point out the following result which is an immediate consequence of their papers.

Corollary 1
All reduced indifference graphs are type 1.

Let $G$ be a simple graph. A total coloring is an assignment of colors to the vertices and edges of $G$ such that no two adjacent or incident elements receive the same color. See Fig. 1.

Figure 1: A total coloring of the Hajós graph.

The minimum number of colors for a total coloring of $G$ is the total chromatic number, $\chi''(G)$. By definition, $\chi''(G) \geq \Delta(G) + 1$. Vizing and Behzad posed the famous Total Coloring Conjecture.

**Theorem 1** [3]
To decide if a cubic bipartite graph $G$ has $\chi''(G) = \Delta(G) + 1$ is NP-complete.

**Total Coloring Conjecture (TCC)** [1, 2]
$\chi''(G) \leq \Delta(G) + 2$

If $G$ has $\chi''(G) = \Delta(G) + 1$, it is type 1, otherwise it is type 2. By Theorem 1, it is NP-complete to decide if a graph is type 1 for the general case.

**Theorem 2** [4]
If $G$ is dually chordal, the TCC holds. Moreover, if $\Delta(G)$ is even, $G$ is type 1.

The proof of Theorem 2 gives a polynomial-time algorithm that yields an optimum total coloring of dually chordal graphs with even maximum degree.

Reduced indifference graphs

$G$ is an indifference graph if and only if its vertices can be ordered such that those that belong to the same maximal clique are consecutive. This order is known as indifference order. Two vertices are true twins if they are adjacent and belong to the same maximal cliques. A graph is reduced if it does not contain true twins. See Fig. 2.

Figure 2: A reduced indifference graph.

Celina Figueiredo, Célia de Mello and Carmen Ortiz [5] presented the following interesting property on indifference graphs.

**Theorem 3** [5]
If $G$ is an indifference graph that does not contain maximum degree true twins, then $G$ is type 1.

Fig. 3 exhibits an indifference graph and a matching that covers its maximum degree vertices.

Figure 3: A matching according to Theorem 3.

We use the same technique presented in the proof of Theorem 2 and the property presented in Theorem 3 to prove Theorem 4 and, consequently, Corollary 1. Our proof also gives a polynomial-time algorithm for an optimum total coloring of reduced indifference graphs.

New result

**Theorem 4**
If $G$ is an indifference graph that does not contain maximum degree true twins, then $G$ is type 1.

**Sketch of proof.** If $\Delta(G)$ is even, $\chi''(G) = \Delta(G) + 1$, by Theorem 2. Suppose that $\Delta(G)$ is odd. Since $G$ does not contain maximum degree true twins, it has a matching $M$ that covers all maximum degree vertices, by Theorem 3. By Theorem 2, $\chi''(G - M) = \Delta(G)$. Consider an optimum total coloring of $G - M$ as in the proof of Theorem 2. Assign a new color for the edges of $M$. If the endvertices of an edge in $M$ receive the same color, $G$ has maximum degree true twins, a contradiction. So, $\chi''(G) = \Delta(G) + 1$. □

Fig. 4 presents a total coloring for the graph of Fig 2.

Figure 4: An optimum total coloring according to Theorem 4.

Corollary 1 is an immediate consequence of Theorem 4.

References

Introduction

The problem of grid embedding is that of drawing a graph $G$ onto a rectangular two-dimensional grid (called simply grid) such that each vertex $v \in V(G)$ corresponds to a grid point (an intersection of a horizontal and a vertical grid line) and the edges of $G$ correspond to paths of the grid. Grid embedding of graphs has been considered with different perspectives [2, 5, 6]. In [5], linear-time algorithms are described for embedding planar graphs having their edges drawn as non-intersecting paths in the grid, such that the maximum number of bends of any edge is minimized, as well as the total number of bends.

Objective

We are interested in embedding trees $T$ with $\Delta(T) \leq 4$ in a rectangular grid, such that the vertices of $T$ correspond to grid points, while edges of $T$ correspond to non-intersecting straight segments of the grid lines. The aim is to minimize the maximum number of bends of a path of $T$. We provide a quadratic-time algorithm for this problem. With this algorithm, we obtain an upper bound on the number of bends of EPG models of VPT∩EPT graphs [3, 4].

Embedding trees in a grid

Let $T$ be a tree such that $\Delta(T) \leq 4$. Consider the problem of embedding such a tree in a grid $G$, so that the vertices must be placed at grid points and the edges drawn as non-intersecting paths of $G$ with no bends, which we will call a model of $T$. See Figures 1-5 for key notations.

Given a model $M$, let $b(p, v)$ be the maximum number of bends of a path in $M$ having as extreme vertices $p$ and a leaf $l \in V(T)$, over all paths that contain $v \in V(T)$. Let $b(M)$ be a model of $T$ and $v \in V(T)$. Let $N(v) = \{u | u \in V(T) \land 1 \leq i \leq \Delta(v)\}$ be the neighborhood of $v$. Then $b(M) \leq b(M') = b(T) - \min \{b(p, v) | v \in V(T)\}$.

Given a model $M$, let $b(M)$ be the maximum number of bends of a path in $M$ having as extreme vertices $p$ and a leaf $l \in V(T)$, over all paths that contain $v \in V(T)$.

Question

Given $T$ and $M$, the algorithm returns $b(M)$.

Algorithm 1: Determining $b(T)$

Input: a tree $T$ such that $\Delta(T) \leq 4$.

Output: a model $M$ of $T$ such that $b(M) = b(T)$.

Let $G = (G_1, G_2, \ldots, G_m)$ be a family of graphs such that $G_i$ is incrementally built by $G_{i-1}$.

Let $M_0$ be a model having a single vertex $v_1$ at some grid point.

For $i = 2, \ldots, m$ do:

Add to $M_0$ the vertices $v_i$ attached to the grid point of $v_i$, in any free horizontal or vertical grid line of $v_i$.

Finally, let $M$ be the model.

Procedure Balance $(M, p, v_i)$

If $v_i$ is not balanced, then this is balanced by recursively calling in $M$ the drawing of the four subgraphs of $v_i$ nested in $u_i$ (for $1 \leq i \leq 5$, if $v_i$ potentially exists and preceding them is $v_i$) (hidden in gray).

Theorem

Given a tree $T$, let $b(T)$ be the maximum number of bends of a path of $T$. Then, $b(M) = b(T)$.

EPG models of VPT∩EPT graphs

We provide an upper bound on the number of bends of an EPG representation of VPT∩EPT graphs. The VPT∩EPT graphs are those that can be represented in host trees with maximum degree at most 3 [3]. In [1], this class is characterized by a family of minimal forbidden induced subgraphs. An EPG model $R = \{P_i | 1 \leq i \leq 10\}$ is shown in Figure 6, obtained from the family $P = \{P_i | 1 \leq i \leq 10\}$.

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Acknowledgment

On Embedding Trees in Grids

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**B₁-EPG representations using block-cutpoint trees**

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**Introduction**

EPG graphs were first introduced by Golumbic et al in [2] motivated from circuit layout problems [1]. In $B₁$-EPG representations, each path has one of the following shapes $\varphi = \{(\uparrow, \downarrow), \ast\}$, besides horizontal or vertical segments. One may consider more restrictive subclasses of $B₁$-EPG by limiting the types of bends allowed in the representation, that is, only the paths in a subset of $\varphi$ are allowed.

Ex.: The $\varphi$-EPG graphs are those in which only the $\uparrow$ or the $\ast$ shapes are allowed.

**Objective**

We show that two superclasses of trees are $B₁$-EPG (one of them being the cactus graphs). On the other hand, we show that the block graphs are $\varphi$-EPG and provide a linear time algorithm to produce $\varphi$-EPG representations of generalization of trees. These proofs employed a new technique from previous results based on block-cutpoint trees of the respective graphs.

**Preliminaries**

We describe a $B₁$-EPG representation of a superclass of trees, inspired on the representation of trees described in [2]. The novelty of our results is the usage of BC-trees to obtain EPG representations, which will be employed to obtain $B₁$-EPG representations of more general classes of graphs.

**Theorem 1**

Let $G$ be a graph such that every block of $G$ is $B₁$-EPG and every cut vertex $v$ of $G$ is a universal vertex in the blocks of $G$ in which $v$ is contained. Then, $G$ is $B₁$-EPG.

**Proof.** (Sketch) The theorem is proved by induction. Actually, we prove a stronger claim, stated as follows: given any graph $G$ satisfying the theorem conditions and a BC-tree $T$ of $G$ rooted at some cut vertex $r$, there exists a $B₁$-EPG representation $R = \{P_r \mid v \in V(G)\}$ of $G$ in which:

i. $P_r$ is a vertical path with no bends in $R$;

ii. all paths but $P_r$ are constrained within the horizontal portion of the grid defined by $P_r$ and at the right of it.

From $T$ (the BC-tree of $G$ shown in Figure 2), build the representation $R$ of $G$ as follows. First, build an arbitrary vertical path $P_r$ in the grid $G_r$ corresponding the root $r$. Next, divide the vertical portion of $G$ defined by $P_r$ and at the right of it into $v$ vertical subgrids $G_1$, $G_2$, $\ldots$, $G_v$, with a row space between them such that the $i$-th subgrid will contain the paths corresponding to the cut vertices that are descendants of $P_r$ in $T$. So, each subgrid $G_i$ is constructed as shown in Figure 3.

We first represent the children of $P_r$ as disjoint $\varphi$-shaped paths, sharing the same grid column in which $P_r$ lies. For each $B₁$ we build the paths in $B₁$'s that correspond to vertices of $B₁$ that are not cut vertices of $G$ (as those in black in Figure 1), and the paths in $T_e$, belonging to $G[T_e]$, for all $1 \leq j \leq f_r$. So, it remains to define how the paths belonging to the regions $B₁$'s and $T_e$'s will be built.

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**Acknowledgment**
Contact L-graphs and their relation with planarity and chordality

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B₀-CPG graphs

- An undirected graph \( G = (V, E) \) is called a VPG graph (1) if one can associate a path in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point.
- An undirected graph \( G = (V, E) \) is then called a \( B₀ \)-VPG graph, for some integer \( k \geq 0 \), if one can associate a path with at most \( k \) bends in a rectangular grid with each vertex such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point.
- An undirected graph \( G = (V, E) \) is said to be \( B₀ \)-CPG if one can associate a horizontal or vertical path in a rectangular grid with each vertex, such that two vertices are adjacent if and only if the corresponding paths intersect on at least one grid-point without crossing each other and without sharing an edge of the grid.

![Figure: On the left, a \( B₀ \)-CPG representation of the graph on the right.](image)

L-Contact graphs

- An \( L \)-graph is a graph with a \( B₀ \)-VPG representation such that all the paths in the representation have the shapes \( \{(V, V')\} \). We will say that the graph is an \( L \)-graph if the paths only have the shape \( \{(V, V')\} \).
- An \( L \)-contact graph is an \( L \)-graph such that all the paths in the representation do not cross each other and do not share an edge of the grid.
- A representation of a \( L \)-contact graph such that no path intersects another in a bend point will be called a basic representation.

![Figure: Two representations of \( K₅ \) as a \( L \)-contact graph.](image)

Relation with planarity

- \( B₀ \)-CPG \( \subseteq \) \( L \)-contact and there are non-planar \( B₀ \)-CPG graphs.

![Figure: On the left, a \( B₀ \)-CPG representation of the non-planar graph on the right.](image)

- As a consequence, there are non-planar \( L \)-contact graphs.
- \( L \)-contact \( \subseteq \) \( B₀ \)-CPG.

**Theorem (13)**

For every \( k \geq 0 \) there is a planar graph \( G \) such that \( G \) is \( B₀ \)-CPG but not \( B₀ \)-CPG.

**Theorem**

If \( G \) is \( L \)-contact then \( G \) is planar.

![Figure: The planar representation obtained from the \( L \)-contact representation of a graph.](image)

Laman graph

- A Laman graph is a graph on \( n \) vertices such that, for all \( k \), every \( k \)-vertex induced subgraph has at most \( 2k - 3 \) edges, and such that the whole graph has exactly \( 2n - 3 \) edges.
- \( \mathbf{An \ Laman \ graph \ is \ a \ graph \ such \ that, \ for \ all \ k, \ every \ k-vertex \ induced \ subgraph \ has \ at \ most \ 2k - 3 \ edges.} \)
- \( \mathbf{An \ Laman \ graph \ is \ a \ graph \ such \ that, \ for \ all \ k, \ every \ k-vertex \ induced \ subgraph \ has \ at \ most \ 2k - 3 \ edges.} \)

**Theorem (14)**

If a graph \( G \) has a maximal \( \mathbf{-L \contact \ representation \ in \ which \ each \ inner \ face \ contains \ the \ right \ angle \ of \ exactly \ one \ , \ \ then \ G \ is \ a \ planar \ Laman \ graph.}\)

- As a consequence, we have the following result.

**Theorem**

Every maximal \( \mathbf{-L \contact \ graph \ is \ a \ planar \ Laman \ graph.}\)

Relation with chordality

**Lemma**

A clique in a \(\mathbf{-L \contact \ graph \ has \ size \ at \ most \ three.}\)

**Theorem**

Let \( G \) be a chordal graph. \( G \) is \( \mathbf{-L \contact \ if \ and \ only \ if \ G \ is \ \mathbf{K₄-free.} \) Moreover, \( G \) admits a basic representation.

Let \( T \) be the family of graphs defined as follows. \( T \) contains \( K⁵ \) as well as all graphs constructed in the following way: start with a tree of maximum degree at most three and containing at least two vertices; this tree is called the base tree; add to every leaf \( v \) in the tree two copies of \( K₄ \) (sharing vertex \( v \)) and to every vertex \( w \) of degree 2 one copy of \( K₄ \) containing vertex \( w \). Notice that all graphs in \( T \) are chordal.

![Figure: On the left the graph \( K₅ \). On the right a typical graph in \( T \).](image)

**Theorem (12)**

Let \( G \) be a chordal graph. \( \mathbf{F} \subseteq \{T \} \). Then, \( G \) is a \( B₀ \)-CPG graph if and only if \( G \) is \( \mathbf{F \free.} \)

- It is immediate that \( \mathbf{-L \contact \ graphs \ are \ \mathbf{K₄free.} \)
- Following the same ideas as in the chordal \( B₀ \)-CPG characterization, all the graphs in \( \mathbf{F} \) are forbidden subgraphs.
- As a consequence, we have the following result concerning block graphs.

**Theorem**

Let \( G \) be a block graph. \( G \) is \( \mathbf{-L \contact \ if \ and \ only \ if \ G \ is \ \mathbf{B₀ \-CPG.} \)

References

Introducción

Todas las gráficas con las que trabajamos son gráficas finitas y simples. Este trabajo fue motivado por un problema abierto del artículo de **Graphs whose complement and square are isomorphic**. Dada una gráfica $G$, el cuadrado de ésta denotado por $G^2$, es la gráfica que consta del mismo conjunto de vértices de $G$, pero $uv \in E(G^2)$ si y solamente si la distancia de $u, v$ en $G$ es $1, 2$. $G$ denota al complemento de la gráfica $G$, en la cual tenemos el mismo conjunto de vértices y $uv \in E(G)$ solo si $uv \notin E(G)$. Decimos que una gráfica $G$ es cuadrado complementaria si cumple $G^2 \cong G$ o equivalentemente $G \cong G^2$. Usaremos el término de **cuadró complementario**, por su abreviatura en inglés, *square-complement*. Algunos ejemplos de gráficas cuadró son la gráfica trivial $K_1$, el ciclo de $7$ vértices $C_7$ y la gráfica de Franklin:

$$
\begin{array}{c}
\text{Figura 1: Ejemplos de gráficas cuadrado-complementarias.}
\end{array}
$$

Objetivo

Dar una respuesta al problema abierto planteado en [1, 4]: Dado un número par $d \geq 4$, ¿existe una gráfica $d$-regular con $d^2 + d + 1$ vértices?

Desarrollo del problema

Por definición si tenemos una gráfica $d$-regular, tenemos que $\forall x \in V(G)$ tenemos que $\deg_G(x) = d$. La longitud del ciclo más pequeño dentro de una gráfica se denomina cuello ($g$) y es denotada por $g(G)$. Notemos que si tenemos una gráfica cuadrado-complementaria no trivial $G$ que sea $d$-regular, entonces $G$ tiene como máximo $d^2 + d + 1$ vértices, debido a que $G$ es regular de grado $d$, sin pérdida de generalidad escogemos un vértice cualquiera llamámosle $v$ el cual tiene $d$ vecinos, a una distancia dos de $v$ tenemos a los más $d(d - 1)$ vecinos más y por último el número de vecinos a una distancia de al menos $3$ tenemos a esto ya que $G$ satisface la condición de ser cuadrado-complementaria, $G \cong G^2$ y estos últimos vecinos a distancia $3$ serán los vértices a distancia $1$ de $G$ en $G^2$ que también tienen que ser $d$-regular. Por lo tanto $G \cong G^2 \leq 1 + d(d - 1) + d = d^2 + d + 1$. Recordamos que la estructura descrita no depende del vértice elegido al inicio pues $G$ es $d$-regular. Cuando consideramos la longitud del ciclo más pequeño, i.e. el cuello de $G$, se satisface que $g(G) \geq 5$ y si sólo si $G^2 = d^2 + d + 1$. Las gráficas que buscamos deben cumplir $g(G) = 3$ y ser $4$-regulares, además de ser cuadrado complementarias lo que implica que $|G| = 21$. Un ejemplo de la estructura buscada es:

$$
\begin{array}{c}
\text{Figura 2: Anadijo para $G$ cuadrado-complementario $4$-regular con $g(G) = 5$.}
\end{array}
$$

El problema se encontraba en encontrar el conjunto de $26$ aristas que completaba a $G$ de la figura $2$, y la hipótesis de ser cuadrado-complementaria de un conjunto total de $108$ posibles aristas; las cuales proviene de las $26 = 108$ a este conjunto le quitamos las aristas que forman un $3$-ciclo en cada conjunto de vértices que se encuentran a distancia $2$ del vértice superior ($12$ aristas), dando como resultado un conjunto de $26$ aristas: $$
\{ (2, 3), (4, 5), (6, 7), (8, 9), (10, 11), (12, 13), (14, 15), (16, 17), (18, 19), (20, 21), (22, 23), (24, 25), (26, 27) \}.
$$

Algoritmo

Después de varias versiones que lograban disminuir el tiempo y el conjunto de posibles soluciones al problema obtuvimos un algoritmo que se compone de varias funciones, las cuales verifican cada una de las características que buscamos verificar que satisfagan las gráficas que buscamos. Entre las cuales se encuentran:

- Que no existan trígonos, ciclos de tamaño $3$, en $G$ y $G^2$.
- Que no existan cuadrados, ciclos de tamaño $4$, en $G$ y $G^2$.
- Función que analiza las simetrías en la gráfica.
- Verifica que cada vértice no exceda el grado $4$.

$$
\begin{array}{c}
\text{Figura 3: Anadijo con configuraciones diferentes pero isomorfas.}
\end{array}
$$

Resultados

Después de las horas empleadas a programar dicho algoritmo que ayuda a saber si existen las gráficas cuadrado-complementarias $4$-regular con cuello $5$, logramos dar respuesta de que dichas gráficas no existen, reduciendo el conjunto de búsqueda considerablemente para dar solución a la interrogante en tan solo $10$ minutos. Además de buscar alguna característica que ayude a reducir más el conjunto de posibilidades para dar una prueba con un número pequeño de casos. Claro el trabajo continúa analizando que pasa para el caso general con gráficas cuadrado-complementarias $d$-regulares con $g(G) \geq 5$, y en particular para $d \geq 6$.

Conclusiones

Después de las horas empleadas a programar dicho algoritmo que ayuda a saber si existen las gráficas cuadrado-complementarias $4$-regular con cuello $5$, logramos dar respuesta de que dichas gráficas no existen, reduciendo el conjunto de búsqueda considerablemente para dar solución a la interrogante en tan solo $10$ minutos. Además de buscar alguna característica que ayude a reducir más el conjunto de posibilidades para dar una prueba con un número pequeño de casos. Claro el trabajo continúa analizando que pasa para el caso general con gráficas cuadrado-complementarias $d$-regulares con $g(G) \geq 5$, y en particular para $d \geq 6$.

Referencias


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Total Coloring in Some Split-Comparability Graphs

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1 Introduction
Let $G$ be a simple graph. For $S \subseteq V(G) \cup E(G)$ and $C \{1, 2, \ldots, k\}$, let $c : S \to C$ be a mapping such that $c(x) \neq c(y)$ for each adjacent or incident elements $x, y \in S$. We say $c$ is a $k$-total coloring when $S = V(G) \cup E(G)$ and a $k$-edge coloring when $S = E(G)$. See Fig. 1 for an example. The least $j$ and the least $k$ for which $G$ has a $j$-total coloring and a $k$-edge coloring are denoted by $\chi'(G)$ and $\chi''(G)$, respectively.

![Figure 1: 9-total coloring for $G$](image)

The Total Coloring Conjecture (TCC) [1, 7] asserts that $\chi''(G) \leq \Delta(G) + 2$ for any $G$. If $\chi''(G) = \Delta(G) + 1$, $G$ is Type 1; otherwise it is Type 2. To decide if $G$ is Type 1 is NP-Complete [6]. A graph $G[Q, S]$ is split if $V(G)$ can be partitioned into $Q, S$ so that $Q$ is a clique and $S$ is an independent set.

**Theorem 1** [2] Let $G$ be a split graph. Then $\chi''(G) \leq \Delta(G) + 2$. In particular, when $\Delta(G)$ is even $G$ is Type 1.

Ortiz and Villanueva [5] characterized the split-comparability graphs.

![Figure 2: 9-edge coloring for $G'$](image)

### 2 Previous Results
When $|E(G)| > \left\lfloor \frac{|V(G)|}{2} \right\rfloor \Delta(G)$ we say $G$ is overfull and if $G$ has a subgraph $H$ with $\Delta(H) = \Delta(G)$ that is overfull, then it is subgraph-overfull. Whenever $G$ is overfull or subgraph-overfull, then $\chi'(G) = \Delta(G) + 1$.

**Theorem 3** [3] A split-comparability graph $G$ has $\chi'(G) = \Delta(G)$ iff $G$ is not subgraph-overfull.

Hilton proved the following result for graphs with a universal vertex, i.e. a vertex with degree $|V(G)| - 1$.

![Figure 3: Extending to a total coloring](image)

### 3 Our Contribution
**Theorem 4** [4] A graph $G$ with a universal vertex is Type 1 iff

$$|E(G)| + \alpha'(G) \geq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor$$

**Theorem 5** A split-comparability graph $G$, with $|Q| \geq |Q_1|$, is Type 1 if

$$|Q| \geq \left( \frac{|S_1|}{|S_1| - 0.5} \right) |Q_1|.$$

**Sketch of proof.** We assume $|S_1| \neq 0$, $|S_1| \neq 0$ and $\Delta(G)$ is odd, otherwise $\chi''(G)$ is known by Theorems 1 and 4. By Theorem 2, $Q_1 \cap Q_1 = \emptyset$. Assume $|Q| \geq |Q_1|$, so $|Q| \leq \frac{|Q_1|}{2}$. We define a split-comparability supergraph $G'$ of $G$ by adding a vertex $v_1$, twin to the largest degree vertex $v_0 \in Q_1$. Since $|Q_1| \geq |Q_1|$, and $|Q| - |Q_1| \geq \frac{|Q_1|}{2}$, $G'$ is not subgraph-overfull. So, it has a $\chi'(G')$-edge coloring $c'$, by Theorem 3. Fig. 2 shows $G'$ obtained from the graph of Fig. 1.

References
Introduction

An L(2,1)-labeling of a simple graph \( G = (V,E) \) is a function \( f: V \to \{0,1,2\} \) such that \( |f(u) - f(v)| \geq 2 \) if \( d(u,v) = 1 \) and \( f(u) \neq f(v) \) if \( d(u,v) = 2 \), where \( d(u,v) \) denotes the distance between two vertices \( u \) and \( v \) of \( G \) and \( t \in \mathbb{N} \). We say that a conflict occurs if any of the necessary conditions to have an \( L(2,1) \)-labeling are not met. The span of an \( L(2,1) \)-labeling \( f \) is the largest integer \( \lambda \) assigned by \( f \) to a vertex of \( G \). The \( \lambda \)-number of \( G \), denoted by \( \lambda(G) \), is the smallest number \( \lambda \) such that \( G \) has an \( L(2,1) \)-labeling with span \( \lambda \). Figure 1 exhibits an \( L(2,1) \)-labeling of the Petersen graph with the smallest span.

The \( L(2,1) \)-labeling problem was introduced by Griggs and Yeh in 1992. The unsolved problem regarding \( L(2,1) \)-labelings is the Griggs and Yeh’s Conjecture, which states that every simple graph \( G \) with maximum degree \( \Delta(G) \geq 2 \) has \( \lambda(G) \leq \Delta(G)^2 \).

Since Griggs and Yeh’s seminal work \( \lambda(G) \) has been determined for various families of graphs [2, 3, 4]. In particular, Georges and Mauro [2] verified Griggs and Yeh’s conjecture for some families of 3-regular graphs and, based on their results, posed Conjecture 1.

**Conjecture 1.** With the exception of the Petersen Graph, every 3-regular graph \( G \) has \( \lambda(G) \leq 7 \).

In this work, we verify Conjecture 1 for a family of Loupekine snarks called \( LP_1 \)-snarks and present a lower bound on \( \lambda(G) \) for its members.

Loupekine Snarks

A snark is a simple, connected, bridgeless 3-regular graph such that its edges cannot be colored with only three colors such that every two adjacent edges are assigned distinct colors. Snarks are related to fundamental problems in graph theory such as the 4-Color Problem and the 5-Flow Conjecture.

Loupekine snarks were originally defined by Loupekine and first presented by Isaacs [1]. \( LP_1 \)-snarks are an infinite family of Loupekine snarks and their construction is presented below.

Let \( k \) be an odd positive integer. A \( k-LP_1 \)-snark \( G \) is constructed from \( k \geq 3 \) subgraphs called blocks, obtained from the Petersen graph \( P \) as follows: given \( k \) copies \( R_0, \ldots, R_{k-1} \) of \( P \), block \( B_i \) is obtained from \( R_i \) by deleting the vertices of an arbitrary path \( P_3 \subset R_i \), for \( 0 \leq i \leq k-1 \). Figure 2 illustrates an arbitrary block \( B_i \) with its vertices named. Vertices \( x_i, u_i, v_i, w_i, y_i \) are called border vertices.

For all \( i \in \{0, \ldots, k-1\} \), the border vertices \( v_i, y_i \) of block \( B_i \) are linked to the border vertices \( u_i, x_i \) of block \( B_{i+1} \) (indices taken modulo \( k \)) by edges called linking edges. The linking edges can be \( \{x_iu_{i+1}, y_iu_{i+1}\} \) or \( \{v_iu_{i+1}, y_iw_{i+1}\} \), but not both.

Any three distinct border vertices \( w_i, w_j, w_k \) are linked to a new vertex \( u_{i,j,k} \) called star vertex, by adding \( u_{i,j,k} \) and three new edges \( w_iu_{i,j,k}, w_ju_{i,j,k}, w_ku_{i,j,k} \) to \( G \). The previous operation can be done an odd number of times, with \( 1 \leq q \leq k \). Since \( k \) is odd, an even number \( k - q \) of border vertices remain. If \( k - q > 0 \), the remaining border vertices are paired up and each pair \( w_i, w_j \) is linked by a new edge \( w_iw_j \), thus concluding the construction of a \( k-LP_1 \)-snark.

Figure 3 shows a 3-\( LP_1 \)-snark with an \( L(2,1) \)-labeling with span 7.

**Results**

**Theorem 1.** Every \( LP_1 \)-snark \( G \) has \( \lambda(G) \leq 7 \).

**Sketch of the proof.** Given a \( k-LP_1 \)-snark \( G \), we construct an \( L(2,1) \)-labeling \( f \) of \( G \) with span 7. Initially, choose a block \( B_0 \) such that its border vertex \( w_0 \) is adjacent to another border vertex \( w_i \) of \( G \). Name this block by \( B_{k-1} \) and name the remaining blocks consecutively from this one. If there is no such block, start the enumeration from any arbitrary block \( i \) with its vertices named. Vertices \( x_i, u_i, v_i, w_i, y_i \) are called border vertices.

If \( k \equiv 1 \mod 3 \), define \( f(u_{i-1}) = f(x_{i-1}) = 1 \) and \( f(v_{i-1}) = f(y_{i-1}) = 3 \). If \( k \equiv 2 \mod 3 \), define \( f(u_{i-1}) = f(x_{i-1}) = 5 \), \( f(v_{i-1}) = f(y_{i-1}) = 3 \) and \( f(t_{i-1}) = 1 \). Other conflicts can occur depending on the adjacency of the star vertices. All of them are resolved so that we finally verify that a valid label can always be assigned for every remaining unlabeled vertex without conflict.

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**References**


Two Level Hamming-Huffman Trees

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Introduction

In information theory, there is a common trade-off that arises in data transmission processes, in which two goals are usually tackled independently: data compression and preparation for error detection. While data compression shrinks the message as much as possible, data preparation for error detection adds redundancy to messages so that a receiver can detect, or fix, corrupted ones. Data compression can be achieved using different strategies, often depending on the type of data being compressed. One of the most traditional methods is the method of Huffman [1], that uses ordered trees, known as Huffman trees, to encode the symbols of a given message. In 1980, Hamming proposed the union of both compression and error detection through a data structure called Hamming-Huffman tree [2], which extends the Huffman tree by allowing the detection of any 1-bit transmission error. Determining optimal Hamming-Huffman trees is still an open problem.

Hamming-Huffman Trees

A Huffman tree (HT) T is a rooted strict binary tree in which each edge (u,v), v being a left (resp. right) child of u, is labeled by 0 (resp. 1) and there is a one-to-one mapping between the set of leaves of T and the set Σ of symbols of the message M to be sent. Given T, each symbol a of M is encoded into a binary string c(a). Such encoding is obtained by the directed path from the root of T to the leaf corresponding to a. Over all possible trees, the HT for M is a tree in which its cost, defined as the sum of p(a)c(a) over all a ∈ Σ, is minimized, where p(a) stands for the probability of occurrence of a and |c(a)| is the length of the string c(a).

A Hamming-Huffman tree (HHT) T is an extension of the HT in which, for each leaf labeled with a ∈ Σ, there exist leaves e1,...,eκ with k = |c(a)| such that each c(ei), 1 ≤ i ≤ κ, differs from c(a) in exactly one position. The leaves e1,...,eκ are called error leaves of a. When c(e) is identified during the decoding process, where e is an error leaf, it means that a transmission error is detected. The cost of HT’s is defined exactly in the same way as the cost of HTs. We define an HHT as optimal if its cost is minimum.

Figure 1 depicts an HT with cost 2.4 and an optimal HHT with cost 3.8, both having 5 symbols with uniform frequencies, that is, symbols with a same probability of occurrence.

Hamming-Huffman trees with leaves in two levels

Consider the problem of finding an optimal HHT for symbols with uniform frequencies. White (resp. black) leaves represent symbol (resp. error) leaves.

References

A graph is a mathematical model used to represent relationships between objects. The main characters that both of these objects and their relationships can assume, allows the construction of the so-called Graph Theory, which has been applied to model problems in several areas, such as Mathematics, Physics, Computer Science, Engineering, Chemistry, Psychology and industry. Most of them are large scale problems.

Fullerene graphs are mathematical models for carbon-based molecules experimentally discovered in the early 1980s by Kroto, Heath, O’Brien, Curl and Smalley. Many parameters associated with these graphs have been discussed to be able to study the stability of fullerene molecules.

By definition, fullerene graphs are cubic, planar, 3-connected with pentagonal and hexagonal faces. The motivation of the present study is to find an efficient method to obtain a 4-total coloring of a particular class of fullerene graphs named fullerene nanodiscs, if it exists.

Teo faces are adjacent if they have a common edge between them in the boundary. We denote the boundary of $f$ by $b(f)$. If $f$ is any face, the degree of $f$ is denoted by $d(f)$ and is the number of edges contained in the closed walk that defines it. In a planar connected graph with $f$ faces, $v$ vertices and $e$ edges, we have that $v + e = 2f - 2$, which is known as Euler’s formula.

### 1. Introduction

A graph is a pair $(V, E)$ where $V$ is a nonempty finite set of vertices and $E$ is a set of edges disjoint from $V$, formed by unordered pairs of distinct elements from $V$. That is, for every $e = (u,v) \in E$, there is a $u \in V$ and $v \in V$ such that $e = (u,v)$, or simply $e = uv$. If $uv \in E$, we say that $u$ and $v$ are adjacent or that $u$ is a neighbor of $v$, and that the edge $e$ is incident to $u$ and $v$. $G$ is an empty graph if $E = \emptyset$. Otherwise, the graph is called disconnected.

### 2. Basic Concepts of Graph Theory

Definition 1. A graph $G = (V(G), E(G))$ is an ordered pair, where $V(G)$ is a nonempty finite set of vertices and $E(G)$ is a set of edges disjoint from $V(G)$, formed by unordered pairs of distinct elements from $V(G)$. That is, for every $e = (u,v) \in E(G)$, there is a $u \in V(G)$ and $v \in V(G)$ such that $e = (u,v)$, or simply $e = uv$. If $uv \in E(G)$, we say that $u$ and $v$ are adjacent or that $u$ is a neighbor of $v$, and that the edge $e$ is incident to $u$ and $v$.

Definition 2. A total coloring of a graph $G$ is a coloring to the set $V(G) \cup E(G)$, $1 \leq k \leq \infty$, such that distinct colors are assigned to:

- Every pair of vertices that are adjacent.
- Every vertex and its incident edges.
- A $k$-total coloring of a graph $G$ is a total coloring of $G$ that uses a set of $k$ colors, and a graph $G$ is a total colorable if there is a $k$-total coloring of $G$. We define the total chromatic number of a graph $G$ as the smallest $k$ such that no $k$-total coloring can be assigned to it. The value of $k$ is denoted by $\chi'(G)$.

Behzad and Vizing independently conjectured the same upper bound for the total chromatic number.

Conjecture 1 (Total Color Conjecture (TCC)).

**TCC**: For every simple graph $G$, $\chi'(G) \leq \Delta(G) + 2$.

The TCC is an open problem, but has been checked for several classes of graphs. Knowing that $\chi'(C_4, C_6) \geq 3$ and from the TCC, we have the following classification: If $\chi'(G) \leq \Delta(G) + 1$, the graph is Type I; if $\chi'(G) = \Delta(G) + 2$, the graph is Type II.

For cubic graphs, the TCC has already been demonstrated, which indicates that these graphs have total chromatic number $\chi'(G) \leq \Delta(G) + 2$. However, the problem of deciding which are Type I or Type II is difficult.

### 3. Fullerene Graphs

#### 3.1 Fullerene: A small history

In 1985 a new carbon allotrope was reported in the scientific community: $C_{60}$. A group of scientists, led by Harry Harwood Harold Walter Kroto and Americans Richard Ewell Smalley and Robert Curl, trying to understand the mechanisms for building long carbon chains observed in interstellar space, discovered a highly symmetrical, stable molecule, composed of 60 carbon atoms different from all the other carbon allotropes.

The $C_{60}$ has a structure similar to a soccer hollow ball (Figure 4), with 32 faces, being 20 hexagonal and 12 pentagonal. Their diameter is called the buckminsterfullerene, in honor of American architect Richard Buckminster Fuller, famous for his geodesic dome constructions, which were composed of hexagonal and pentagonal faces. At the end of the 1980s, other carbon allotrope molecules with similar spatial structure to the $C_{60}$, were reported called fullerene molecules (Kroto et al., 1985).

The buckminsterfullerene was the first new allotrope form discovered in the 20th century, and earned Kroto, Curl and Smalley the Nobel Prize in Chemistry in 1996. Nowadays fullerene molecules are widely studied by different branches of science, from medicine to mathematics. These molecules are supposed to contribute to transport chemotherapy, antibiotics or antitoxins agents and released in contact with deficient cells.

#### 3.2 Fullerene Graphs

Each fullerene molecule can be described by a graph where the atoms and the bonds are represented by the vertices and edges of the graph, respectively. In addition, fullerene graphs preserve the geometric properties of fullerene molecules, so fullerene graphs are planar and connected. Moreover, all vertices have exactly 3 incident edges and all faces are pentagon or hexagon (Nicodemos, 2017).

#### 3.3 Fullerene Nanodiscs

The fullerene nanodiscs, or nanodiscs of radius $\geq 2$, are structures composed of two identical flat covers connected by a strip along their borders. While in the nanodiscs there are only hexagonal faces, in the connecting strip, 12 pentagonal faces are arranged side by side.

A nanodisc of radius $\geq 2$, represented by $D_r$, can be obtained through its flattened. The idea is to arrange the faces in layers around the nearest previous layer starting from a hexagonal face (Nicodemos, 2017).

The sequence $[1, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12]$ provides the amount of faces on each layer of nanodisc planning $D_r$, while $r \geq 2$. In addition, this sequence satisfies the petals that a $D_r$ nanodisc has ($r^2 + 2r + 12$) vertices and $(r^2 + 12r + 36)$ edges. The $2(r + 1)$ pentagonal faces will always be distributed in the same layer with other $(12r + 36)$ hexagonal faces. This is the key property of fullerene nanodiscs.

### 4. Goals

To prove that a cubic graph is Type I, it suffices to show a total coloring with $k$ colors. However, to demonstrate that a cubic graph is Type II, we need to show that it has no total coloring with $k$ colors. Thus, finding Type II cubic graphs is more complicated.

We define the girth of a graph $G$ as the length of its shortest cycle. Until now, every Type II cubic graph we know has squares or triangles. So, we could think that there are no Type II cubic graphs with girth at least 7. Thus, we investigate the following question.

Motivated by this question, we analyze the family of fullerene nanodiscs, in search of evidence that can positively or negatively contribute to this question. In this context, we look for an efficient algorithm to find a $k$-total coloring of the fullerene nanodiscs, if this coloring exists.

### 5. Results

After a few attempts using the brute force method, we were able to obtain a $k$-total coloring of the $D_r$ nanodiscs, with $r = 2$. Therefore, $D_2$ is Type I, which contributes to the evidences that the previously proposed question has a negative answer.

### 6. Conclusion

We will continue the study of total coloring of nanodiscs, looking for an algorithm that gives a total coloring of the graphs of the infinite family of fullerene nanodiscs, also seeking to answer the question previously proposed.

### References

Alliance and Domination on Uniformly Clique-expanded Graphs

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Introduction

This work aims at presenting the uniformly clique-expanded graphs and its results on global defensive alliance and total dominating set problems. Those graphs are related to Sierpiński graphs [5] and subdivided-line graphs [1]. We show the minimum cardinality of the global defensive alliance for some particular situations of uniformly clique-expanded graphs, and we also relate that cardinality to the total dominating set number for graphs having a path or cycle as the root.

Basic Definitions

Consider $G = (V, E)$ a finite, simple, and undirected graph. We write $P_n$, $C_n$, and $K_n$ for a path, cycle, and clique of the order $n$, resp. For the closed (resp. open) neighborhood of a vertex $v \in V$, we denote it by $N[v]$ (resp. $N(v)$). Analogously, we use $N[S]$ (resp. $N(S)$) for the closed (resp. open) neighborhood of a vertex subset $S \subseteq V$. A vertex subset $S \subseteq V$ is said a dominating set if $N[S] = V$. Moreover, we call the subset $S$ by total dominating set only for $N(S) = V$. Now, $S$ is a defensive alliance if it satisfies $|N[v] \cap S| \geq |N(v) \cap (V/S)|$ for every $v \in S$. When $S$ is both a defensive alliance and a dominating set, we say $S$ is a global defensive alliance. We denote $\gamma_1(G)$ (and $\gamma_2(G)$) as the minimum cardinality of a total dominating set (and global defensive alliance) of $G$.

The Main Definition & an Example

We say that a graph $H$ is a uniformly clique-expanded graph if there exist a graph $G$ and a clique $K_n$ with $n \geq \Delta(G)$ (maximum degree of $G$) satisfying:
1. $V(H)$ consists of vertices from $K_n$; each vertex $v$ of $G$, and $E(H)$ contains edges of all clique copies, and every edge $(u)(v)$ linking a vertex $(u) \in K_n$ to some $(v) \in K_n$ since $uv \in E(G)$ and no edges coincide end-vertices in $H$ besides the ones inside of cliques. $G$ is the so-called root of $H$. See an example in Figure 1.

2. $H$ is a uniformly clique-expanded graph

Figure 1: The graph $H$ can be obtained from the root $G$ and the clique $K_n$, and so it is a uniformly clique-expanded graph.

Conclusions & Remarks

The uniformly clique-expanded graphs are particular line graphs of bipartite graphs since we can verify that they are (claw,diamond,odd-hole)-free. Thus, we presented preliminary results that somehow are important to the well-known superclass.

References

[3] HEDENSTROM, M. A. Graphs with large total domination number. J. Graph Theory 56:3 (2007), 21–45 (Sep 2008)

Acknowledgment
INTRODUCTION

The annihilation number is a graph invariant used as a sharp upper bound for the independence number. In this paper, we present bounds and Nordhaus-Gaddum type inequalities for the annihilation number.

We also investigate the extremal behavior of the invariant and showed that both parameters satisfy the interval property. In addition, we characterize some extremal graphs, ensuring that the bounds obtained are the best possible.

ANNIHILATION NUMBER

The independence number of a graph is the cardinality of a largest set of mutually non-adjacent vertices. It is not always possible to determine the number of independence of a graph, since this is a well-known widely-studied NP-hard problem, and for this reason the approximation of the independence number through inequalities represents a relevant research topic.

The annihilation number is a polynomial time computable upper bound for the independence number introduced by R. Pepper and S. Fajtlowicz [1,2].

Definition

The annihilation number of $G$, denoted by $a(G)$, can be defined as the largest integer $k$ such that the sum of the smallest $k$ degrees of graph $G$ was at most its number of edges $e(G)$, that is:

$$a(G) = \max \left\{ k \in \mathbb{N} : \sum_{i=1}^{k} d_i \leq e(G) \right\},$$

where $d_i$ is the $i$-th smallest degree of $G$.

The annihilation number and the independence number are used to investigate the relationships between the reactivity of an organic molecule, represented by a graph, and its independence number. More precisely, the research states that, for a fixed number of vertices, molecules with a lower independence number are, in general, less reactive than molecules with a greater independence number. This study is known in organic chemistry as the independence-stability hypothesis [2].

Acknowledgment:

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Un algoritmo lineal para el problema de la k-upla dominación en grafos web

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Notación:
\[ n, m \in N : n = c(2m + 1) + r, \quad c \in N, \quad 0 \leq r \leq 2m, \]
\[ M := \text{mod}(2m + 1, r), \quad [1, x]_N := \{ z \in N : 1 \leq z \leq x \}. \]

- Para cada \( i \in [1, M] : [i]_M \rightarrow \) clase de equivalencia de \( i \mod M \),
- \( S_i := [i]_M \cap [1, n]_N. \)

Teorema

\[ \gamma_k(W_n^m) = \frac{n}{M}, \quad \forall k \leq 2m. \]

Una subclase de grafos arco-circulares:

**Grafo web:** \( W_n^m \) es un conjunto -upla dominante en \( W_n^m \) con \( n \equiv 0 \mod 2m + 1, l \in \{1, \ldots, m\}. \)

**Definición:** Dado \( W_n^m \), presentar un algoritmo que devuelva un conjunto k-upla dominante en \( W_n^m \) de tamaño \( \gamma_k(W_n^m) \).

**Notación:** \( n, m \in N : n = c(2m + 1) + r, \quad c \in N, \quad 0 \leq r \leq 2m, \]
\[ M := \text{mod}(2m + 1, r), \quad [1, x]_N := \{ z \in N : 1 \leq z \leq x \}. \]

- Para cada \( i \in [1, M] : [i]_M \rightarrow \) clase de equivalencia de \( i \mod M \),
- \( S_i := [i]_M \cap [1, n]_N. \)

**Lema**

- \( \{S_i\}_{i=1}^M \) es una partición de \([1, n]_N\).
- \( |S_i| = \frac{n}{M} \quad \forall i \in [1, M]. \)

Identificamos:

- \( j \in S_i \iff v_{m+j} \in V(W_n^m) \) (suma mod n en los subíndices, en \([1, n]\).)

**Corolario**

\( \{S_i\}_{i=1}^M \) es una partición de \( V(W_n^m) \) en conjuntos de tamaño \( \frac{n}{M} \).

Ejemplo sobre \( V(W_{15}^9) \):

\[ 2m + 1 = 9, \quad r = 6, \quad \text{mod}(9, 6) = 3 = M. \]

**Definición:** Dado \( W_n^m \), para cada \( r \in V(W_n^m) \) y cada \( i \in [1, M] : [i]_M \) se tiene \( [N[r] \cap S_i]_N = l, \) i.e. \( S_i \) es conjunto -upla dominante en \( W_n^m \), donde \( 2m + 1 = lM, \quad l \in N. \)

**Proposición 1:**

- Dado \( W_n^m \) con \( r \in V(W_n^m) \) y \( i \in [1, M] : [i]_M \), se tiene \( [N[r] \cap S_i]_N = l, \) i.e. \( S_i \) es conjunto -upla dominante en \( W_n^m \), donde \( 2m + 1 = lM, \quad l \in N. \)

**Proposición 2:**

- Dados \( W_n^m \) y \( l \in N \) tal que \( 2m + 1 = lM \), se tiene \( \gamma_k(W_n^m) = \frac{n}{M}. \)

**Proposición 3:**

- Para cada \( i \in [1, M] : [i]_M \) se tiene \( S_i = \bigcup_{t \in [0, n/M-1]} \{ w \in [1, n] : w \equiv i + t(2m + 1) \mod n \}. \)

**Definición:** para \( i, j \in V(W_n^m), \) \( j \) es 1-contiguo a \( i \) si

\[ j \equiv i + 2m + 1 \mod n. \]

La 1-contigüedad induce en cada \( S_i \) un ordenamiento tal que, empezando por \( i, \) cada vértice se obtiene del anterior, como un «movimiento circular» de \( 2m + 1 \) posiciones.

**Procedimiento PROC(n,m,i) → devuelve \( \langle S_i \rangle \) \( S_i \) con el ordenamiento.**

**Procedimiento DOM(n, m, (S_i), \alpha) → devuelve un conjunto α-upla dominante en W_n^m donde α ∈ N, \( \alpha \equiv \gamma_k(W_n^m) \leq 2m + 1 \mod M. \)

\[ S_i = \{ v \in V(W_n^m) : j \equiv i + \alpha \mod n \}. \]

**Referencias**

1 Introdução

As rotulações $\mathcal{L}(h,k)$ foram introduzidas como uma generalização natural das rotulações $\mathcal{L}(2,1)$ [1], estas conhecidas por sua importância para o problema de atribuir canais em redes [2].

Rotulação $\mathcal{L}(h,k)$

Sejam $h, k \in \mathbb{Z}_{\geq 0}$ e $G$ um grafo simples. Uma rotulação $\mathcal{L}(h,k)$ de $G$ é uma função $\sigma : V(G) \to \mathbb{Z}_{\geq 0}$ tal que:

(i) $|\sigma(u) - \sigma(v)| \geq h$, $\forall uv \in E(G)$;
(ii) $|\sigma(u) - \sigma(v)| \geq k$, $\forall uw, vw \in E(G)$, $u \neq v$.

Span $\lambda_{h,k}$

Sendo $\sigma$ uma rotulação $\mathcal{L}(h,k)$ de $G$: 

$$\lambda_{h,k}(C_{n}) = \lambda_{h,k}(G) = \min_{\sigma} \{\lambda_{h,k}(\sigma)\};$$

$$\lambda_{h,k}(C_{3}) = 2h + k.$$

O span foi estudado apenas em classes de grafos básicos, como ciclos e caminhos [3], ou classes em contextos muito restritos [1, 4]. Neste trabalho, determinamos o span dos Sunlets $C_{n}$, obtidos a partir do $C_{n}$ adicionando-se um pingente a cada vértice do ciclo.

Outras classes relacionadas que estão sob investigação são os Caterpillars e os Multisunlets, os últimos obtidos adicionado-se possivelmente mais de um pingente a cada vértice do ciclo.

2 O span dos Sunlets

Teorema

Sejam $h, k, n \in \mathbb{Z}_{\geq 0}$ tais que $h \geq k$ e $n \geq 3$. Então:

$$\lambda_{h,k}(C_{n}) = \begin{cases} 
    h + 3k & \text{se } n = 5 \text{ e } h < 2k; \\
    h + 3k & \text{se } n \equiv 0 \pmod{4} \text{ e } h \geq 2k; \\
    h + 4k & \text{se } n \equiv 2 \pmod{4} \text{ e } h > 3k; \\
    2h + k & \text{nos demais casos.}
\end{cases}$$

Esboço de demonstração.

Por contradição, suponha que exista $\sigma$ com span menor do que o enunciado pelo teorema. Os rótulos são particionados em três conjuntos. Por exemplo, nos casos em que $\lambda_{h,k}(C_{n}) = 2h + k$,

$$\mathcal{X}_1 = \{0, 1, ..., h - k - 1\},$$
$$\mathcal{X}_2 = \{h - k, h - k + 1, ..., h + 2k - 1\},$$
$$\mathcal{X}_3 = \{h + 2k, h + 2k + 1, ..., 2h + k - 1\}.$$

Por um lado, mostramos que os rótulos dos vértices do ciclo não podem pertencer a $\mathcal{X}_2$ e, por outro, que não é possível utilizar apenas rótulos de $\mathcal{X}_1$ e $\mathcal{X}_3$ para o ciclo.

(≤) Construa a rotulação por blocos pré-definidos a partir de casos-base.

(Caso $n \equiv 0 \pmod{3}$)

(Caso $n \equiv 1 \pmod{3}$)

Referências

Emparelhamentos perfeitos no produto cartesiano de árvores

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Resumo
Neste trabalho, investiga-se a existência de emparelhamento perfeito no produto cartesiano de duas árvores sem emparelhamento perfeito, focando-se no caso de árvores do tipo caterpillar. Especificamente, é descrita uma família infinita de caterpillars com um número par de vértices e sem emparelhamento perfeito tais que o produto cartesiano de duas quaisquer destas árvores possui emparelhamento perfeito. Palavras-chave: Produto cartesiano de grafos; Emarelhamento perfeito. Caterpillar.

Introdução
Sejam $G_1, G_2$ grafos com conjuntos de vértices $V_1 = \{u_1, \ldots, u_l\}$ e $V_2 = \{v_1, \ldots, v_s\}$, respectivamente. O produto cartesiano de $G_1$ por $G_2$, denotado $G_1 \square G_2$, é o grafo com conjunto de vértices $V = V_1 \times V_2$ no qual $(u_i, v_j)$ e $(u_r, v_t)$ são adjacentes quando $u_i$ é adjacente a $u_r$ em $G_1$ e $j = t$ ou $i = l$ e $v_j$ é adjacente a $v_t$ em $G_2$, $1 \leq i, l \leq r$, $1 \leq j, t \leq s$.

Um emparelhamento em um grafo $G = (V, E)$ é um subconjunto $M$ do conjunto de arestas $E$ tal que nenhum par de elementos de $M$ possui vértice em comum. Dizemos que o emparelhamento $M$ satura um vértice e de $G$ quando alguma aresta de $M$ que incide em $v$. Dizemos que $M$ é um emparelhamento perfeito quando $M$ satisfaz todos os vértices de $G$. Se um grafo $G$ com $n$ vértices admite um emparelhamento perfeito $M$, então $n$ é par e $M$ tem cardinalidade $n/2$. Um grafo que admite um emparelhamento perfeito é chamado perfeitamente emparelhável.

É conhecido [1] que se $G_1$ ou $G_2$ é perfeitamente emparelhável então $G_1 \square G_2$ também é. Em 2015, A. R. Almeida ([2]), exibiu um grafo $G$ sem emparelhamento perfeito tal que $G \square G$ possui emparelhamento perfeito e levanta a questão: como caracterizar grafos $G$ sem emparelhamento perfeito tais que $G \square G$ possua emparelhamento perfeito?

Dizemos que uma árvore $T$ é do tipo caterpillar (ou, brevemente, uma caterpillar) se ao retirarmos todos os vértices pendentes, resta um caminho, chamado corpo da caterpillar.

Neste trabalho, investigamos a questão acima proposta na família das caterpillars.

Uma última definição a ser usada em nosso resultado é dada a seguir.

Definição.[3] Dado $G = (V, E)$, uma partição $P = \{V_i, V_2, \ldots, V_k\}$ de $V$ é dita uma partição por estrelas induzidas de $G$ quando para cada $i$, $1 \leq i \leq k$, o subgrafo induzido $G[V_i]$ de $G$ for isomorfo a uma estrela.

Teorema 1 Seja $C$ uma caterpillar que admite uma partição por estrelas induzidas que, da esquerda para a direita, é descrita como: uma quantidade ímpar de $K_{1,2}$ e sujos centros coincidem com o corpo, seguido por um número par de $K_{1,1}$ e, por fim, outra quantidade ímpar de $K_{1,2}$, então $C \square C$ é perfeitamente emparelhável.

Corolário Sejam $C_1$ e $C_2$ caterpillars tais como a descrita no Teorema 1. Então $C_1 \square C_2$ é perfeitamente emparelhável.

Conclusões
Descrevemos uma família infinita de caterpillars sem emparelhamento perfeito tais que o produto cartesiano de qualquer par delas possui emparelhamento perfeito.

Referências

Agradecimentos

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Connected Matchings

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Introduction

Graph matching problems are well known and studied, in which we want to find sets of pairwise non-adjacent edges[1]. This work focuses on the study of matchings that induce subgraphs with special properties [2][3]. For this work, we consider the property of being connected, also studying it for weighted or unweighted graphs. For unweighted graphs, we want to obtain a matching with the maximum cardinality, while, for the weighted graphs, we look for a matching whose sum of the edge weights is maximum.

Objective

The problem of maximum connected matching is polynomial[1]. We show ideas that lead to two linear algorithms. One of them, having a maximum matching as input, determines a maximum unweighted connected matching. The complexity of the maximum weighted connected matching problem is unknown for general graphs. However, we present a linear algorithm that solves it for trees.

Unweighted Connected Matchings

For a graph G and a matching M, we denote G[M] as the subgraph induced by the vertices of M and N(v) as the set of neighbors of v in G. Note that, in the same graph, the cardinalities of a maximum connected matching and of a maximum weighted connected matching are not always the same. We exemplify in Figure 1. Therefore, we expect that these problems have different computational treatments.

Theorem 1

If G is connected and M is an unweighted maximum matching in G, then the unweighted maximum connected matching has cardinality |M| [2].

An algorithm can dynamically build a maximum connected matching M as follows. From an arbitrary articulation r elected as root, two searches are made. The first computes the vertices from the leaves to the root r. It obtains, for each vertex u, a child vertex s_u of u that maximizes B_u. In addition, B_M is calculated from the sum of B_u for all its children w. The second search is responsible for building M, computing the vertices from r to the leaves, so that, when a vertex v is processed, if v is not part of M yet, we add (s_u, u) to M. In the end, M will be a maximum weighted connected matching.

Acknowledgment

References

Number of spanning trees of a subclass of matrogenic graphs

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Introduction

Laplacian Matrix of a Graph

Definition ([1]) Let \( G = (V, E) \) be a simple graph with \( n \) vertices. The adjacency matrix of \( G \) is the matrix \( A(G) = (a_{ij}) \) with order \( n \), whose entries are given by

\[
a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E \text{ for } v_i, v_j \in V; \\ 0, & \text{otherwise}.\end{cases}
\]

Let \( D(G) \) be the diagonal matrix given by the degree of vertices of \( G \). The Laplacian matrix of \( G \) is the matrix \( L(G) \) defined by

\[
L(G) = D(G) - A(G).
\]

For example,

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
L(G) =
\begin{pmatrix}
2 & -1 & 0 & -1 \\
-1 & 1 & -1 & 0 \\
0 & -1 & 2 & 0 \\
-1 & 1 & -1 & 1
\end{pmatrix}
\]

Preliminaries

Matrix-Tree Theorem

Theorem 1.1 ([3]) The number of spanning trees of a graph \( G \) with order \( n \) is equal to any co-factor of \( L(G) \). In symbols

\[
\text{adj}(L(G)) = \tau(G) \frac{\text{adj}(A(G))}{\text{adj}(D(G))},
\]

where \( \text{adj}(L(G)) \) is the classical adjoint of \( L(G) \), \( \tau(G) \) is the number of spanning trees of \( G \) and \( \text{adj}(A(G)) \) is the matrix with order \( n \times n \) whose entries are all equal to one.

We emphasize that this counting does not disregard isomorphic trees, that is, the number of non-isomorphic spanning trees is less than or equal to the number of spanning trees.

Corollary 1.2 ([1]) Let \( G \) be a connected graph which \( n \) vertices. If \( \mu_1, \mu_2, \ldots, \mu_{n-1} \) are the non-zero eigenvalues of \( L(G) \), then

\[
\tau(G) = \frac{\prod_{i=1}^{n-1} \mu_i}{n}.
\]

This is the spectral version of the Matrix-Tree Theorem, which is very useful, since we've solved the problem of finding the number of spanning trees in a graph to a problem of characterization of Laplacian eigenvalues. For more reference see [4] and [5].

Matrogenic Graphs

Theorem 1.1 ([3]) The number of spanning trees of a graph \( G \) is given by \( \det(A(G)) / \det(D(G)) \). Let

\[
A(G) = (a_{ij}) \quad \text{and} \quad D(G) = \text{diag}\{d_1, d_2, \ldots, d_n\}
\]

where \( d_i \) is the degree of vertex \( v_i \). The number of spanning trees of \( G \) is

\[
\tau(G) = \det(A(G)) / \det(D(G)) = \prod_{i=1}^{n} \frac{d_i^{d_i}}{i^{d_i}}.
\]

Theorem 2.2 ([2]) A graph \( G = (V, E) \) is matrogenic if and only if for any incomparable vertices \( u \) and \( v \) in \( G \), we have that the cardinality of symmetric difference between the sets \( N_G(u) \setminus N_G(v) \) and \( N_G(v) \setminus N_G(u) \) is 2.

Proposition 2.1 The split complete graph is matrogenic.

A Subclass of Matrogenic Graphs

From Theorem 2.2 every matrogenic graph of order \( n \) can be denoted by \( G_d(K \cup S \cup C) \), where \( K \), \( S \) and \( C \) are defined in the same theorem.

Given the non-negative integers \( r \), \( s \), \( t \), we consider the class of graphs \( G_d(K \cup S \cup C) \) such that \( \tau(G) \) can be obtained by

\[
\tau(G) = \frac{\prod_{i=1}^{n} \mu_i}{n}.
\]

The figure below shows the graph \( G_d(CS(3,2), 2K_2) \).

Application

Theorem 3.1 Let \( G = G_d(CS(r, s), tK_2) \), then \( \tau(H) = (r + s + 2t)^{-1}(r + 2t)^{r-1} \).

Sketch of proof. We have

\[
L(H) = \begin{pmatrix}
0 & -1 & 0 & -1; \\
-1 & 0 & -1 & 0; \\
0 & -1 & 0 & -1; \\
-1 & 1 & -1 & 1
\end{pmatrix}
\]

where \( D(K_2) \) is the diagonal matrix of the induced subgraph by \( K_2 \), \( D(S) \) is the diagonal matrix of induced subgraph by \( S \). \( L(S) \) is the identity matrix with order \( r \times r \) and \( 0_{2 \times 2} \) is the matrix with order \( 2 \times 2 \) with all entries is equal to 0.

From eigenvalue calculation techniques we obtain \( r \in \text{Spec}(L(H)) \) with \( m(r) \geq s+1 \), where \( m(r) \) is the algebraic multiplicity of \( r \) as eigenvalue. On other hand, \( r + s + 2t \in \text{Spec}(L(H)) \) with \( m(r + s + 2t) \geq r - 1 \). In addition, we obtain \( r + 2 \in \text{Spec}(L(H)) \) with \( m(r + 2) \geq t \).

By a result about reduced matrices, we obtain that \( r + s + 2t, r, 0 \in \text{Spec}(L(H)) \). So, \( \text{Spec}(L(H)) = \{r + s + 2t, r, 0\} \).

By the Corollary 1.2, the number of spanning trees of \( H \) is

\[
\tau(H) = (r + s + 2t)^{-1}(r + 2t)^{r-1} \).
\]

Corollary 3.2 The number of spanning trees of \( H \) depends of the cardinality of each cell of the partition of vertices in \( H \).

For example, if \( H = G_d(CS(3,2), 2K_2) \), then \( \tau(H) = (3 + 2 + 4)^{-1}(3 + 2)\sum_{i=0}^{3} (3 + 2) = 54/57 \).

References

Arc-disjoint Branching Flows: a study of necessary and sufficient conditions

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Introduction

In this work, we investigate a conjecture by [1] that aims to characterize networks admitting \( k \)-arc-disjoint \( s \)-branching flows, generalizing a result from [2] that provides such characterization when all arcs have capacity \( n−1 \), based on Edmonds’ branching theorem [3].

- **Network:** \( \mathcal{N} = (D, c) \), where \( D = (V, A) \) is a digraph and \( c : A(D) \to \mathbb{Z}_+ \) is the capacity function. For an integer \( \lambda \geq 0 \), we write \( \equiv \lambda \) to state that \( c(\alpha) = \lambda \). An arc \( \alpha \in A(D) \) with tail \( u \) and head \( v \), we may refer to \( \alpha \) as \( uv \).

- **A flow** \( f \) on a network \( \mathcal{N} \) is a function \( f : A(D) \to \mathbb{Z}_+ \) such that \( f(\alpha) \leq c(\alpha), \forall \alpha \in A(D) \).

- **Two flows** \( f_1 \) and \( f_2 \) on a network \( \mathcal{N} \) are arc-disjoint flows if \( f_1(\alpha) \times f_2(\alpha) = 0, \forall \alpha \in A(D) \).

- **The balance** of a vertex \( v \) with respect to a flow \( f \) is \( bal_f(v) = \sum_{\alpha \in A(D)} f(\alpha) - \sum_{\alpha \in A(D)} f(\alpha)^- \). That is, \( bal_f(v) \) is the sum of the flow entering \( v \) minus the sum of the flow entering \( v \).

- **\( s \)-branching flow:** flow \( f \) such that \( bal_f(s) = n−1 \) and \( bal_f(v) = 0 \) for all \( v \in V(D) \setminus \{s\} \).

The hardness of the problem of finding \( k \) arc-disjoint \( s \)-branching flows in a network \( \mathcal{N} = (D, c) \) where \( c \equiv \lambda \), in general, depends on the choice of \( \lambda \). Table 1 summarizes those results.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Hardness</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda \geq n−\ell )</td>
<td>Poly-time solvable for fixed ( \ell ) [6]</td>
</tr>
<tr>
<td>((\log n)^{3+\varepsilon} \leq \lambda \leq n−(\log n)^{3+\varepsilon} )</td>
<td>No poly-time algorithm (unless ETH fails) [1, 5]</td>
</tr>
<tr>
<td>( \lambda \leq n−v )</td>
<td>( \mathcal{N}^\ell )-complete [5]</td>
</tr>
</tbody>
</table>

Table 1: Summary of known hardness and algorithmic results for the problem of finding \( k \) arc-disjoint \( s \)-branching flows in a network \( \mathcal{N} = (D, c) \) with \( c \equiv \lambda \). Here, \( \ell \) is a non-negative integer, \( \varepsilon > 0 \), and \( v = |V(D)| \).

In [1], the authors showed that the following property is a necessary condition satisfied by any network admitting \( k \) arc-disjoint \( s \)-branching flows.

\[
d_i(X) \geq k : \left\lceil \frac{|X|}{\lambda} \right\rceil, \forall X \subseteq V(D) \setminus \{s\}. \tag{Property 1}\]

They also conjectured that Property 1 is a sufficient condition for the existence of \( k \) arc-disjoint \( s \)-branching flows in a network \( \mathcal{N} = (D, c) \) with \( c \equiv \lambda \), for any choices of \( k, \lambda, \) and \( s \). In this work, we prove that their conjecture is true for some graphs, but false in general. An out-branching with root \( r \) is a digraph \( D \) formed by adding parallel arcs to an out-branching with root \( r \). Observe that the underlying simple graph of \( D \), constructed by discarding the orientation of the edges of \( D \) and removing parallel edges, is a tree. See Figure 1 for an example of a multi out-branching with root \( r \) and its underlying simple graph.

Figure 1: Example of a multi out-branching with root \( r \) and its underlying simple graph.

Arc-disjoint branching flows on networks satisfying Property 1

We now state our results.

**Theorem 1.** Let \( \mathcal{N} = (D, c) \) be a network, where \( D \) is a multi out-branching with root \( s \) and \( c \equiv \lambda \). If Property 1 holds for \( D \) with respect to \( k, \lambda \), then \( \mathcal{N} \) admits \( k \) arc-disjoint \( s \)-branching flows.

Figure 2 shows a network satisfying Property 1 for \( k = \lambda = 2 \) that does not contain 2 arc-disjoint \( s \)-branching flows. This statement is formalized by Theorem 2.

**Theorem 2.** Let \( D \) be the digraph shown in Figure 2 and \( \mathcal{N} = (D, c) \) be a network with \( c \equiv \lambda \). Then Property 1 holds for \( \mathcal{N} \) with respect to \( \lambda = 2, s \), and \( k = 2 \), and there are no 2 arc-disjoint \( s \)-branching flows in \( \mathcal{N} \).

Figure 2: A network for which Property 1 holds with respect to \( k = \lambda = 2 \) and the vertex \( s \), but not containing 2 arc-disjoint \( s \)-branching flows.

Future works

In future works, it will be interesting to consider whether there is a version of Theorem 1 for larger classes of digraphs, or whether there is a stronger necessary and sufficient condition that applies to all cases. We remark that, by the results shown in Table 1, we do not expect this condition to be easily verifiable in a given digraph.

In [6] the authors left open the question of whether the problem of finding \( k \) arc-disjoint \( s \)-branching flows in a network \( \mathcal{N} = (D, c) \) with \( c \equiv n−\ell \) is fixed-parameter tractable with respect to \( \ell \). To our knowledge, this question remains open.

References


Acknowledgement
Maximal Independent Sets in Graphs of Girth at Least 6

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1. Introduction

For a graph $G$ we denote by $\alpha(G)$ the maximum size of an independent set in $G$ and by $\nu(G)$ the minimum size of a maximal independent set in $G$. The independence gap of a graph $G$, denoted by $\mu(G)$, is the difference $\alpha(G) - \nu(G)$. Well-covered graphs have independence gap zero. We present characterizations of some graphs with independence gap at least 1 that are of girth at least 6, including graphs with independent gap $r - 1$, for $r \ge 2$, with $r$ distinct and consecutive sizes of maximal independent sets.

Finbow et al. [3] define the set $\mathcal{M}_r$, for every positive integer $r$, to be the set of graphs that have maximal independent sets of exactly $r$ different sizes. If the $r$ different sizes of its maximal independent sets are consecutive, then it is also a member of $\mathcal{I}$, defined by Barbosa and Hartnell [1].

We present results related to the number of trees with specific maximum and minimum sizes of maximal independent sets (MIS). For a graph $G$, $\text{mis}(G) = \{|I| : I \text{ is a MIS of } G\}$. See Figure 1. A vertex is said to be of type $r$ if it is adjacent to exactly $r$ leaves.

2. Results

Before we show some results regarding trees, we present in Table 1 the distribution in the set $\mathcal{I}$ of trees with $n$ vertices, where $0 \le n \le 20$. Not all trees in $\mathcal{M}_r$ belong to $\mathcal{I}$. The data were obtained via a computational program.

In Theorem 1, we show the number of non-isomorphic trees having specific sizes of MIS and prove that there are exactly $\left\lceil \frac{n}{2} \right\rceil - 1$ non-isomorphic trees $T$ with $n$ vertices having $\mu(T) = n - 4$.

3. Theorem 1

Let $n \geq 3$ and $T$ be a tree with $n$ vertices.

1. There are exactly $n - 3$ trees with $\alpha(T) = n - 2$.

2. There are exactly $n - 4$ trees with $\nu(T) = n - 2$.

3. There are exactly $\left\lceil \frac{n}{2} \right\rceil - 1$ trees $\mu(T) = n - 4$.

4. Corollary 3

Let $r \geq 3$ and $G$ be a connected graph of girth at least 6, with exactly two vertices $u_1$ and $u_2$ of type $r$ and with no type $k$ vertices for $k \neq r + 1$. Then $\mu(G) = r - 1$ and only if $u_1$ and $u_2$ are adjacent, any other support vertex of $G$ is type 1, and one of the following two conditions holds:

1. $V(G_0) = \emptyset$.

2. $G_0 \cong K_2$, neither of $u_1$ and $u_2$ has a neighbor in $G_0$, and the two vertices of $G_0$ are of degree 2 in $G$ and are contained in an induced 6-cycle containing $u_1$ and $u_2$.

Moreover, if $V(G_0) = \emptyset$, then $\text{mis}(G) = \{\nu(G) + 1, |V(G)| + 2r\}$ otherwise $\text{mis}(G) = \{\nu(G) + r + 1, |V(G)| + 2r\}$.

5. Proof (Sketch)

Let $F_1$ and $F_2$ be the sets of leaves, respectively, of vertices $u_1$ and $u_2$. Suppose $\mu_u(G) = r - 1$. We claim that the other neighbors of vertices $u_1$ and $u_2$ are vertices of type 1, and $u_1$ and $u_2$ are adjacent. Suppose $V(G_0) \neq \emptyset$. Let $L_1 = N_0(u_1) - (F_1 \cup \{u_2\})$ and $L_2 = N_0(u_2) - (F_2 \cup \{u_1\})$. Let $I'$ be the set of leaves adjacent to vertices of $L_i$, $i = 1, 2$. Now, let $I = F_1 \cup F_2 \cup I_1' \cup I_2'$ and let $G' = G - N_0[I']$. See Figure 2. We also claim that: 1) graph $G'$ is well-covered and has a perfect matching formed by its pendant edges. 2) $G_0$ has only one component that is isomorphic to $K_2$ and their vertices are under a 6-cycle containing $u_1$ and $u_2$. For the converse, we show all possible sizes of MIS considering the two cases: $V(G_0) = \emptyset$ and $V(G_0) \neq \emptyset$. If $V(G_0) = \emptyset$, then $\text{mis}(G) = \{|V(G)| + 1, |V(G)| + 2r\}$ otherwise $\text{mis}(G) = \{|V(G)| + r + 2, |V(G)| + 2r, |V(G)| + 2r + 1\}$. Therefore $\mu_u(G) = r - 1$.

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References


The 3-flow conjecture for almost even graphs with up to six odd vertices

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1. Integer flows

Let \( G = (V(G), E(G)) \) be an undirected graph. Let \( D \) be an orientation for \( E(G) \), and \( f \) an assignment of non-negative integer weights to each edge of \( E(G) \). We say that \( (D, f) \) is a \( k \)-flow for \( G \) if:

1. \( 0 < f(e) < k \), for each \( e \in E(G) \);
2. the flow balance \( \sum_{e \in \partial^+(v)} f(e) - \sum_{e \in \partial^-(v)} f(e) = 0 \), for each \( v \in V(G) \).

where \( \partial^+(v) \) (\( \partial^-(v) \)) is the set of edges leaving (entering) vertex \( v \).

In a mod-\( k \) flow, the flow balance at each vertex \( v \) is \( \sum_{e \in \partial^+(v)} f(e) - \sum_{e \in \partial^-(v)} f(e) \equiv 0 \) (mod \( k \)). Figure 1 shows two graphs that admit a mod-3 flow.

A graph \( G \) admits a \( k \)-flow if and only if it admits a mod-\( k \) flow. Also, if \( G \) admits a mod-\( k \) flow, then it admits a mod-\( k \) flow for any given orientation. See [1], [2] and [3] for more on \( k \)-flows.

Figure 1: Examples of mod-3 flows for graphs \( K_3 \) and \( K_4 \) plus an edge. In both cases, all weights are equal to 1.

2. Tutte’s 3-flow Conjecture and equivalent formulations

A 3-cut is an edge cut of size three. A bridge is an edge cut of size one. Tutte’s 3-flow conjecture is

\textbf{Conjecture (Tutte’s 3-flow conjecture)}

Every bridgeless graph with no 3-cuts admits a 3-flow.

Two equivalent forms of this conjecture are:

- Every bridgeless 5-regular graph with no 3-cuts admits a 3-flow.
- Every bridgeless graph with at most three 3-cuts admits a 3-flow.

3. Objective

In this work, our objective is to characterize classes of graphs with up to four 3-cuts that admit a 3-flow. \( K_4 \), the complete graph on four vertices, is the smallest bridgeless graph that does not admit a 3-flow. We focus on essentially 4-edge connected graphs, i.e., whose edge cuts of size less than four are associated with vertices of degree three (3-vertices). Also, our graphs are almost even, i.e., having at most six odd vertices.

We obtain a characterization for such graphs with up to four odd vertices. We also obtain a partial characterization for graphs with up to four 3-vertices and two odd vertices of higher degree.

4. Motivation

Our motivation is to provide tools for a possible inductive approach to prove Tutte’s 3-flow conjecture.

5. Graphs with exactly four vertices of odd degree

Let \( G \) be an essentially 4-edge connected, almost even, graph having at most four odd vertices, with \( S \) its set of odd vertices. We say that \( G \) has a forbidden configuration if:

(i) the vertices of \( S \) all have degree three; (ii) \( G \) contains \( K_{3,3} \); and (iii) every even-degree vertex \( v \) of \( G \) is separated from \( S \) by an edge cut of size at most four. We abuse this definition by saying that \( K_4 \) has a forbidden configuration.

\textbf{Theorem 1}

An essentially 4-edge connected, almost even, graph \( G \) with at most four odd-degree vertices admits a 3-flow, if and only if \( G \) does not have a forbidden configuration.

6. Graphs with exactly six vertices of odd degree

We give a partial characterization of almost even graphs with six odd-degree vertices that admit a 3-flow. By using the same definition of forbidden configuration to graphs with four 3-vertices and two odd-degree vertices of degree greater than 3, we obtain

\textbf{Theorem 2}

Let \( G \) be an essentially 4-edge-connected, almost even, graph with 4 3-vertices and two other odd vertices of degree greater than 3, and assume \( G \) has a forbidden configuration. Then, \( G \) admits a 3-flow if and only if there are no 4-cuts separating the 3-vertices from the remaining odd vertices.

\textbf{Sketch of proof:} (if) We contract a set \( X \) that contains the two odd vertices with degree higher that three, and having an associated edge-cut of size six (e.g. \( V(G) \) minus the vertices of degree three). By Theorem 1, the resulting graph admits a 3-flow, that can be extended to \( G/X \). This is a 3-flow for \( G \).

(only if) We contract a set \( X \) that contains the two odd vertices of degree higher than three, with an associated edge-cut of size four. By the previous theorem, \( G/X \) does not admit a 3-flow, and so neither does \( G \).

\textbf{References}


\textbf{Acknowledgements}

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Two representations of the Petersen graph, the C5oC5 graph.

Figure 1: Two representations of the Petersen graph, the C5oC5 graph.

Haynes et al. [4] show upper and lower bounds for the maximum cardinality of 1-independent sets and for the minimum cardinality of 1-dominating sets. For a graph G, Chellali et al. [1] present a survey with relations and bounds between α1(G), i(G), γ(G) and Γ(G). Duarte et al. [2] prove that finding α1 of a complementary prism G ¯ is an NP-complete problem.

We present sharp lower and upper bounds for maximum 2-independent sets in complementary prism of any graph, characterize the graphs for which the upper and lower bound holds, and present closed formulas for the complementary prism of paths, cycles and complete graphs.

Relationships between the parameters

Since every set which is both 1-independent and 1-dominating is a minimal 1-dominating set of G, it is easy to see that

γ1(G) ≤ i1(G) ≤ α1(G) ≤ Γ1(G)

for any graph G.

Favaron [3] shows that, for any graph G and positive integer k, γ(k)(G) ≤ α1(G) and i(k)(G) ≤ Γ1(G).

Some general properties:

• Every k-dominating set of a graph G contains at least k vertices and all vertices of degree less than k; so γ(k)(G) ≥ k when n ≥ k.
• Every set with k vertices is k-independent, so i(k)(G) ≥ k when n ≥ k.
• Every set S that is both k-independent and k-dominating is a minimal k-independent set and a minimal k-dominating set.
• Every (k + 1)-dominating set is also a k-dominating set.
• Every k-independent set is also a (k + 1)-independent set.

Results on 2-independent sets in complementary prisms

Haynes et al. [4] show that, for any graph G, α1(G) + α1( ¯ G) ≥ 1 − α1(G ¯ ) + α1(G ¯ ) + α1(G) and both these bounds are sharp. In Theorem 1, we generalize this result for α2(G)Γ(G).

Figure 2: Graph with a maximum 2-independent set highlighted (red vertices) with α2(G Γ(G)) = α2(G) + α2( ¯ G).

Theorem 1

For any graph G,

α1(G) + α1( ¯ G) ≤ α2(G Γ(G)) ≤ α2(G) + α2( ¯ G),

and both these bounds are sharp.

The graph G whose complementary prism G ¯ is shown in Figure 2 attains the lower bound of Theorem 1, and the graph C5, attains the upper bound. In the following result, we characterize the graphs for which the upper bound holds.

Theorem 2

A graph G has α2(G Γ(G)) = α2(G) + α2( ¯ G) if and only if there exist disjoint vertex sets S and T in V(G) such that S is α2(G)-set and T induces a maximum multipartite graph in G such that every partition has size at most two.

Now we show exact values for α2 for some particular graph classes.

Theorem 3

Let n ≥ 5. Then, α2(K(n,n)) = n + 1,

α2(Pn) = \begin{cases} 2 \lfloor n/3 \rfloor + 4, & n \equiv 2 \pmod{3} \\ 2 \lfloor n/3 \rfloor + 3, & \text{otherwise} \end{cases}

α2(C3n) = \begin{cases} 2 \lfloor n/3 \rfloor + 3, & n \equiv 2 \pmod{3} \\ 2 \lfloor n/3 \rfloor + 2, & \text{otherwise} \end{cases}

Future work

As future work, we plan to characterize graphs attaining the lower bound on Theorem 1, to extend the presented results for αk for k ≥ 3, and to study k-dominating sets in complementary prisms.

References

LEXICOGRAPHIC BREADTH-FIRST SEARCH AND EXACT ALGORITHMS FOR THE MAXIMUM CLIQUE PROBLEM

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CONCLUSION

With algorithms modified by LexBFS the search space is significative smaller, but the runtime was significative longer.

References

INTRODUCTION
In this work five exact algorithms for the maximum clique problem (MC) were modified with Lexicographic Breadth-first Search (LexBFS) algorithm. Also, Experimental Analysis of Algorithms and hypothesis test were used to evaluate the changes.

OBJECTIVES
1. Study branch and bound algorithms that use vertex coloring to solve MC modified with the LexBFS algorithm.
2. Apply Experimental Analysis of Algorithms.
3. Apply Statical Inference Theory.

MAXIMUM CLIQUE ALGORITHMS
Branch and bound algorithms for MC evaluate a search space. A small search space may or may not result in shorter runtime.

RESULTS

Comparison of the search space evaluated by two algorithms and their respective modifications. For each vertex set size there is a sample size of 10.

- 80 instances from DIMACS Second Implementation Challenge
- 150 instances: random graphs, chordal graphs and cographs
- C++ implementation in https://gitlab.c3sl.ufpr.br/apzuge/maxcliquebb

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Kernelization lower bounds for Multicolored Independent Set
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1. Introduction
Pre-processing algorithms are frequently employed when solving large problems and are often fundamental to do so. Until recently, however, these algorithms were designed without theoretical guarantees, and measuring their effectiveness was a completely empirical process. Parameterized complexity offers a sound theoretical framework that allows us to prove lower and upper bounds for these kernelization algorithms, as they came to be known in the community [2]. Given an instance \((x, k)\) of a parameterized problem \(\Pi\), we say that \(\Pi\) admits a kernel of size \(g(k)\) when parameterized by \(k\) if we can build an equivalent \(\Pi\) instance of size at most \(g(k)\). Motivated by the fact that Multicolored Independent Set is a central problem in parameterized complexity, we prove the following theorem, where a class \(G\) is non-trivial if, for every \(t \in \mathbb{N}\), \(G\) contains a graph on \(t\) vertices; we point out that Independent Set does admit a polynomial kernel [3] under vertex cover.

2. The theorem
For every fixed non-trivial graph class \(G\), Multicolored Independent Set does not admit a polynomial kernel when jointly parameterized by vertex deletion distance to \(G\) and size of the solution, unless \(\text{NP} \subseteq \text{coNP/poly}\).

3. Multicolored Independent Set
An instance is a pair \((G, \varphi)\) where \(G\) is a graph, \(\varphi\) is a partition of \(V(G)\), and the goal is to find an independent set of \(G\) that hits each part of \(\varphi\) exactly once.

4. Cross-composition
We use the cross-composition framework of Bodlaender et al. [1] to show that 3-COLORING OR-cross-composes into Multicolored Independent Set parameterized by distance to \(G\) and size of the solution. That is, it is a one-to-many reduction with the following constraints:

\[
\begin{align*}
\text{t instances of} & \quad 3\text{-COLORING} \\
V(H_i) & = \{n\} \\
\text{H_i} & \quad \text{Parameters must be bounded by poly}(n + \log t) \\
(G, \varphi) & \quad \text{YES} \text{ if some } H_i \text{ is YES} \\
\end{align*}
\]

5. Instance Selector Gadget
Begin by adding to \(G\) a set \(Y = \{y_1, \ldots, y_t\}\) that induces a graph of \(G\), and add \(Y\) as a part of \(\varphi\).

6. Vertex Gadget
For each \(v \in [n]\), add to \(G\) a gadget \(G_v\) containing an independent set \(A(v) = \{v(a) \mid a \in [n] \setminus \{v\}\}\) and a copy \(K(v)\) of the complete tripartite graph \(K_{n-1,n-1,n-1}\). For each \(G_v\), add parts \(\{p(v, a) \mid a \in [n] \setminus \{v\}\}\) to \(\varphi\); each \(p(v, a)\) contains \(v(a)\) and three non-adjacent vertices \(v_1(a), v_2(a), v_3(a)\) of \(K(v)\). For each \(H_i\), if \(av \notin E(H_i)\), add edges \(\{y_j, v_j(a)\} \in [3]\) and \(\{y_j, a_1(v)\}\). For each \(av \in \bigcup_{i \in \mathbb{N}} E(H_i)\), add \(v_j(a) a_1(v)\) for every \(j \in \{1, 2, 3\}\).

7. Intuition
- For each \(v \in [n]\) we can only choose vertices of one color class of \(K(v)\) to \(v_1(\cdot)\) in the solution \(I\) if and only if we color \(v \in [n]\) with color \(i\).
- If \(v_1(a) \in I\), then \(a_1(v) \notin I \Rightarrow v\) and \(a\) cannot have the same color.
- If \(v_1(a) \in I\), we can ignore edge \(av\) if \(v_2(a) \in I\) and \(a\) cannot have the same color.
- There is a unique \(y_i \in I\) and, for every \(av \in E(H_i)\), \(v_2(a) \notin I\), so some \(v_1(a)\) must be in \(I\) and \(a_1(v)\) must not \(\Rightarrow\) if \(y_i \notin I\), vertices that are adjacent in \(H_i\) cannot have the same color.

References
Partitioning Graphs into Monochromatic Trees
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Introduction

This work presents complexity results about the NP-Completeness of Partition edge-coloured Graphs into vertex disjoint Monochromatic Trees (PGMT) when we restrict the frequency with each color occurs at the edges of the graph. Jin and Li [3] defined the the PGMT problem as follows:

THE PGMT PROBLEM

Instance: An edge-coloured graph G and a positive integer k.
Question: Are there k or less vertex disjoint monochromatic trees which cover the vertices of the graph G?

Figure 1 shows an example of a graph partitioned into monochromatic trees, even in the colored graph with 3 colors, only two trees are sufficient to cover the vertices.

Related Works

• In their work, Jin and Li [3] showed that PGMT is NP-Complete and there is no constant factor approximation algorithm.
• Jin and Li [4] considered a more restricted version of the problem. In this version, the number of distinct colors of G is fixed, and this version is known as r-PGMT, where r is the number of colors. For all r ≥ 5, they showed that r-PGMT is also NP-Complete.
• Jin et al. [2] showed that, for r = 2, r-PGMT is also NP-Complete for bipartite graphs. For complete bipartite and complete multipartite graphs, however, they presented algorithms that solve the problem in polynomial time.

The /MAX-PGMT Problem

Jin and Li [4] considered a version of PGMT where the number of different colors of the graph is fixed. In this work we consider another kind of restriction to the input graph. In this version, instead of fixing the number of different colors, we only guaranteed that each color appears at most f times. We define this version as follows:

THE /MAX-PGMT PROBLEM

Instance: An edge-coloured graph G, where each color occurs at most f times, and a positive integer k.
Question: Are there k or less vertex disjoint monochromatic trees which cover the vertices of the graph G?

NP-Completeness Results

We now show that fMAX-PGMT is NP-Complete, when f = 3, by reducing from Exact Cover by 3-Sets - X3C [1], which is defined as follows:

The X3C Problem

Instance: An set X = {v1, ..., vn}, |X| = 3k; an family of subsets F = {S1, S2, ..., Sq}, Si ⊆ X and |Si| = 3, i ∈ {1, 2, ..., |F|}.
Question: Is there F ′ ⊆ F, such that |∪F ′| = 3k?

We build an instance (G, k + m − 2) of fMAX-PGMT that is equivalent to an instance (X, F) of X3C as follows: The set of vertices is V(G) = {v1, ..., vn, S1, ..., Sn, v1, ..., vn}. The set of edges is

\[ E(G) = \begin{cases} v_iS_j, & \text{if } v_i \in S_j \\ v_iS_k, & \text{if } v_i \not\in S_j \end{cases} \]

for all i ∈ {1, ..., n}, j ∈ {1, ..., m}, p ∈ {i, i + 1, i + 2}. And coloring the edges as follow:

\[ c(e) = \begin{cases} c_j, & \text{if } e = v_iS_j \\ c_{p+1}, & \text{if } e = v_iS_k \end{cases} \]

for all e ∈ E(G), i ∈ {1, ..., n}, j ∈ {1, ..., m}, p ∈ {1, ..., m − 2}, q ∈ {p, p + 1, p + 2}. Figure 2 shows an example of the transformation described: (a) X3C instance and (b) colored graph from that instance.

References


\textbf{A}_α\text{-Spectrum of some Matrogenic Graphs}

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\section{A}_α\text{-Spectral Theory}

\textbf{Definition 1.} Let \(G = (V, E)\) be a simple graph. The matrix \(A_α(G)\) is defined by
\[
A_α(G) = α \cdot D(G) + (1 - α) \cdot A(G), \quad α \in [0, 1],
\]
where \(A(G)\) denotes the adjacency matrix of \(G\) and \(D(G) = (d_{ij})\), is a matrix of order \(n\), where \(d_{ij} = 0\), if \(i \neq j\) and \(d_{ij} = d(v_i)\), if \(i = j\).

\section{Matrogenic Graphs}

\textbf{Definition 2.} A graph \(G\) is matrogenic if for any two vertices \(u\) and \(v\), incomparable in \(G\), we have \(|\{N_G(u) - \{v\}\} \sqcup \{N_G(v) - \{u\}\}| = 2\), where the symbol \(\sqcup\) denotes the symmetric difference.

\textbf{Definition 3.} A graph \(G\) is matrogenic if every vertex of \(C\) induces a perfect matching in \(G\) and \(C\) is a independent set; \(C\) is a independent set.

\textbf{Definition 4.} A split graph \(S(r, s)\) is a graph whose vertices can be partitioned into a clique of size \(r\), and an independent set of size \(s\). A split graph is complete if every vertex in the independent set is adjacent to every vertex in the clique; it is denoted by \(CS(s, r)\).

\textbf{Definition 5.} A graph \(G\) is threshold if for \(u, v \in V(G)\), either \(u\) dominates \(v\) or \(v\) dominates \(u\).

\section{Properties of Matrogenic Graphs}

\textbf{Definition 6.} A perfect matching, \(tK_2\), is the union of \(t\) copies of \(K_2\) and a cocktail party graph, \(CP(2t)\), is the complement of a perfect matching.

\begin{center}
\begin{tabular}{c|c|c}
3\(K_2\) & \(CP(6)\) & \(G_{11}(CS(3, 2), CP(6))\) \\
\end{tabular}
\end{center}

Some properties of the matrogenic graphs: all induced subgraphs of a matrogenic graph are matrogenic; the complement of a matrogenic graph is matrogenic and the class of matrogenic graphs contains the class of threshold graphs. In particular, as the split complete graph is threshold, it is matrogenic.

\textbf{Theorem 1.} A graph \(G = (V, E)\) of order \(n\) is matrogenic if and only if \(V\) can be partitioned into three distinct sets \(K, S, C\) such that
\begin{enumerate}[(i)]
\item \(K \cup S\) induces a matrogenic split subgraph in which \(K\) is a clique and \(S\) is a independent set,
\item \(C\) induces a perfect matching, or a cocktail party, or a \(C\) \\
\item every vertex of \(C\) is adjacent to every vertex of \(K\) and to no vertex in \(S\).
\end{enumerate}

Theorem 1 gives us a way to characterize matrogenic graphs from a partition of its vertex set \(V\). Thus, we can denote every matrogenic graph as \(G_α([K \cup S], [C])\). In the previous figure we show the matrogenic graph \(G_{11}(CS(3, 2), CP(6))\).

\section{A}_α\text{-Spectrum}

In this work, we analyze the \(A_α\)-spectrum of a subclass of matrogenic graphs.

\textbf{Theorem 2.} If \(H = G_α(CS(k, α), CP(2t))\) then \(A_α\)-characteristic polynomial of \(H\) is given by
\[
F_{A_α(H)}(x) = (x - α(2τ + k) + 2α^{-1}((x - α + 1)^{k-1} + (x - α(2τ + k - 2))^2))T(α(k + 2τ - 2) + (1 - α)(2τ - 2)),
\]
where \(f(x) = det(xI - A_α(H))\).

\textbf{Sketch of proof.} There is a labeling of the vertices of the graph \(H\) so that the matrix \(A_α\) can be written \(A_α(H) = B_α + C_α\), where we denote the all-ones matrix by \(J\), the all-zeros matrix by \(0\) and the identity matrix by \(I\). Define \(e_1\) the vector with \(i\)-th coordinate equal to 1 and the others entries are zero.

\begin{center}
\begin{tabular}{c|c|c}
\(B_α\) & \(C_α\) & \(A_α(H)\) \\
\hline
\((1 - α)J_{2t \times k} \quad 0_{2t \times s}\) & \((1 - α)J_{k \times k} \quad 0_{k \times s}\) & \(A_α(H)\) \\
\end{tabular}
\end{center}

For each \(j, i, l\) and \(s\), consider the vectors \(v_j = (v_{j+1} - v_{j+2})T\), \(v_s = (v_{s+1} - v_{s+2})T\) and \(v_{2t} = (0 - v_{2t+1} - v_{2t+2})T\). We have,
\[
A_α(H)v_j = α(j + 2τ - 2)\alpha^j, \quad A_α(H)v_s = (αs - 1)\alpha^s, \quad A_α(H)v_{2t} = αk\alpha^s.
\]

Now, consider the vector \(v_{2t} = (v_{2t+1} - v_{2t+2})T\). Some calculations show that the \(t + 1\) vectors of the form \(v_{2t} = (v_{2t+1} - v_{2t+2})T\), \(2 \leq i \leq t\), are the eigenvectors of \(A_α(H)\) associated with the eigenvalue \(α(k + 2τ - 2) - 2\).

As it was claimed in \([3]\), the matrix \(A_α\) can underpin a unified theory of the spectral study of the adjacency and singless Laplacian matrices of a graph. In this work, we obtain a partial factorization of the \(A_α\)-characteristic polynomial of a subfamily of matrogenic graphs which explicitly gives some eigenvalues of the graph.

\section{References}

Decomposing cubic graphs into locally irregular subgraphs
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Introduction

A decomposition of a graph $G$ is a set $\mathcal{D} = \{H_1, \ldots, H_k\}$ of edge-disjoint subgraphs of $G$ such that $\bigcup_{i=1}^{k} E(H_i) = E(G)$. A locally irregular graph is a graph in which adjacent vertices have distinct degrees.

A locally irregular decomposition (or locally irregular coloring) of a graph $G$ is a decomposition in which every element is locally irregular. We say that $G$ is decomposable if it admits a locally irregular decomposition. Equivalently, a locally irregular decomposition is a coloring of $E(G)$ in which every color class induces a locally irregular subgraph in $G$. If $k$ colors are used, then we say locally irregular $k$-edge-coloring or $k$-LIC for short.

Given a decomposable graph $G$, the irregular chromatic index of $G$ is the smallest number for which $G$ admits a $k$-LIC. We denote the irregular chromatic index of $G$ by $\chi_{irr}(G)$. The problem of computing the irregular chromatic index was proven to be an NP-complete problem [2].

In this work we explore the following conjecture posed by Baudon et al. [1].

Conjecture 1 (O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak, 2015). For every decomposable graph $G$, we have $\chi_{irr}(G) \leq 3$.

Results toward confirming Conjecture 1 include that graphs whose set of vertices can be partitioned into a clique and an independent set admit a 3-LID [3] and graphs with maximum degree at most 3 admit a 4-LID [4]. We explore Conjecture 1 for graphs in which all vertices have degree 3, which are called cubic graphs.

Let $H$ be the graph obtained from $G \setminus E(G[V_{P}])$ by identifying vertices in the same pair or triple, and keeping parallel edges. Note that each path of $P$ has exactly one edge in $H$. The graph $H$ is a bipartite graph with maximum degree exactly 3. It is not hard to prove that $G$ admits a proper edge-coloring with three colors.

Now, we use the proper edge-coloring above to obtain a locally irregular coloring of $E(G)$. By construction every path in $P$ has precisely one edge already colored in $H$, and we color its remaining edge (which is in a cycle of $G[V_{P}]$) with the same color. Since each vertex of $G[V_{P}]$ is in the same pair or triple of at least one of its neighbors in $G[V_{P}]$, each path of $P$ is colored with the same color of at most one path with which it shares a vertex. Therefore each color consists of vertex-disjoint paths of length 2 and trees with four edges and one vertex of degree 3, and hence, is a locally irregular graph.

Locally irregular coloring of some cubic graphs

A proper edge-coloring of a graph $G$ is an assignment of colors to the edges of $G$ in which edges that share a vertex are colored with different colors. A $P_2$-decomposition of a graph $G$ is a decomposition of $G$ into paths of length 2. Let $G$ be a cubic graph, and let $P$ be a $P_2$-decomposition of $G$. Given a vertex $v \in V(G)$, let $d_P(v)$ denote the number of paths $P \in \mathcal{P}$ for which $d_P(v) = 1$, and let $V_P^0$ be the set of vertices $v \in G$ for which $P(v) = i$.

Theorem 1. If $G$ is a cubic graph that admits a $P_2$-decomposition $\mathcal{P}$ for which $G[V_{P}^0]$ is a set of vertex-disjoint cycles, then $\chi_{irr}(G) \leq 3$.

Proof: First note that $\mathcal{P}(v) \in \{1, 3\}$ for every $v \in V(G)$. In particular every vertex of $V_{P}^0$ is the interior vertex of precisely one path of $P$. Since $G[V_{P}^0]$ is a set of vertex-disjoint cycles, every vertex in $V_{P}^0$ is adjacent to precisely one vertex of $V_{P}$ and two vertices of $V_{P}^0$. Given a cycle $C \in G[V_{P}^0]$ we partition the vertex set of $C$ into pairs and at most one triple of consecutive vertices.

In order to prove that some cubic graphs have locally irregular chromatic index at least 3, we define the gadget below which we call a strip. So we have the following theorem.

Theorem 2. If $G$ has a strip $S$ whose vertices with degree 3 are not adjacent to vertices in $V(G) \setminus S$, then $\chi_{irr}(G) > 2$.

Proof: The proof follows from the fact that any 2-LIC of a “half strip” must be as in the figure below, and then the two “half strips” of the same strip cannot be colored in a compatible manner.

By replacing one edge by strip, we can prove that there are infinitely many graphs that do not admit a 2-LIC. In particular, there are an infinite number of cubic graphs with chromatic index 4 and planar graphs that do not admit an 2-LIC, and hence the upper bound of Conjecture 1 is tight for these classes of graphs.

References

Some results on Vertex Separator Reconfiguration

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1. Introduction

Graph reconfiguration problems have been studied extensively in the literature, with INDEPENDENT SET RECONFIGURATION [3] being by far the favorite research topic. Nevertheless, reconfiguration problems of other graph structures, such as vertex covers [4] and vertex colorings [1], have also been investigated. No previous work, however, has dealt with the reconfiguration of vertex separators. In this work, we begin this study in the form of the VERTEX SEPARATOR RECONFIGURATION problem. In this problem, we are given a graph \( G \) and two st-separators \( S_a \) and \( S_b \) of \( G \), and the goal is to reconfigure \( S_a \) into \( S_b \). We prove its complexity on a subclass of bipartite graph under the three most common reconfiguration rules: token sliding (TS), token jumping (TJ), and token addition/removal (TAR); being \( \text{PSPACE}-\text{hard} \) under TS and \( \text{NP}-\text{hard} \) under the other two. We also show that TS and TAR computationally equivalent.

2. Token Sliding

Replace \( v \in A \) with some vertex in \( N(v) \setminus A \).

3. Token Jumping

Replace a vertex of \( A \) with any other vertex of \( G \) not in \( A \).

4. \( k \)-Token Add./Rem.

Add/remove a vertex from \( A \), so long as the resulting \( A' \) satisfies \( |A'| \leq k \).

5. TAR/TJ are equivalent

Let us assume that \( |S_a| \geq |S_b| \) and \( S_a \neq S_b \). We can easily simulate a TJ instance \((G, S_a, S_b)\) just create the TAR instance \((G, S_a, S_b, k + 1)\), where \( k = \max(\{|S_a|, |S_b|\}) \). For the converse, given a TAR instance \((G, S_a, S_b, k)\), if \( |S_a| \neq k \) and \( S_b \) is minimal, then we answer negatively. Otherwise, pick any two st-separators \( S'_a \subseteq S_a \) and \( S'_b \subseteq S_b \) of same size and with at most \( k \) – 1 vertices; it follows that \((G, S'_a, S'_b, k)\) is equivalent to \((G, S'_b, S'_a, k)\) and that we can reconfigure \( S'_a \) and \( S'_b \) into \( S_a \) and \( S_b \) respectively. We can also show that any reconfiguration sequence between \( S'_a \) and \( S'_b \) can be made into an alternating reconfiguration sequence, i.e. it simulates a TJ reconfiguration sequence.

6. Complexity on bipartite graphs

Let \( G \) be a bipartite graph with partition \( A, B \) and \( H \). Our reduction is from INDEPENDENT SET RECONFIGURATION which is \( \text{NP}-\text{complete} \) on bipartite graphs under TJ and \( \text{PSPACE}-\text{hard} \) under TS [3]. Its correctness follows from a simple but powerful observation: A set \( I \subseteq V(G) \) is independent in \( G \) if and only if \( V(G) \setminus I \) is an uv-separator of \( H \). Formally, if \((G, I_a, I_b)\) is the INDEPENDENT SET RECONFIGURATION instance, we construct the equivalent VERTEX SEPARATOR RECONFIGURATION instance, where \( H \) is defined as before.

7. Final Remarks

We investigated the complexity of the reconfiguration of vertex separators under three commonly studied reconfiguration rules. We also showed that TAR and TJ are computationally equivalent. In the arXiv version of this work [2], we also presented polynomial time algorithms for various classes, including series-parallel graphs and graphs with a polynomial number of minimal separators, which have been omitted here.
General Packing Functions of Graphs with few $P_4$’s

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Abstract

We introduce a concept of packing of graphs which generalizes all those previously defined in the literature and we study the computational complexity of computing the associated parameter, the generalized packing number of the graph. We find that this new packing parameter comes to be much more complicated to handle than those previously defined, even on particular graph classes such as quasi-spider graphs. Nevertheless, we prove that the associated optimization problem can be solved in linear time for some graph classes with few $P_4$’s.

General packing functions

Let $G = (V, E)$ be a graph and $k, \ell, u \in \mathbb{Z}_+^\ell$ with $\ell \leq u$. A $(k, \ell, u)$-packing function of $G$ is a function $f: V \to \mathbb{Z}_+$ satisfying the following conditions for all $v \in V$: $f(v) \leq f(e) \leq u(e)$ and $f(N[v]) = k(v)$. In addition, we define $L_{k,\ell, u}(G) = \{ f: f \text{ is a } (k, \ell, u) \text{-packing function of } G \}$. Then, the $(k, \ell, u)$-generalized packing number of $G$ is $L_{k,\ell, u}(G) = \max\{ f(V) : f \in L_{k,\ell, u}(G) \}$. Reduction to instances with $\ell = 0$: $L_{k,0, u}(G) = \ell(V) + L_{k,0,0}(G)$ for $\ell(k) = \ell(N[v])$, $\ell(u) = u(e) - \ell(e)$ $L_{k,0,0}(G) \to L_{k,0,0}(G)$.

Given $f \in L_{k,0,0}(G)$ such that $f(V) = L_{k,0,0}(G)$ we say that $f$ is an optimal $(k, u)$-packing function of $G$. The Packing Function Problem (PFP) has a graph $G$ and vectors $k, u \in \mathbb{Z}_+^\ell(G)$ as input and the objective is to obtain an optimal $(k, u)$-packing function of $G$.

Modular decomposition

The modular decomposition of a graphs involves two operations, union $(\cup)$ and join $(\triangledown)$. If a graph is not connected, it is the union of two graphs and if the complement of a graph is not connected, the graph is the join of two graphs (figure below). A graph is modular if it is connected and its complement is connected. The parameter for these two operations can be computed as follows. Let $k = (k_1, k_2, u = (u_1, u_2), \ell_1 = \min\{k_1(u) u \in V_1\}$ for $i = 1, 2$, $\ell_2(r) = \min\{L_{k_2, \ell_1, u_2}(G_1), k_2\}, \ell_2(r) = \min\{L_{k_2, \ell_1, u_2}(G_2), \ell_2\},$ and $\Delta(s) = \min\{\Delta : s \leq \ell_2(s) - \Delta, \Delta \in [0, \ell_2(s)]\}$. Then, $L_{k_1, u_1}(G_1 \cup G_2) = L_{k_1, u_1}(G_1) + L_{k_1, u_1}(G_2), L_{k_1, u_1}(G_1 \triangledown G_2) = \max\{s + \ell_2(s) - \Delta(s) : s \in [0, \ell_2(0)]\}$.

Example: $G_k = (3,3,3,3,3,3,2,2), u(1,2,1,1,1,1,1,2), k_1(3,3,3,3,3,3,2,2)$

For a graph class $F$, denote by $M(F)$ the class of graphs in $F$ which are modular. From the previous formulas we have the following result.

Lemma 1. Let $F$ be a hereditary family of graphs such that the PFP can be solved in polynomial (resp. linear) time for graphs in $M(F)$. Then, the PFP can be solved in polynomial (resp. linear) time for every graph in $F$.

Thus, let us study the graphs in $F$ for graph classes with few $P_4$’s, such as $P_4$-sparse graphs and $P_4$-tardy graphs.

Spider graphs and $P_4$-sparse graphs

A spider is a graph $G = (V, E)$ such that $V$ is partitioned into sets $S, C, H$ and $L$, where $S = \{ s_1, s_2 \}$ is a stable set, $C = \{ c_1, c_2 \}$ is a clique, $n \geq 2$, and the head $H$ is allowed to be empty. Moreover, all vertices in $H$ are adjacent to all vertices in $C$ and no vertex in $S$. Besides, in a thin spider graph, $s_1$ is adjacent to $c_2$ if and only if $i = j$, and in a thick spider graph, $s_1$ is adjacent to $c_2$ if and only if $i \neq j$. We denote a spider by $(C, S, H)$.

Lemma 2. Let $G = (C, S, H)$ be a thin spider graph. Then

$L_{k,0,0}(G) = u(S) + L_{k,0,0}(G[C] \cup G[H])$.

where $\tilde{k}(h) = k(h)$ for all $h \in H$, $\tilde{k}(c) = k(c) - u(s)$, and $\tilde{u}(c) = \min\{u(c), \tilde{k}(s) - u(s)\}$.

The approach to study the problem in thick spiders is based on technical lemmas. They allow us to reduce the general problem to a particular instance $(k = u)$ in thick spiders with empty head. Then, we apply a transformation from a thick spider with empty head to a particular graph $H_{u,s}$, which has even order and the edges missing form a perfect matching as is shown in the next example.

For a spider graph $G$ and $k = (6,3,5,6,7,5,3)$ we have $k = (7,6,3,5), u = (6,3,5,4) \Rightarrow L_{k,0,0}(G) = L_{k,0,0}(H_u)$

Finally, this process derives the following result.

Proposition 1. If the PFP is polynomial (linear) time solvable on a graph family $F$, then the PFP can be solved in polynomial (linear) time on spider graphs such that the graph induced by the head is in $F$.

From the decomposition results [1, 2], if $F$ is the class of $P_4$-sparse graphs, we know that the graphs in $M(F)$ are the trivial graph and spider graphs such that the graph induced by the head is $P_4$-sparse. Lastly, considering the previous results and Lemma 1, we obtain the next theorem.

Theorem 1. The PFP is linear time solvable for $P_4$-sparse graphs.

Particular case in $P_4$-tardy graphs

A partner of a path $P$ on four vertices in a graph $G$ is a vertex $v \in V(G) \setminus V(P)$ such that the subgraph induced by $V(P) \cup \{v\}$ has at least two paths on four vertices. A graph $G$ is $P_4$-tardy if every path on four vertices has at most one partner.

Considering the problem, a particular case of the PFP is obtained when $u(v) = k(v) = k \forall v \in V$, for $k \in \mathbb{Z}_+$ fixed. In this case the problem is denoted $(k)$-PFP.

Concerning this restricted problem, the linearity result can be settled in $P_4$-tardy graphs, a graph class larger than $P_4$-sparse. Regarding modular decomposition, it is known that a $P_4$-tardy graph $G$ is modular if and only if $G$ is in the trivial graph, $C_4$, $P_4$, $T_k$, or a quasi-spider graph such that the graph induced by the head is not $P_4$-tardy. A quasi-spider graph is a graph obtained from a spider by replacing at most one vertex in $S$ or $C$ by a $K_2$ or a $S_2$.

Based on modular decomposition and applying the results obtained for the mentioned $P_4$-tardy modular graphs, we derive the following.

Theorem 2. The $(k)$-PFP is linear time solvable for $P_4$-tardy graphs.

References

Introdução

Neste trabalho, consideraremos grafos simples, finitos e não direcionados. Considere a situação em que um grafo $G$ modela uma rede de multiprocessadores com dispositivos de detecção colocados em vértices escolhidos de $G$. O objetivo desses dispositivos é detectar e identificar com precisão a localização de um processador defeituoso que pode estar presente em qualquer vértice. Às vezes, esse dispositivo pode determinar se um processador defeituoso está em sua vizinhança, mas não pode detectar se o defeito está em sua própria localização. Como é caro instalar e manter dispositivos de detecção, nessa rede cada detector terá no máximo um dispositivo detector em sua vizinhança. Então, queremos determinar a localização do número mínimo de dispositivos que podem, entre eles, determinar com precisão uma falta em qualquer local. Essa situação é uma aplicação direta do conceito de conjuntos dominantes e independentes abertos. O principal objetivo desse trabalho é a determinação do número de dominação total e independência aberta, $\gamma_{OIND}^{OP}$ para o produto lexicográfico de grafos.

Conceitos Básicos

- **Produto Lexicográfico**
  Sejam os grafos $G$ e $H$, onde os conjuntos de vértices é dado por $V(G) = \{g_1, g_2, \ldots, g_n\}$ e $V(H) = \{h_1, h_2, \ldots, h_m\}$. O **produto lexicográfico** [1, 3], desses dois grafos, representado por $G \circ H$, terá o conjunto de vértices, $V(G \circ H) = V(G) \times V(H)$ e o conjunto de arestas, $E(G \circ H) = \{(g_i, h_j) | g_i \in V(G) \text{ ou } h_j \in V(H)\}$.

- **Conjuntos Dominantes Totais e Independentes Abertos**
  Um conjunto dominante de um grafo $G = (V, E)$, é um subconjunto $D$ de $V(G)$ onde cada vértice que não pertence a $D$ é adjacente a pelo menos um vértice de $D$. O conjunto $D$ é **dominante total** se $\cup_{x \in V} N(x) = V(G)$, ou seja, se $|\{v \in V \cap D\}| > 1$, para todo $v \in V(G)$. Um conjunto independente $S$ de vértices de $G$, tal que não existem dois vértices adjacentes contidos em $S$. O conjunto $S$ de $V(G)$ é **independente aberto** se para cada $v \in S$, $|N(v) \cap S| < 1$.

Denoteremos por $\gamma_{OIND}^{OP}(G)$ a cardinalidade mínima de um conjunto dominante total e independente aberto de um grafo $G$, quando existir.

Trabalhos Relacionados

Seo e Slater [2] definem conjuntos dominantes totais e independentes abertos e consideram propriedades adicionais para esse parâmetro em classes específicas de grafos. Em [4], os autores apresentam resultados de conjuntos dominantes totais para produtos lexicográficos e produtos lexicográficos generalizados. Samodivkin, em [5], mostrou resultados de conjuntos dominantes em subgrafos de um grafo, e demonstrou que os conjuntos dominantes são em alguns casos completos.

Alguns resultados básicos

**Proposição 1** Seja $K_n$ um grafo completo, para $n \geq 2$, então $\gamma_{OIND}^{OP}(K_n) = 2$.

**Proposição 2** Seja $P_n$ um grafo caminho, para $n \geq 2$, então $\gamma_{OIND}^{OP}(P_n) = 2$.

**Proposição 3** Seja $C_n$ um ciclo, com $n \geq 2$ e $n \neq 5$, então $\gamma_{OIND}^{OP}(C_n) = 2$.

**Produtos Lexicográficos**

**Teorema 4** Sejam $G$ e $H$ dois grafos quaisquer. Se $G$ admite um conjunto dominante total e independente aberto, então $\gamma_{OIND}^{OP}(G \circ H) = \gamma_{OIND}^{OP}(G)$.

**Idea da prova:** Seja $G$ um grafo qualquer com $n$ vértices e com conjunto dominante total e independente aberto $D$. Considere $V(H) = \{h_1, h_2, \ldots, h_m\}$ e a componente $G = G \circ H = \{g_1, g_2 \ldots, g_n\}$. Seja $D' = \{(g_i, h_j) | g_i \in V(G) \text{ e } h_j \in D \in \{1, 2, \ldots, m\}\}$. É possível verificar que $D'$ é um conjunto dominante total e independente aberto em $G \circ H$ e $\gamma_{OIND}^{OP}(G \circ H) \leq \gamma_{OIND}^{OP}(G)$.

**Corolário 5** Seja $K_n$ um grafo completo, $C_n$ o grafo ciclo e $P_n$ o grafo caminho com $n$ vértices. Então,

- $\gamma_{OIND}^{OP}(K_n) = 2$, para $n \geq 2$.
- $\gamma_{OIND}^{OP}(C_n) = 2$, para $n \geq 2$ e $n \neq 5$.
- $\gamma_{OIND}^{OP}(P_n) = 2$, para $n \geq 2$.

Agradecimentos

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Referências


Mutually Included Biclique Graphs of Interval Containment Bigraphs and Interval Bigraphs
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Abstract

The recognition of biclique graphs in general is still open. Recently, Groshaus and Guedes introduced the mutually included biclique graph as an intermediate operator to define the biclique graph. Also, we previously studied the biclique graph of interval bigraph and proper interval bigraphs. In this work, we extend the results to a superclass, the interval containment bigraphs, in the context of the mutually included biclique graphs.

Introduction

Given a graph $G$, its biclique graph ($K_{B}(G)$) is the intersection graph of the biclique of $G$. It was introduced by Groshaus and Szwarcfiter in 2010 [5]. They presented a characterization of biclique graphs and a characterization of bigraphs of biclique graphs, but the time complexity of the problem recognizing biclique graphs remains open.

Bicliques in graphs have applications in various fields, for example, biology: protein-protein interaction networks, social networks: community discovery, genetics, medicine, information theory. More applications (including some of these) can be found in the work of Liu, Sim, and Li [9].

A graph $G$ on the real line such that two vertices are adjacent if and only if they are of different parts and the interval containment model $G$. A bipartite graph $G$ is an interval containment bigraph if its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if they are of different parts and one of the corresponding intervals contains the other. Call that family of intervals a interval containment model of $G$.

A bipartite graph $G$ is an interval containment bigraph if its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if they are of different parts and the corresponding intervals intersect. Call that family of intervals a bipartite interval containment model of $G$.

A bipartite graph $G$ is an interval containment bigraph if its vertices can be represented by a family of intervals on the real line such that two vertices are adjacent if and only if they are of different parts and the corresponding intervals intersect. Call that family of intervals a bipartite interval containment model of $G$.

Definitions

1. **CGI**: Containment graph of intervals [2].
2. **ICB**: Interval containment bigraphs [8].
3. **IBG**: Interval bigraphs (IBG $\subseteq$ ICB) [6].
4. **PG**: Permutation graphs = CGI $\cap$ comparability $\cap$ co-comparability [1].

Mutually Included Biclique Graphs of Interval Bigraphs

**KB** and **KB** of ICB and IBG

- **KB** of ICB is $\subseteq$ PG.
- For every $H \in$ PG, there is a $G \in$ IBG such that $H \subseteq KB(G)$.

References

Introduction to the Token Swap (TS) Problem

Let $G = (V, E)$ be a graph with $|V|$ vertices and $|E|$ edges, with distinct tokens placed on it’s vertices. The objective is to reconfigure this initial token placement called $f_0 : V \rightarrow V$ into the identity token placement $f_1$, that maps every node to itself, through a sequence of pairs of adjacent graph vertices that swap the tokens between these vertices. The aim is to know if it is possible to have a swap sequence $S$ that achieve the objective in $k$ or less swaps, with $k \in \mathbb{N}$.

Applications of the TS problem encompass a wide range of fields. From computing efficient interconnection network structures [1], computational biology [2, 3], modelling Wireless Sensor Networks (WSS) [4], protection routing [5] to qubit allocation for quantum computers [6, 7].

Token Swap as a Integer Linear Program

The novel TS problem model given by Formulation (1)-(10) tests all possible configurations of the problem with a given upper-bound in the number of swaps $T$, allowing at maximum one swap per step $t \in [T]$. Each step is composed of a set of variables that describe the current configuration, which swap is being selected and Equation 8 checks if a swap sequence solves the current instance. The constant $T$ can be calculated by using any of the best approximation algorithms, or by using the trivial upper-bound $O(n^2)$ on the size of an optimal swap sequence. Binary variables $x_{iut}$ determine if a token $i$ is at node $u$ in step $t$. The binary variables $y_{uvt}$ flags if a swap happened between nodes $u$ and $v$ in step $t$.

Some techniques are being used in this model to try to achieve a better overall performance, and they will be explained in detail in future papers. The performance measurement, improvements and other models for the problems of Colored Token Swap and Parallel Colored Token Swap are planned for future research. Note that the problem of Parallel Token Swap currently has a model, but it was omitted here for the sake of conciseness. These models differentiate from the usual TS problem by allowing swaps to be done in parallel or by removing the uniqueness property of a token, assigning a color for a set of tokens instead of a single label.

Token Swap on Specific Graph Classes

A Conflic graphical CG = (V, E) is a di-graph that, for a token placement $f$ of a graph $G$, an edge $(u,v) \in E$ if and only if $f(u) = v$. Each node has outdegree 1 and the digraph may contain self-loops.

![Example of a cograph.](image1)

A cograph is defined recursively as follows: a graph on a single vertex is a cograph; if $G_1, G_2, \ldots, G_k$ are cographs, then so is their disjoint union; if $G$ is a cograph, then so is its complement $\overline{G}$. A cotree $T(G)$ of a cograph $G = (V, E)$ is a rooted tree representing it’s structure. The leaves of $T(G)$ are exactly $V$, and each internal node is either a 0-node and 1-node. The children of an 1-node are 0-nodes or leaves and the children of a 0-node are 1-nodes or leaves. Two vertices are adjacent in a cograph if and only if their lowest common ancestor is an 1-node.

We define $CS(CG) = \{C_1, C_2, \ldots, C_k\}$ as the set of permutation cycles of $CG$ for $f$. Let $C^1 \subseteq CS$ be the set of cycles that have a lowest common ancestor of all vertex pairs of $V(C)$ as an 1-node in the cotree or is a cycle of size one and let $C^0 = CS \backslash C^1$. The Cycle Matching Graph $H$ of a cograph $G$ has each cycle on $C^0$ as vertex set and two vertices are adjacent if the lowest common ancestor of all vertex pairs in the vertex union in $T(G)$ is an 1-node. Let $\mu(H)$ be the maximum matching in this graph. The following theorem implies the polynomial time solvability of Token Swap for cographs.

![Cotree and conflict graph joint representation.](image2)

Theorem. Let $G$ be a cograph with an initial token placement $f_0$. The minimum number of required swaps is given by $|V(G)| + |C^0| - |C^1| - 2 \times \mu(H)$.

This result came from two observations: Each independent cycle $C \in CS$ can be solved in $|C| + 1$ or $|C| - 1$ swaps depending on whether this cycle is part of $C^0$ or $C^1$, respectively. Also, it is possible to show that cycle interaction is restricted in the best-case scenario and the best improvement on swaps can be calculated on the value of the maximum matching of the cycle matching graph $H$. This behavior is also being used to find more efficient algorithms in other graph classes like bipartite chain, wheel and gel.

References


On the edge-$P_3$ and edge-$P_3^*$-convexity of grid graphs

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Introduction

Consider a graph $G = (V, E)$ and a subset $C$ of $V$. The $P_3$-convex hull (resp. $P_3^*$-convex hull) of $C$ is obtained by iteratively adding vertices with at least two neighbors in $C$ (resp. two non-adjacent neighbors in $C$). A subset $S$ of $V$ is $P_3$-Helly-independent (resp. $P_3^*$-Helly-independent) when the intersection of the $P_3$-convex hulls (resp. $P_3^*$-convex hulls) of $S \setminus \{v\}$ ($\forall v \in S$) is empty. The $P_3$-Helly number (resp. $P_3^*$-Helly number) is the size of a maximum $P_3$-Helly-independent (resp. $P_3^*$-Helly-independent).

The line graph $L(G)$ of a graph $G$ is the intersection graph of the edges of $G$, i.e., $V(L(G)) = E(G)$ and there is an edge between two vertices in $L(G)$ if the edges they represent in $G$ share a common endpoint. The edge counterparts of $P_3$-Helly-independent and $P_3^*$-Helly-independent of a graph follow the same restrictions applied to its edges instead of its vertices, i.e., the edge $P_3$-convexity (resp. edge $P_3^*$-convexity) of a graph $G$ is described by $P_3$-convexity (resp. $P_3^*$-convexity) of its line graph $L(G)$.

Each vertex of a grid graph $G_{pxq}$ is related to a pair $(x, y)$ that defines its position on the grid with $1 \leq x \leq p$, $1 \leq y \leq q$. There is an edge between two vertices of a grid graph if they share a same coordinate $x$ (or $y$) and the other coordinate differ only by one unit.

In this work, we established the edge $P_3^*$-Helly number of grid graphs $G_{pxq}$ when both $p$ and $q$ are equal or larger than 16. Moreover, we give partial results on forbidden configurations of the edge $P_3$-Helly independent sets of these grid graphs.

![Figure 1: The red edges are an (a) edge $P_3$-Helly independent and (b) edge $P_3$-Helly-independent set of a $G_{3x3}$ and a $G_{4x4}$ grid graphs](image)

Motivations and Related Works

There are many applications for graph convexity on distributed systems[7], social networks, and marketing strategies[6]. Moreover, the excellent survey [5] describes several results of the Helly property on graphs. The problem we address considers the Helly property on the edge $P_3$-convexity and edge $P_3^*$-convexity of grid graphs.

The first results about the $P_3$-Helly number on grid graphs appeared in [1]. Later, several results about $P_3$-Helly number, $P_3^*$-Helly number and their edge counterparts were established in [2] and [3].

Results

In [3] the authors established that the edge $P_3^*$-Helly number of a graph occurs between $|V(G)| - \iota(G)$ and $|V(G)| - \gamma(G)$, where $\iota(G)$ is the minimum independent dominating set and $\gamma(G)$ is the minimum dominating set of a graph $G$.

For grid graphs $G_{pxq}$ with $p \geq 16$ and $q \geq 16$, we have that $\iota(G) = \gamma(G)$. Moreover, the value of $\gamma(G)$ is given by $\lfloor (p+2)(q+2)/4 \rfloor$ [4]. Therefore, we know the $P_3^*$-Helly number of grid graphs $G_{pxq}$ with $p \geq 16$ and $q \geq 16$. The other values for the cases when $p < 16$ and $q < 16$ are computationally obtained. Now, we aim to use these values to obtain the edge $P_3^*$-Helly numbers when $p < 16$ and $q \geq 16$ or $p \geq 16$ and $q < 16$.

We also consider the edge $P_3$-Helly independent of grid graphs by establishing several forbidden configurations that allow us to reduce the patterns of possible optimal solutions. Note that in both cases (a) and (b), the edge $P_3$ convex hull of the red edges without the horizontal edge contains this edge.

![Figure 2: Two forbidden configurations for edge $P_3$-Helly independent sets.](image)

Ongoing works

We are currently trying to establish the edge $P_3$-Helly number of grid graphs. Moreover, we want to extend this study to include the other two types of regular grids: the triangular grids and the hexagonal grids.

References


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Recognition of Biclique Graphs: What we know so far
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Abstract

The recognition of biclique graphs in general is still open. In recent years we presented some results on the characterization of biclique graphs of graphs of certain graph classes, along with the complexity associated to the recognition problem. These results introduced some intermediate operators, which we call now as "functors". In this work we summarize all these results and organize the different approaches using the functors.

Introduction

The biclique graph of a graph $G$, denoted by $B(G)$, is the intersection graph of the bicliques of $G$. The biclique graph was introduced by Groshaus and Szwarcfiter [8], based on the concept of clique graphs. They gave a characterization of biclique graphs (in general) and a characterization of biclique graphs of bipartite graphs. The time complexity of the problem of recognizing biclique graphs remains open.

Bicliques in graphs have applications in various fields, for example, biology: protein-protein interaction networks; social networks: web community discovery, genetics, medicine, information theory. More applications (including some of these) can be found in the work of Liu, Sim, and Li [10].

The efforts since the definition of the problem of recognizing biclique graphs, similarly to what have been done for other graph classes, are mainly focused on understanding the class $KB(A)$, for some graph class $A$. In this work we summarize what is known about recognition of $KB(A)$ for a collection of graph classes.

Classes Studied

- $G$: All graphs
- $G_k$: Graphs with girth at least $k$
- $P_n$: Path with $n$ vertices
- $C_n$: Cycle with $n$ vertices
- $K_n$: Complete graph of order $n$
- $co-CG$: Co-comparability graphs
- $1IC$: Interval intersection closed comparability graphs [6, 7]
- $1IC-PC$: 11IC-Permutation Graphs [6, 7]
- $1BC$: Interval bigraphs
- $HIB$: Helly interval bigraphs [4]
- $PIB$: Proper interval bigraphs
- $PIB-ASG$: Proper interval bigraphs having acyclic simplification graph [1]
- $PIG$: Proper interval graphs
- $3-PIC$: 3-Propers interval graphs [1]
- $BBHCG$: Bipartite biclique-Helly graphs with no dominated vertices [9]

Operators and Functions

- $G^2$: Square of graph $G$
- $K(G)$: Clique graph of graph $G$
- $K_{BG}(G)$: Mutually included biclique graph of graph $G$ [6, 7]
- $L(G)$: Line graph of graph $G$
- $J(G)$: Set of leaves (vertices of degree 1) of graph $G$
- $S(G)$: Simplification graph of graph $G$ [1]

Functors

The idea behind the techniques used in most of the results on $KB(A)$ is to characterize $KB(A)$ using some other operator (or a composition of operators). That is, $KB(G) = F(G)$, for $G \in A$ and some operator $F$.

We say that such scheme with more than one way to compute an operator is a "Functor".

<table>
<thead>
<tr>
<th>Class $A$</th>
<th>$KB(G), G \in A$</th>
<th>$KB(A)$</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(G)$</td>
<td>$L(G)$ complete</td>
<td>$L(G)$</td>
<td>$\mathbb{P}$ (linear)</td>
</tr>
<tr>
<td>$G$</td>
<td>$G$ leaves $G$</td>
<td>$G$</td>
<td>$\mathbb{P}$ (linear)</td>
</tr>
<tr>
<td>$K(G)$</td>
<td>$K(G)$</td>
<td>$K(G)$</td>
<td>$\mathbb{P}$ (linear)</td>
</tr>
<tr>
<td>$K_{BG}(G)$</td>
<td>$K_{BG}(G)$</td>
<td>$K_{BG}(G)$</td>
<td>$\mathbb{P}$ (linear)</td>
</tr>
<tr>
<td>$J(G)$</td>
<td>$J(G)$</td>
<td>$J(G)$</td>
<td>$\mathbb{P}$ (linear)</td>
</tr>
<tr>
<td>$S(G)$</td>
<td>$S(G)$</td>
<td>$S(G)$</td>
<td>$\mathbb{P}$ (linear)</td>
</tr>
</tbody>
</table>

Table 1: At column "$KB(G), G \in A$", a brief description of $KB(G)$; at column "class $KB(A)$", class that is equal to (or a super-class of) $KB(A)$, at column "complexity", complexity (of knowledge of recognizing $KB(A)$).

Acknowledgments

(*) Note that to decide if $G$ is the square of a graph of girth $\geq 5$ is $NP$-complete [2].

References

Abstract
A bipartite graph is a circular arc bigraph if there exists a bijection between its vertices and a family of arcs on a circle such that vertices of opposing partite sets are neighbors precisely if their corresponding arcs intersect. This class is a relatively unexplored subject, with most results on it and its subclasses being quite recent. In our work, we provide a full exploration of the containment relations and interactions between seven subclasses of circular arc bigraphs.

Introduction
The class of Circular arc bigraphs is a bipartite variation of the class of circular arc graphs. A circular arc graph \( G = (V, E) \) is a circular arc graph if there exists a bijection \( V \uplus W \rightarrow A \) such that \( A \) is a family of arcs on a circle and two vertices \( v \in V, w \in W \) are neighbors if and only if \( (v, w) \in E \).

Unlike its non-bipartite counterpart, circular arc bigraphs are a mostly unexplored topic of research, with most results on it being relatively recent. In 2013, Basu et al. [1] published matrix-based characterizations for the class of circular arc bigraphs, which are proper subclasses of both Helly and proper circular arc bigraphs. We also showed that non-bichordal Helly circular arc bigraphs are a proper subclass of proper circular arc bigraphs, and that Helly interval bigraphs are a proper subclass of proper interval bigraphs. We also showed that proper and Helly circular arc bigraphs, which contain a non-empty intersection, are not comparable.

Findings

Subclass Hierarchy on Circular Arc Bigraphs
A study of graph class containments
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The class of Circular arc bigraphs is a bipartite variation of the class of circular arc graphs. A circular arc graph \( G = (V, E) \) is a circular arc graph if there exists a bijection \( V \uplus W \rightarrow A \) such that \( A \) is a family of arcs on a circle and two vertices \( v \in V, w \in W \) are neighbors if and only if \( (v, w) \in E \).

Unlike its non-bipartite counterpart, circular arc bigraphs are a mostly unexplored topic of research, with most results on it being relatively recent. In 2013, Basu et al. [1] published matrix-based characterizations for the class of circular arc bigraphs, which are proper subclasses of both Helly and proper circular arc bigraphs. We also showed that non-bichordal Helly circular arc bigraphs are a proper subclass of proper circular arc bigraphs, and that Helly interval bigraphs are a proper subclass of proper interval bigraphs. We also showed that proper and Helly circular arc bigraphs, which contain a non-empty intersection, are not comparable.

The results provide a full understanding of the containment hierarchies of the classes mentioned, allowing us to present a comprehensive Venn diagram of them.

Future Research
Future research includes looking into relationships between other subclasses of circular arc bigraphs, such as unit circular arc bigraphs (graphs that admit a bi-circular-arc model such that all arcs are of the same length), cross-proper circular arc bigraphs (graphs that admit a bi-circular-arc model where no two arcs corresponding to vertices of opposing partite sets are comparable), and normal circular arc bigraphs (graphs that admit a bi-circular-arc model where no union of two arcs equals the entire circle). It also includes looking into the recognition problems of circular arc bigraphs, and any important subclasses of circular arc bigraphs for which no efficient recognition algorithm is known.

References
Efficient characterizations and algorithms of tree t-spanners

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Introduction

The t-admissibility problem has been widely studied specially because determining if a graph G is 3-admissible is still an open problem since it was proposed [2]. Although recognizing if a graph is 2-admissible is a polynomial time solvable problem, we realized that for some classes could be easier. Hence, in this work we present simple and efficient algorithms in order to characterize 2 and 3-admissible graphs for some graphs classes as cographs, split graphs, P₄-sparse and other superclasses.

Tree t-spanners

A tree t-spanner of a graph G is a spanning tree T of G in which the distance between adjacent vertices of G is at most t in T. In this case, we say that G is a t-admissible graph and the t-admissibility problem concerns in deciding if G is t-admissible. The minimum t for which G is t-admissible is the stretch index of G.

The t-admissibility problem has been widely studied specially because deciding if a graph G is 3-admissible is still an open problem since it was proposed [2]. Although recognizing if a graph is 2-admissible is a polynomial time solvable problem, we realized that for some classes could be easier. Hence, in this work we present simple and efficient algorithms in order to characterize 2 and 3-admissible graphs for some graphs classes as cographs, split graphs, P₄-sparse and other superclasses.

Deciding whether G is 2-admissible can be solved in O(n+m) time, where n and m are the number of vertices and edges of G, respectively. t-admissibility is NP-complete for t ≥ 4, and 3-admissibility remains an open problem.

Our goal is to provide simple and fast characterizations of tree t-spanners for graph classes in order to check 2- or 3-admissibility for them.

3-admissibility has been already efficiently solved for some graph classes, such as cographs, split graphs, cycle-power graphs and (2,1)-chordal graphs [1,3].

2-admissible P₄-sparse graphs and (0,2)-graphs

For P₄-sparse graphs (graphs obtained from trivial graphs, by applying in any order union, join and spider operations), we have that, if G is not a thin spider (Figure 1) and has not a universal vertex, its stretch index is equal to 3.

Moreover, given a P₄-sparse graph G, G is 2-admissible if and only if either G has universal vertex; or G is a thin spider.

Figure 1: Thin spider graph and its tree 2-spanner T. Two parallel lines represent a join operation between the touched parts. Each vertex in the spider’s body is connected to all other vertices in the spider’s body and the vertices on the spider’s head R. Thus there is a spanning star with respect to the body and the head R with any vertex v of the spider’s body as the center of the star. The paw that is not adjacent to v is placed in any of the leaves of the spider’s body. And, thus, the stretch index of the graph is 3.

We present a linear time algorithm to decide 2-admissibility for P₄-sparse graphs. The algorithm consists in verifying the existence of a universal vertex and if the given graph is a thin spider. For this second part, we calculate its spider partition (S, K, R) and check the degrees of the vertices in order to differ the thin form the thick spider (Figure 2), which is not 2-admissible.

Considering (0,2)-graphs (graphs that can be partitioned into 0 independent set and 2 cliques) we also present a linear time algorithm to check the 2-admissibility. Given a (0,2)-graph G, G is 2-admissible if and only if G has a universal vertex, a cut-vertex or between the parts of the (0,2)-partition is a strict 2-connected graph that has not an induced C₄.

Further work

In addition to the results presented above, we determined linear time algorithms to check 2-admissibility for P₄-tidy graphs, graphs that generalize P₄-sparse graphs, as described above.

We also considered the t-admissibility problem for a superclass of (0,2)-graphs, the (k,l)-graphs. Specifically: split graphs (i.e. (1,1)-graphs) and (0,l)-graphs. We presented linear time algorithms to verify the existence of a tree 2-spanner.

As future work, we intend to extend this study to other graph classes and to deal with a recent study that is a variation of t-admissibility, called edge admissibility [4], concerning in obtaining a spanning tree of the line graph of G in which the distance between adjacent edges of G is at most t.

References

Hamiltonicity of Token Graphs of Some Fan Graphs

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Abstract
In this poster we present some recent results about the Hamiltonicity of the 2-token graph $F_2(G)$ and the 2-multiset graph $M_2(G)$ of some fan graphs $G$. In particular, we exhibit an infinite family of graphs for which $F_2(G)$ and $M_2(G)$ are Hamiltonian.

Introduction
As far as we know, token graphs have been defined, independently, at least four times since 1988. Since then, several combinatorial parameters of token graphs have been studied, such as connectivity, regularity, planarity, Hamiltonicity, Eulerianicity, and, chromatic, clique, independence and packing numbers; as well as their automorphism group and spectrum. Also, several connections between token graphs and other research areas have been discovered, such as Quantum Mechanics and Coding Theory. For example, token graphs model the following system in Quantum Mechanics: consider a cluster of $n$ interacting qubits (two-level atoms) represented by a graph $G$ (where the qubits are interacting via an (excitation)-exchange Hamiltonian), in which, at each moment, exactly $k$ qubits are in the excited state and the remaining in the ground state; this system corresponds to the $k$-token graph of $G$. In Coding Theory, the packing number of the $k$-token graph of $F_n$ corresponds to the largest code of length $n$ and constant weight $k$ that can correct a single adjacent transposition; also the $k$-token graph of the Complete graph $K_n$ is isomorphic to the Johnson graph $J(n, k)$, which have several applications in Coding Theory. Besides, token graphs have been used to study the Isomorphism Problem of Graphs.

Motivation
Besides the possible applications of token graphs, one of our motivations to study the Hamiltonicity of token graphs was to extend our result of 2018 [5]:

Theorem 1 If $n \geq 3$, and $1 \leq k \leq n - 1$, then the $k$-token graph of the fan graph $F_{1,n-1}$ is Hamiltonian.

Definitions
For two disjoint graphs $G_1$ and $G_2$, the join graph $G = G_1 + G_2$ of graphs $G_1$ and $G_2$ is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and its edge set is $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$, a simple example is the fan graph $F_{n-1} = E_n + F_1$, where $E_n$ denotes the graph of $n$ isolated vertices and $F_1$ denotes the path graph of $n$ vertices.

Let $G$ be a simple graph of order $n$. The $k$-token graph $F_k(G)$ of $G$ is the graph whose vertices are the $k$-subsets of $V(G)$, where two of such vertices are adjacent if their symmetric difference is a pair of adjacent vertices in $G$. The $k$-multiset graph $M_k(G)$ of $G$ is the graph whose vertices are the $k$-multisubsets of $V(G)$, and two of such vertices are adjacent if their symmetric difference (as multisets) is a pair of adjacent vertices in $G$. See an example of these constructions in the figure below. The $2$-token graph is usually called the double vertex graph and the $2$-multiset graph is called the complete double vertex graph.

A Hamiltonian path (resp. a Hamiltonian cycle) of a graph $G$ is a path (resp. cycle) containing each vertex of $G$ exactly once. A graph $G$ is Hamiltonian if it contains a Hamiltonian cycle.

Results
These results were obtained by Luis Adame and the authors of this poster.

Theorem 2 Let $m \geq 1$ and $n \geq 2$. Then, $F_2(F_{m,n})$ is Hamiltonian if and only if $m \leq 2n$, and $M_2(F_{m,n})$ is Hamiltonian if and only if $m \leq 2(n - 1)$.

This theorem implies the following result.

Corollary 3 Let $G_1$ and $G_2$ be two graphs of order $m \geq 1$ and $n \geq 2$, respectively, such that $G_2$ has a Hamiltonian path. Let $G = G_1 + G_2$. If $m \leq 2n$ then $F_2(G)$ is Hamiltonian, and if $m \leq 2(n - 1)$ then $M_2(G)$ is Hamiltonian.

Open Questions
1. To study the Hamiltonicity of $F_k(F_{m,n})$ and $M_k(F_{m,n})$, for $k > 2$.
2. Given two graphs $G$ and $H$, to study the Hamiltonicity of $F_k(G \square H)$ and $M_k(G \square H)$.
3. To find other families of non-Hamiltonian graphs for which their $k$-token graph and $k$-multiset graph are Hamiltonian.
4. What is the smallest Hamiltonian graph $G$ for which $F_k(G)$ and $M_k(G)$ are Hamiltonian?

Previous results
It is well known that the Hamiltonicity of $G$ does not imply the Hamiltonicity of $F_k(G)$. For example it is known that if $n = 4$ or $n \geq 6$, then $F_2(C_n)$ is not Hamiltonian. On the other hand, there exist non-Hamiltonian graphs for which its double vertex graph is Hamiltonian, for example $F_2(K_{1,3})$ is Hamiltonian. Next, we list the known results about the Hamiltonicity of $F_k(G)$ or the existence of a Hamiltonian path in $F_k(G)$, when $k$ may be greater than two.

- If $n \geq 3$ and $1 \leq k \leq n - 1$, then $F_k(K_n)$ is Hamiltonian, see for example [3].
- If $m \geq 2$, then $F_2(K_{m,m})$ has a Hamiltonian path if and only if $k$ is odd [4].
- If $G$ is a graph containing a Hamiltonian path and $n$ is even and $k$ is odd, then $F_k(G)$ has a Hamiltonian path [4].
- If $n \geq 3$ and $1 \leq k \leq n - 1$, then $F_k(F_{1,n-1})$ is Hamiltonian [5].

In addition to these results, the following are some known results for the double vertex graph ($k = 2$).

- $F_2(C_n)$ is non-Hamiltonian [2].
- If $G$ is a cycle with an odd chord, then $F_2(G)$ is Hamiltonian [2].
- $F_2(K_{m,m})$ is Hamiltonian if and only if $(m - n)^2 = m + n + 2$ [2].

More results about the Hamiltonicity of double vertex graphs can be found in the survey of Alavi et. al. [1].

References

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