Network Science

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Summary

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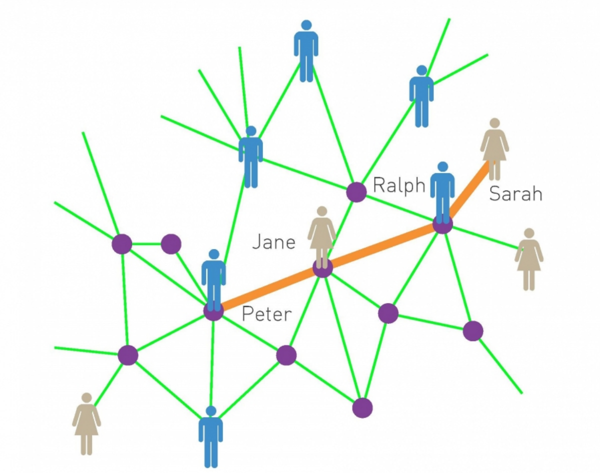
The ‘Small World’ Property

Six Degrees of Separation



Source: L.-A. Barabási, *Network Science*, `http://networksciencebook.com`, accessed 2020-08-22

Brief Historical Account

1929	Hungarian writer F. Karinthy used the idea in some of his books.
1958	Mathematician M. Kochen and political scientist I. de Sola Pool wrote first mathematical analysis on the subject, published in 1978, but widely circulated since 1958.
1967	Social psychologist S. Milgram performed first experiment testing the idea. Found 5.2 mean.
1991	Brodaway play "Six degrees of separation" by John Guare popularized the idea.
1998	Physicists D. J. Watts and S. Strogatz *Nature* paper: how a few changes in regular graph make it small-world.
1999	Physicists H. Jeong, R. Albert and A. L. Barabási *Nature* paper: estimated 19 degrees for WWW.
2011	Facebook Data Team found 4.7 mean separation for their social network at the time.

Are Small Worlds Common?

- Random Networks:

$$\langle d \rangle \propto \frac{\ln N}{\ln \langle k \rangle}$$

- ℓ-dimensional Lattices:

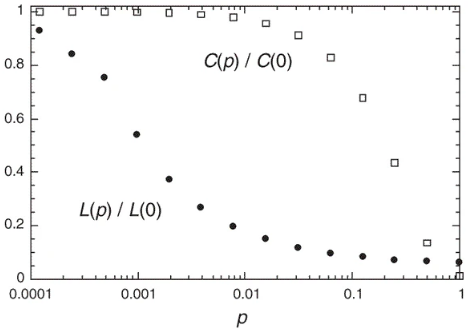
$$\langle d \rangle \propto N^{1/\ell}$$

- Cycle powers $C_N^{k/2}$:

$$\langle d \rangle \propto \frac{N}{2k}$$

Watts-Strogatz (Nature 1998)

- Switch links at random in a sparse cycle power with probability p
- Rapid drop in $\langle d \rangle$ $(= L(p))$ as p increases



Source: D. J. Watts, S. Strogatz, *Nature*, **393**, pp. 440–442 (1998)

Most real networks are small-world

Network	N	$\langle k \rangle$	$\langle d \rangle$	$\ln N / \ln \langle k \rangle$
Internet Routers	192,244	6.34	6.98	6.58
WWW Documents	325,729	4.60	11.27	8.31
Power Grid Stations	4,941	2.67	18.99	8.66
Mobile Phone Calls	36,595	2.51	11.72	11.42
Email Messages	57,194	1.81	5.88	18.40
Science Collaboration	23,133	8.08	5.35	4.81
Paper Citations	449,673	10.43	11.21	5.55
Actor co-staring	702,388	83.71	3.91	3.04
E. coli Metabolism	1,039	5.58	2.98	4.04
Protein Interaction	2,018	2.90	5.61	7.14

Source: L.-A. Barabási, *Network Science*, `http://networksciencebook.com`, accessed 2020-08-24

Scale-free networks

Is WWW a random network?

Both small-world

Random Network: two equivalent models

$G(N, L)$	Erdös-Rényi
$G(N, p)$	Gilbert

Gilbert's model easier to work with

What do we know about random networks?

Random Networks

What do we know about $G(N, p)$ random networks?

- Average degree

$$\langle k \rangle = p(N-1)$$

- Average number of edges

$$\langle L \rangle = p\frac{N(N-1)}{2}$$

- Degree distribution
  - Binomial form

$$p_k = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

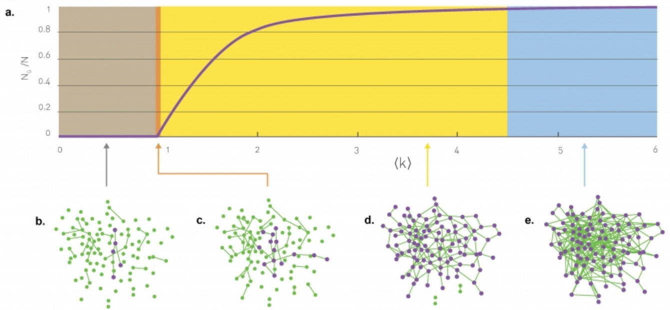
  - Poisson form

$$p_k \sim e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

- Connectivity as $\langle k \rangle$ varies

Connectivity of $G(N, p)$

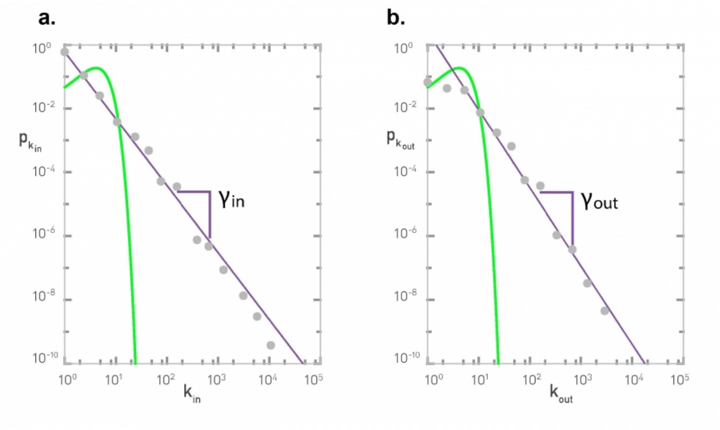
Size of giant component (largest component) N_G as $\langle k \rangle$ varies



$\langle k \rangle < 1$	$1 < \langle k \rangle < \ln N$	$\ln N < \langle k \rangle$
$N_G \sim \ln N$	$N_G \sim N^{2/3}$	$N_G \sim N/(\langle k \rangle - 1)$

WWW not a random network

Degree distribution follows power law rather than Poisson (Binomial)



p_k = probability of having degree k

Scale-free networks

- **Definition**: degree distribution follows a power law

$$p_k \sim k^{-\gamma}$$

- **No scale**: 1st moment exists, but 2nd moment diverges

- **Noticeable** even in finite setting. WWW:

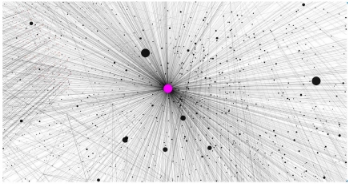
$$k_{in} = 4.60 \pm 1546$$

Most real networks are scale-free

Network	N	$\langle k \rangle$	γ_{in}	γ_{out}	γ
Internet Routers	192,244	6.34			3.42
WWW Documents	325,729	4.60	2.00	2.31	
Power Grid Stations	4,941	2.67			(*)
Mobile Phone Calls	36,595	2.51	4.69	5.01	
Email Messages	57,194	1.81	3.43	2.03	
Science Collaboration	23,133	8.08			3.35
Paper Citations	449,673	10.43	3.03	4.00	
Actor co-staring	702,388	83.71			2.12
E. coli Metabolism	1,039	5.58	2.43	2.90	
Protein Interaction	2,018	2.90			2.89

(*) Exponential degree distribution. Not scale-free.

Scale-free implications



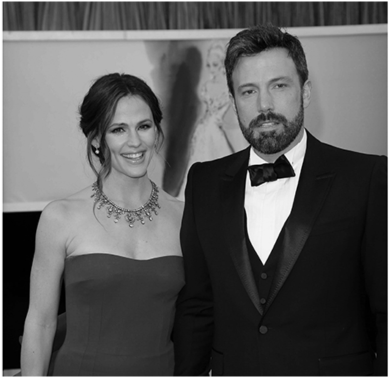
- Existence of **hubs**: nodes with very high degree
- Robustness against random failure
- Vulnerability to attacks
- Small world: fast spreading phenomena (news, disease, etc.)

Degree Correlations

Celebrity marriages

Degree correlations:

- Tendency to link nodes of similar degree



Source: L.-A. Barabási, *Network Science*, `http://networksciencebook.com`, acc. 2020-08-29

Types of network

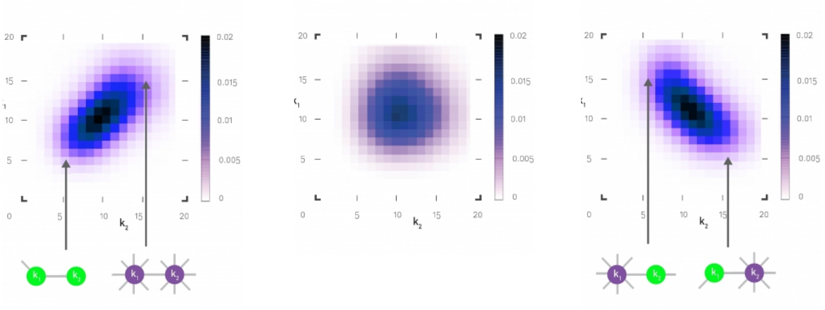
- **Assortative**:
  favors links between nodes with similar degrees
- **Neutral**:
  no particular bias in links with respect to endpoint degrees
- **Disassortative**:
  favors links between nodes with widely different degrees

How to measure

- degree correlation matrix (size k_{max}^2)
- degree correlation function (size k_{max})
- degree correlation coefficient (size 1)

Degree correlation matrix

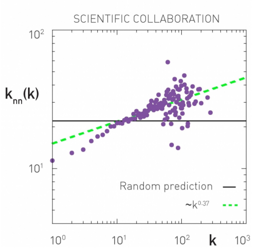
Definition: e_{ij} = probability of finding a node with degrees i and j at the two ends of a randomly selected link



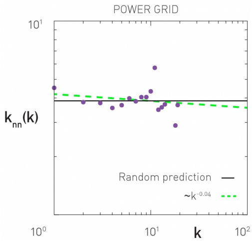
Assortative | Neutral | Disassortative

Source: L.-A. Barabási, *Network Science*, `http://networksciencebook.com`, acc. 2020-09-10

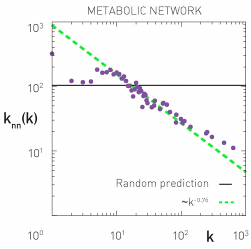
Definition: $k_{nn}(k)$ = average degree of the neighbors of all degree-k nodes



Assortative



Neutral



Disassortative

Source: L.-A. Barabási, *Network Science*, `http://networksciencebook.com`, acc. 2020-09-11

Degree correlation coefficient

Definition:

$$r = \sum_{jk} \frac{jk(e_{jk} - q_j q_k)}{\sigma^2},$$

where:

$$\sigma^2 = \sum_k k^2 q_k - \left[\sum_k k q_k\right]^2 \quad \text{and} \quad q_k = \frac{k p_k}{\langle k \rangle}.$$

We have:

$$-1 \leq r \leq +1$$

$r > 0$: assortative network
$r = 0$: neutral network
$r < 0$: disassortative network

Source: L.-A. Barabási, *Network Science*, `http://networksciencebook.com`, acc. 2020-10-14

Degree correlation in real networks

Network	n	r
Physics coauthorship (a)	52 909	0.363
Biology coauthorship (a)	1 520 251	0.127
Mathematics coauthorship (b)	253 339	0.120
Film actor collaborations (c)	449 913	0.208
Company directors (d)	7 673	0.276
Internet (e)	10 697	−0.189
World-Wide Web (f)	269 504	−0.065
Protein interactions (g)	2 115	−0.156
Neural network (h)	307	−0.163
Marine food web (i)	134	−0.247
Freshwater food web (j)	92	−0.276
Random graph (u)		0
Callaway *et al.* (v)		$\delta/(1+2\delta)$
Barabási and Albert (w)		0

Social networks are *assortative*

Biological, technological networks are *disassortative*

Source: L.-A. Barabási *et al.*, slides for Chapter 7 of *Network Science* book, `http://networksciencebook.com/slides-2017.zip`, accessed on 2020-11-13

Network Robustness

Network Robustness

- How resilient is a real network to **RANDOM FAILURES**?

  (i.e., removal of random nodes)

- How easy is it to break it with **PLANNED ATTACKS**?

  (i.e., removal of hubs)

"break": disconnect the giant component

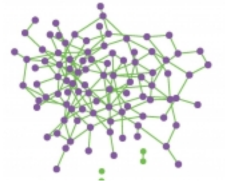
Conditions for the Existence of a Giant Component

- For random networks: $\langle k \rangle > 1$
- BUT, scale-free networks are not random!
- **Molloy-Reed**: condition for giant component in random networks that follow a given degree distribution

$\langle k^2 \rangle / \langle k \rangle < 2$

All components small

$\langle k^2 \rangle / \langle k \rangle > 2$



Giant component

Spreading Phenomena

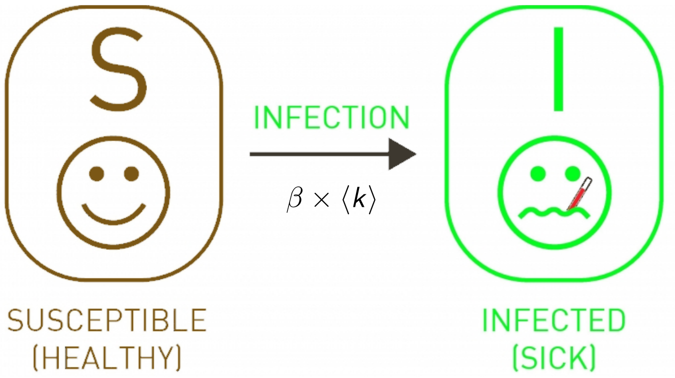
Examples of spreading phenomena

- Not restricted to diseases

Phenomena	**Agent**	**Network**
Venereal Disease	Pathogens	Sexual Network
Rumor Spreading	Information, Memes	Communication Network
Innovation Diffusion	Ideas, Knowledge	Communication Network
Computer Viruses	Malwares	Internet
Mobile Phone Virus	Mobile Viruses	Social/Proximity Network
Bedbugs	Parasitic Insects	Hotel-Traveler Network
Malaria	Plasmodium	Mosquito-Human Network

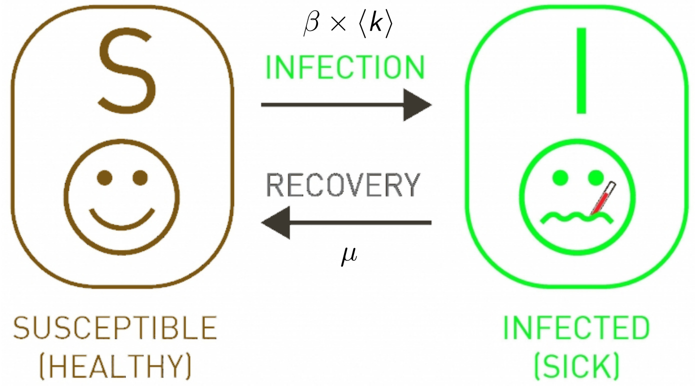
Source: L.-A. Barabási, *Network Science*, `http://networksciencebook.com`, acc. 2020-11-02

Classical epidemic modeling: SI model



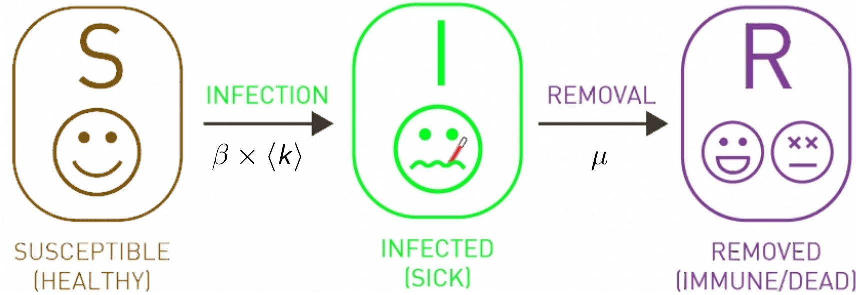
$$\frac{di}{dt} = +\beta \langle k \rangle \, i \, s$$
$$\frac{ds}{dt} = -\beta \langle k \rangle \, i \, s$$

Classical epidemic modeling: SIS model



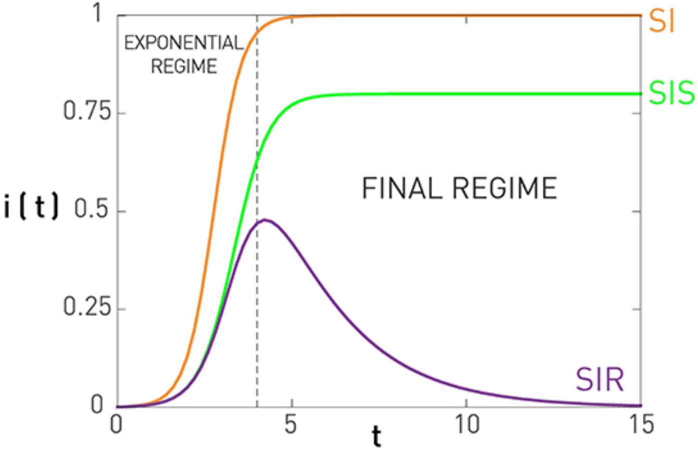
$$\frac{di}{dt} = +\beta \langle k \rangle \, i \, s - \mu \, i$$
$$\frac{ds}{dt} = -\beta \langle k \rangle \, i \, s + \mu \, i$$

Classical epidemic modeling: SIR model



$$\frac{di}{dt} = +\beta \langle k \rangle \, i \, s - \mu \, i$$
$$\frac{ds}{dt} = -\beta \langle k \rangle \, i \, s$$
$$\frac{dr}{dt} = +\mu \, i$$

Classical epidemic modeling: Outcome Summary



S = susceptible I = infected R = recovered

Network epidemic modeling

Model	Continuum Equations	Epidemic Condition
SI	$\frac{di_k}{dt} = \beta[1 - i_k]k\theta_k$	$\beta > 0$
SIS	$\frac{di_k}{dt} = \beta[1 - i_k]k\theta_k - \mu i_k$	$\frac{\beta}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle}$
SIR	$\frac{di_k}{dt} = \beta[1 - i_k - r_k]\theta_k - \mu i_k$ $\frac{dr_k}{dt} = \mu i_k$	$\frac{\beta}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$

How to halt an Epidemic?

- **Transmission-reducing interventions**:
  - Face masks, gloves, hand washing: airborne or contact-based pathogens
  - Condoms: sexually transmitted pathogens
- **Contact-Reducing Interventions**:
  - Quarantine patients
  - Close schools
  - Limit access to frequently visited public spaces
- **Vaccinations**:
  - Permanently remove vaccinated nodes
  - Reduce the spreading rate

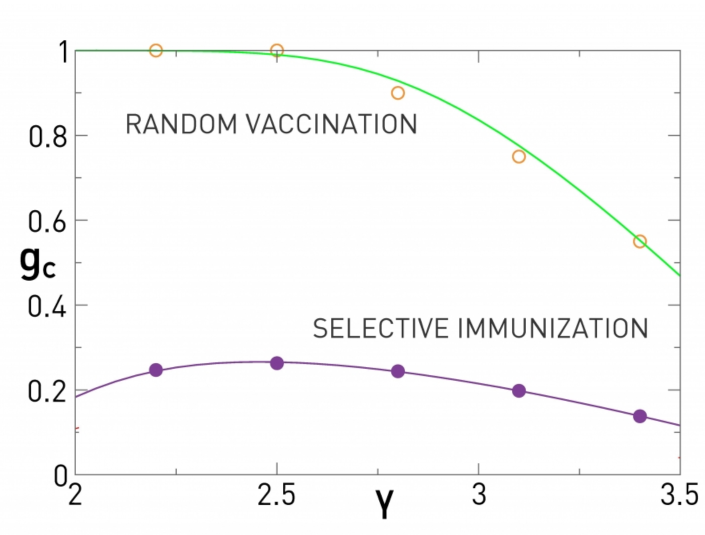
Selective Immunization

Target hubs. But how to find hubs?

Friendship paradox: the fact that on average the neighbors of a node have higher degree than the node itself

- Group 0: randomly chosen p fraction of population
- Select a random link from every node in Group 0
- Group 1: other side of links selected in previous step
- Immunize the Group 1 individuals

Selective Immunization



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