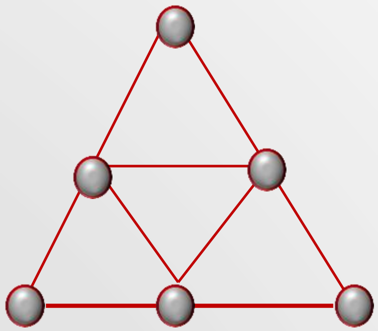
# CONJUNTOS INDEPENDENTES EM GRAFOS



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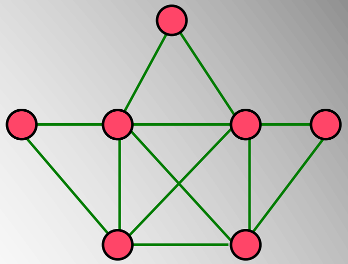
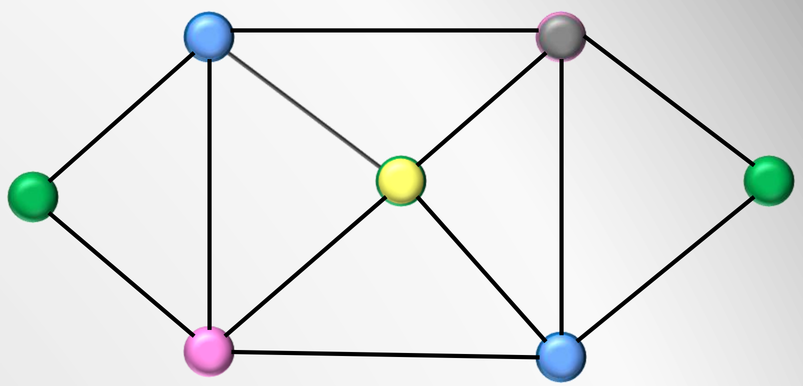


- DEFINIÇŌES
- CONJUNTOS INDEPENDENTES
- CONJUNTOS INDEPENDENTES MAXIMAIS

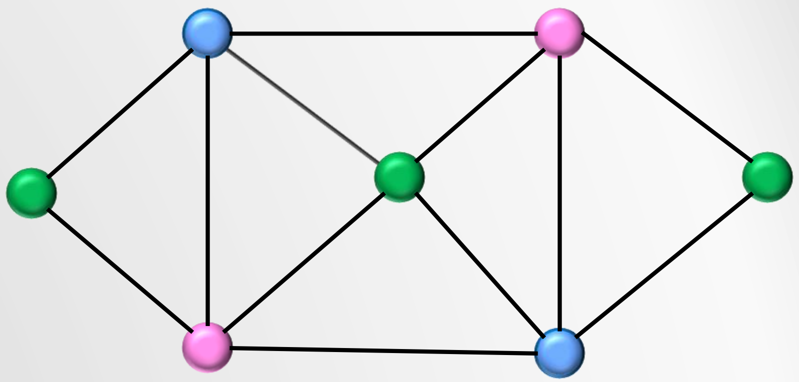
$G = (V,E)$ grafo simple conexo não direccionado

Conjunto independente:

$S \subseteq V \quad \forall\, u, v \in S{:}\ (u,v) \notin E$



Conjunto independente **máximo**



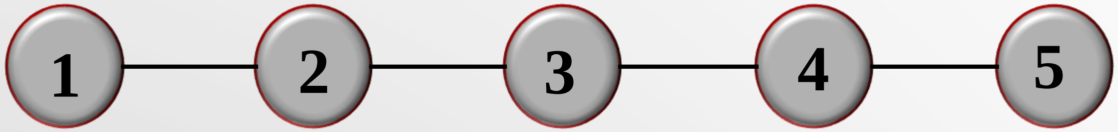
Conjunto independente **maximal**

*Tudo conjunto independente máximo é maximal*

# Problemas

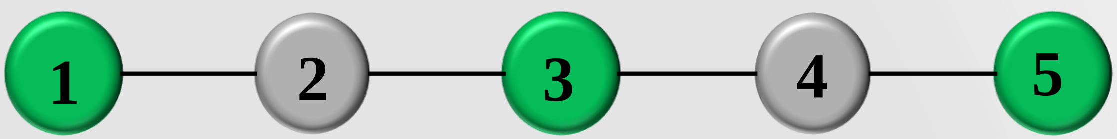
- Cardinal de um conjunto independente máximo
- Número de conjuntos independentes
- Número de conjuntos independentes maximais
- Enumerar tudos os conjuntos independentes
- Enumerar tudos os conjuntos independentes maximais
- Grafo independente: *grafo interseção de conjuntos independentes maximais*

$P_k$ caminho sem cordas com $k$ vértices

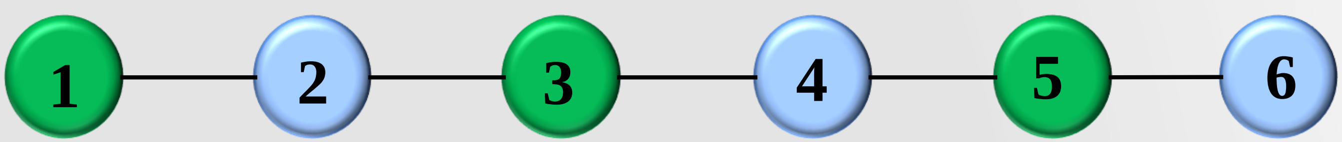



- conjunto independente máximo

$$\alpha(P_k) = \begin{cases} k/2 & \text{si } k \text{ par} \\ \lceil k/2 \rceil & \text{si } k \text{ impar} \end{cases}$$

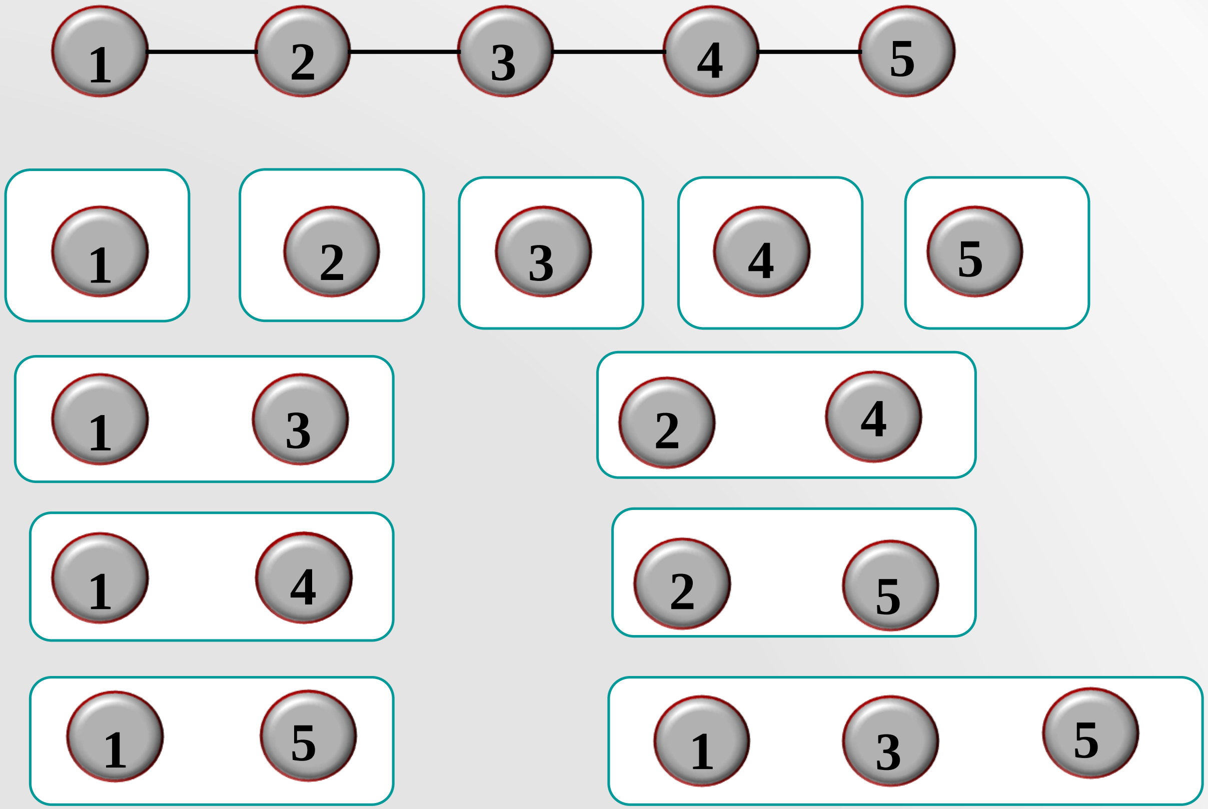



$\alpha(P_5) = 3$




$\alpha(P_6) = 3$

- Número de conjuntos independentes




$$f(P_k) = F_{k+2}$$

Prodinger & Tichy, 1982

= Merrifield Simmons Index ≈ *Ponto ebulição de hidrocarburos*

# CONJUNTOS INDEPENDENTES (CI)

#P-completo (Valiant, 1979)

- # Número de conjuntos independentes

## COTAS

| | Grafo | Referência |
|---|---|---|
| 1982 | Grafo geral<br><br>Árvore | Prodinger &Tichy |
| 1996 | Foresta | Lin & Lin |
| 2000 | *k*-regular | Sapozhenko |
| 2006 | Bipartido<br>Uniciclo<br>Regular<br>Livre de garras (*claw-free*) | Pedersen & Vestergaard |

- Número de conjuntos independentes

## ECUAÇÕES DE RECORRÊNCIA

| | Grafo | Referência |
|---|---|---|
| 1982 | Ciclo | Prodinger &Tichy |
| 1982 | Caminho | Prodinger &Tichy |
| 1997 | Reticulado | Calkin & Wilf |
| 2014 | Girino (*tadpole*) | De Maio & Jacobson |

- # Número de conjuntos independentes

## ALGORITMOS

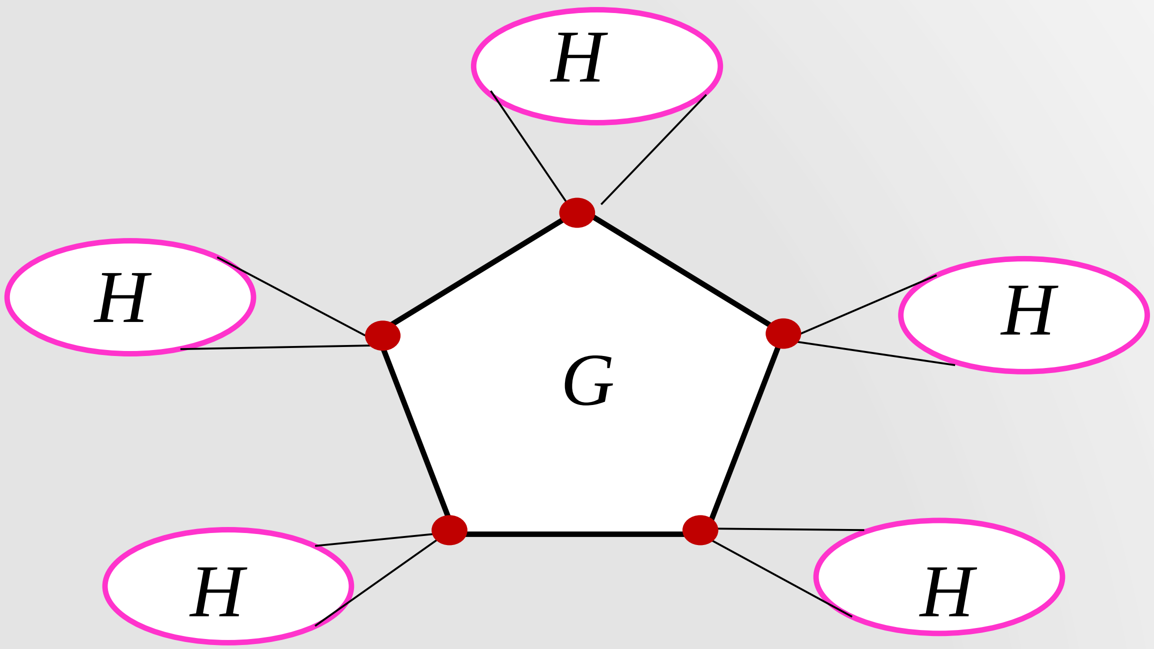
| | Grafo | Referência |
|---|---|---|
| 1982 | Árvore | Prodinger &Tichy |
| | | Dutton, Chandrasekaran Knopfmacher, Tichy, Wagber & Ziegler (2007) |
| 1988 | Produto cartesiano de caminhos, ciclos e árvores | Forbes & Ycart |
| 2008 | Cordal | Okamoto et al. |
| 2014 | Tolerância | Lin & Su |
| 2015 | Circulante | Dosal-Trujillo & Galeana-Sanchez |

- Número de conjuntos independentes de $G \odot H$

Grafo coroa $G \odot H$ (Frucht & Harary, 1970)

Uma copia de $G$ e $|V(G)|$ copias de $H$

Cada vértice $v_i$ de $G$ é adjacente a tudo vértice na copia $i$ de $H$

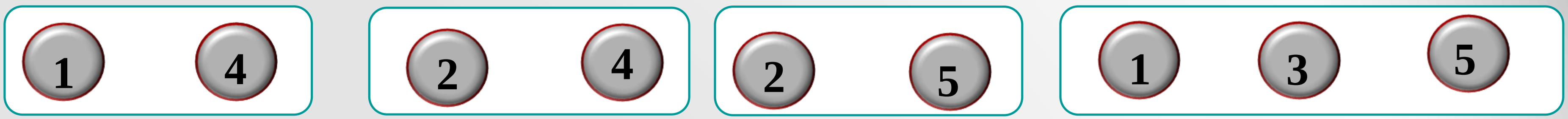


## • Número de conjuntos independentes $G \odot H$

| | Grafo | Referência |
|---|---|---|
| 2012 | $P_n \odot K_1$<br>$C_n \odot K_1$<br>$G \odot K_{1;}$ $G \odot K_2$<br>$G \odot \overline{\boldsymbol{K}}_i$<br>$P_n \odot \overline{K}_i$<br>$C_n \odot \overline{K}_i$ | Reyhani, Alikhani & Iranmanesh |
| 2012 | $P_n \odot K_2$<br>$C_n \odot K_2$ | Reyhani, Arikhani & Iranmanesh<br>Wu, Yang & Cheng |
| 2020 | $K_{1,n} \odot G;$<br>$K_n \odot G$ | Ortiz & Villanueva |

# CONJUNTOS INDEPENDENTES MAXIMAIS (CIM)

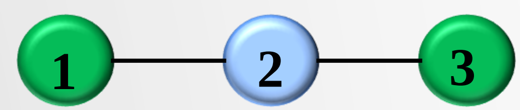
- Número de conjuntos independentes maximais

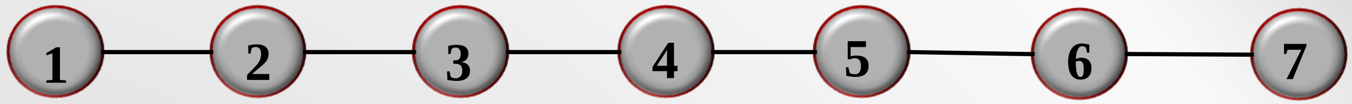

$$i\,[P_k] = i\,[P_{k-2}] + i\,[P_{k-3}] \quad k \geq 4$$

Füredi (1987)

$i\,[P_1]=1$ $i\,[P_2]= 2$ $i\,[P_3]= 2$

- **Enumerar tudos os conjuntos independentes maximais**




{1, 3, 5,7}

{1, 3, 6}

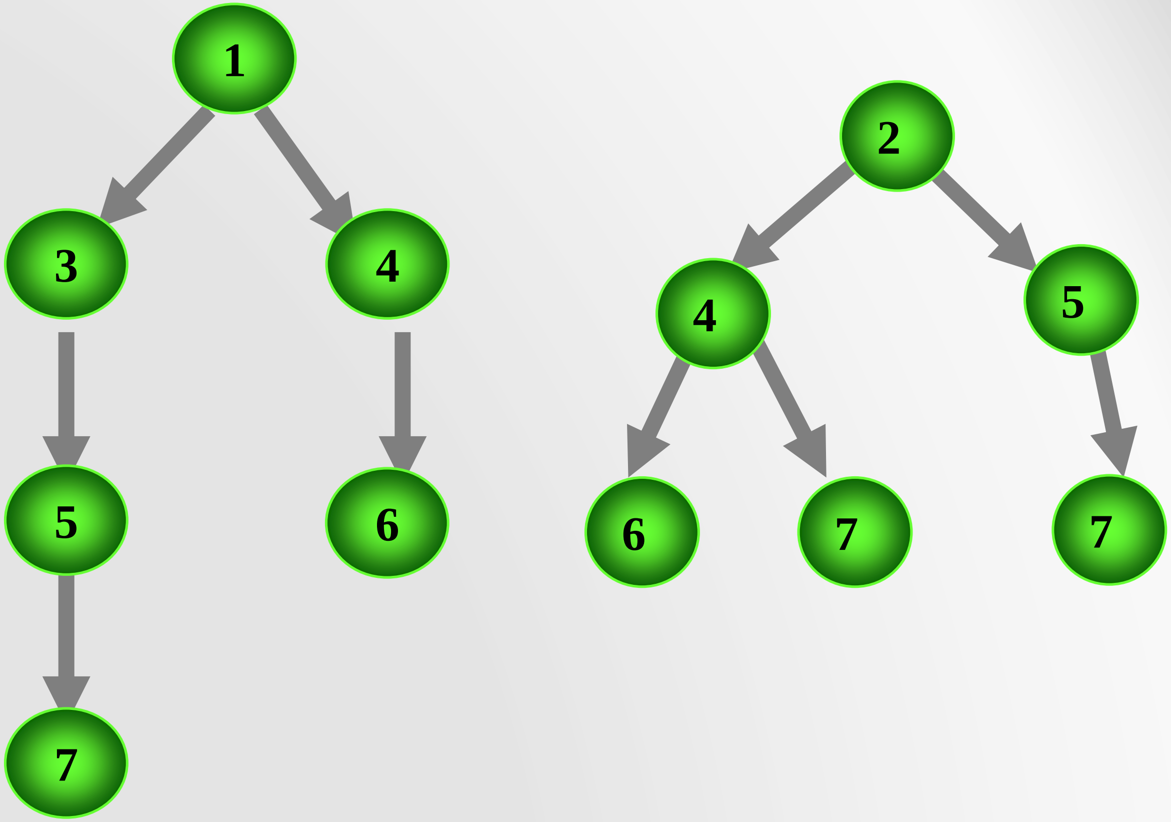
{1, 4, 6}

{1, 4,7}

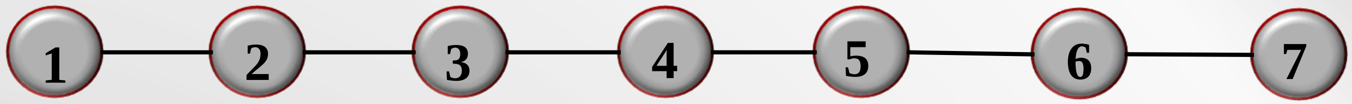
{2, 4, 6}

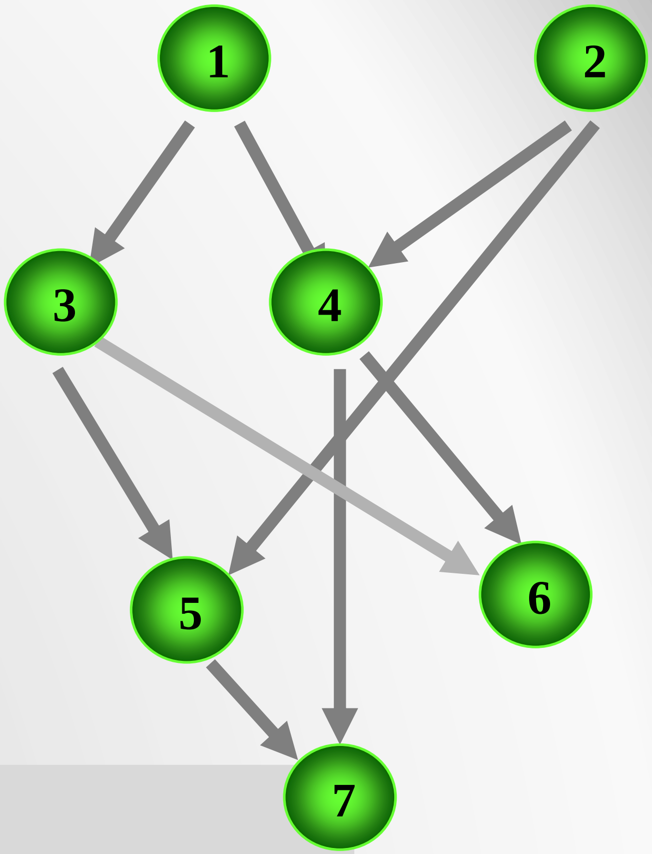
{2, 4,7}

{2, 5,7}



- Enumerar tudos os conjuntos independentes maximais

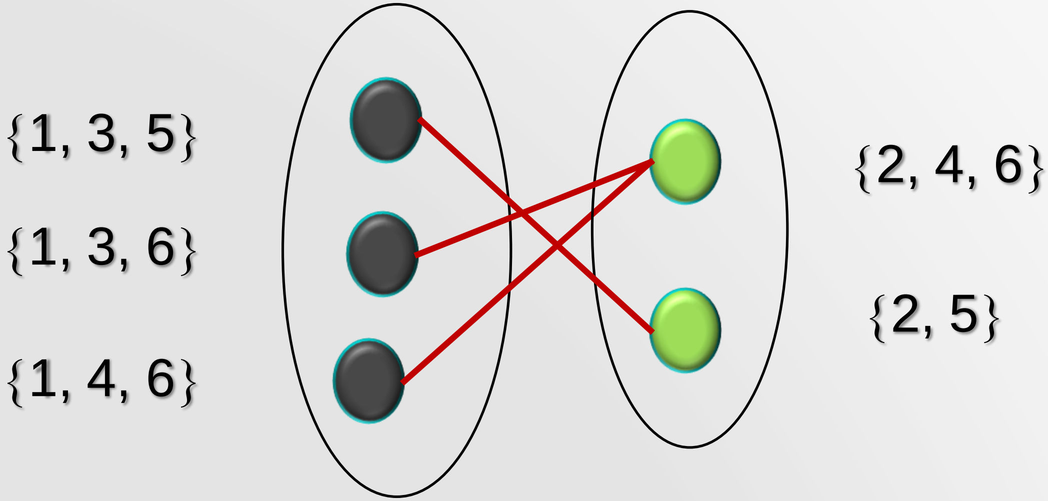



<u>Teorema:</u>

cim de $P_k$ $\leftrightarrow$ caminho fonte sumidouro de $D$

# Grafo independente: grafo interseção de conjuntos independentes maximais


$I(P_k)$: grafo co-bipartido

$$V(I(P_k)) = M_{v_1}(P_k) \cup M_{v_2}(P_k) \text{ com } b_k \textit{ arestas}$$

$$b_k = ½ [\mu_{v_1} \cdot (\mu_{v_1} - 1) + \mu_{v_2} \cdot (\mu_{v_2} - 1)] + \mu_{v_1} \cdot \mu_{v_2} - d_k$$

# • Número de conjuntos independentes maximais

#P-completo Valiant (1979)

Cotas:

Erdos & Moser (1960), Moon & Moser (1965),

Erdos, Hobbs & Payan (1982)

Füredi (1987), Griggs et al (1988), Liu (1994)

Algoritmo para contar cim:

Gaspers, Krastch & Liedloff (2008)

Junostza & Tucsinzki (2015)

- # Número de conjuntos independentes maximais

## COTAS

| | Grafo | Referência |
|---|---|---|
| 1986 | Árvore | Wilf, Meir & Moon (1988), Sagan & Vattter (2005), Wloch (2007), Zhang (2010) |
| 1991 | Bipartido | Alon, Liu (1993) |
| 1993 | Sem triângulos | Hujter & Tuza, Chang & Jou (1999) |
| 1997 | Máximo 1 ciclo | Jou & Chang |
| 2001 | Máximo 2 ou 3 ciclos | Goh & Koh |
| 2006 | Máximo *r* ciclos | Goh & Koh, Sagan & Vatter |
| 2008 | Uniciclo | Koh, Goh & Dong |
| 2013 | Bipartido com, pelo menos, 1 ciclo | Li, Zhang & Zhang |
| 2014 | Co-comparabilidade | Su & Lin |

- # Número de conjuntos independentes maximais

## ECUAÇÕES DE RECORRÊNCIA

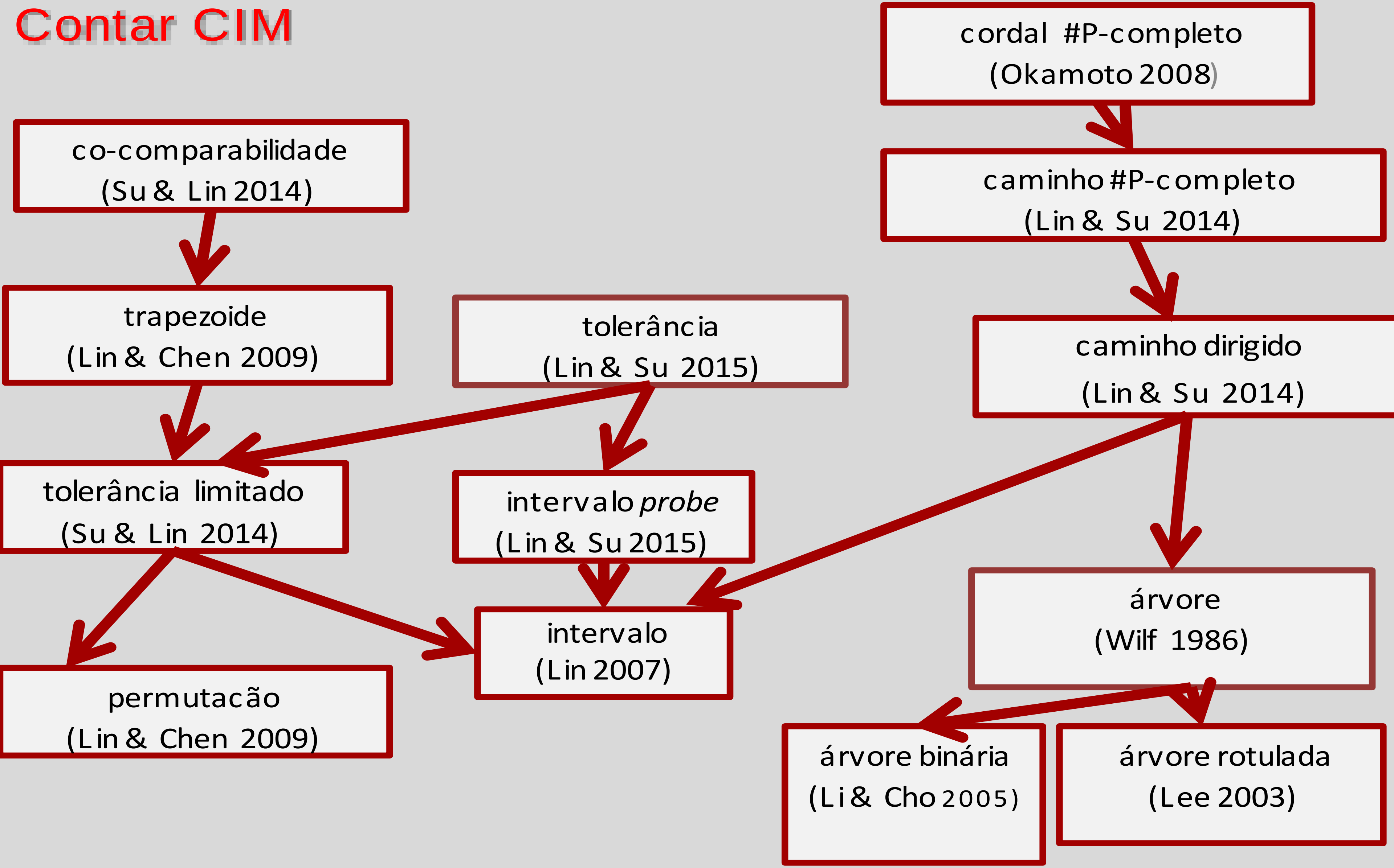
| | Grafo | Referência |
|---|---|---|
| 1987 | Caminho $P_k$ | Füredi |
| 1987 | Ciclo $C_k$ | Füredi |
| 1997 | Reticulado | Calkin & Wilf |
| 2005 | | Euler,<br>Euler et al. (2013) |
| 1995 | Reticulado de 2 linhas | Szwarcfiter & Villanueva |
| 2001 | *Starlike ladders* | Stevanovic |
| 2005 | Árvore binária | Lee & Cho |
| 2015 | Tolerância | Lin &Su |
| 2020 | Coroa $G \odot H$ | Ortiz & Villanueva |

- # Número de conjuntos independentes maximais

## ALGORITMOS

| | Grafo | Referência |
|---|---|---|
| 1986 | Árvore | Wilf |
| 2009 | Trapezoide | Lin & Chen |
| 2014 | Caminho dirigido | Lin & Su |
| 2015 | Subcúbico | Junosza & Tuczynski |
| 2015 | Tolerância | Lin &Su |

# Contar CIM




Lin, M.-S & Su, S.-H. (2015) , “Counting independent sets in a tolerance graph”, DAM 18, 174-184.

# Enumerar tudos os conjuntos independentes maximais

- Tsukiyama et al (1977) : *backtracking*
- Loukakis & Tsouros (1981): ordem lexicográfico
- Loukakis (1983)
- Johnson et al (1988): ordem lexicográfico

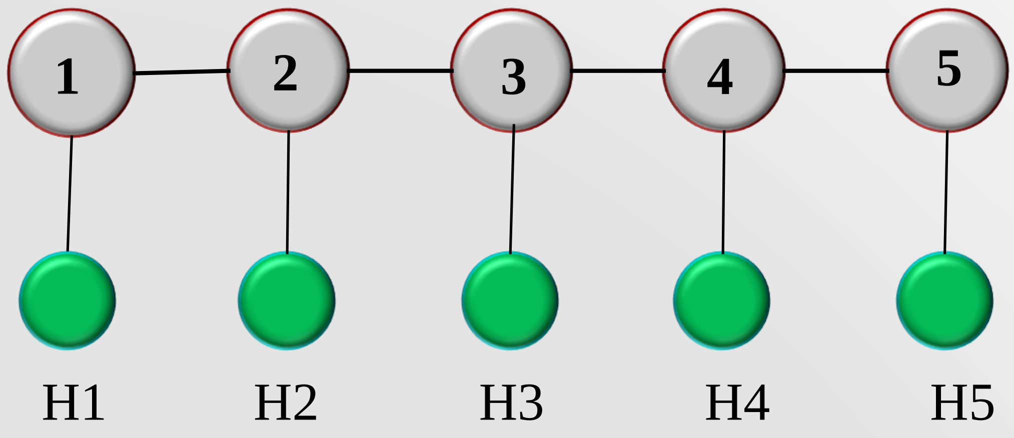
- **Enumerar tudos os cim**

## ALGORITMOS

| | Grafo | Referência |
|---|---|---|
| 1984 | Cordal, intervalo, arco-circular | Leung |
| 1993 | Permutacão | Yu & Chen |
| | | Saha, Pal & Pal (2005) |
| 1994 | Árvore | Chang, Wang, Lee |
| 1993 | Intervalo | Szwarcfiter & Villanueva |
| 1999 | Trapezoide | Hota, Pal & Pal |
| 2004 | *Threshold* | Ortiz & Villanueva |
| 2004 | Meyniel | Inostroza & Villanueva |
| 2012 | *Caterpillar* | Ortiz & Villanueva |
| 2012 | *Bisplit* | Stuardo & Villanueva |
| 2017 | Reticulado com 2 linhas | Ortiz & Villanueva |

# GRAFO INDEPENDENTE

## Grafo Caterpillar completo: $CC(P_k)$



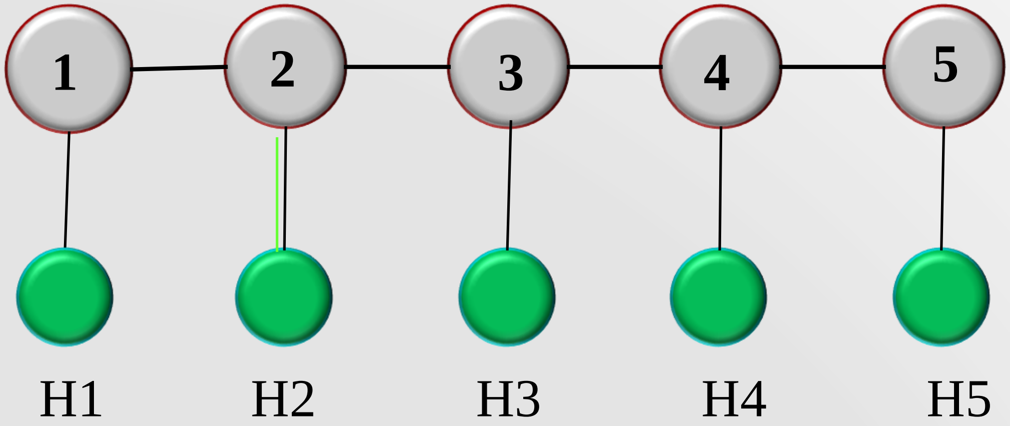

{1, **H2**, 3, **H4**, 5}
{1, **H2**, 3, **H4**, **H5**}
{1, **H2**, **H3**, **H4**, **H5**}
{1, **H2**, **H3**, **H4**, 5}
{1, **H2**, **H3**, 4, **H5**}

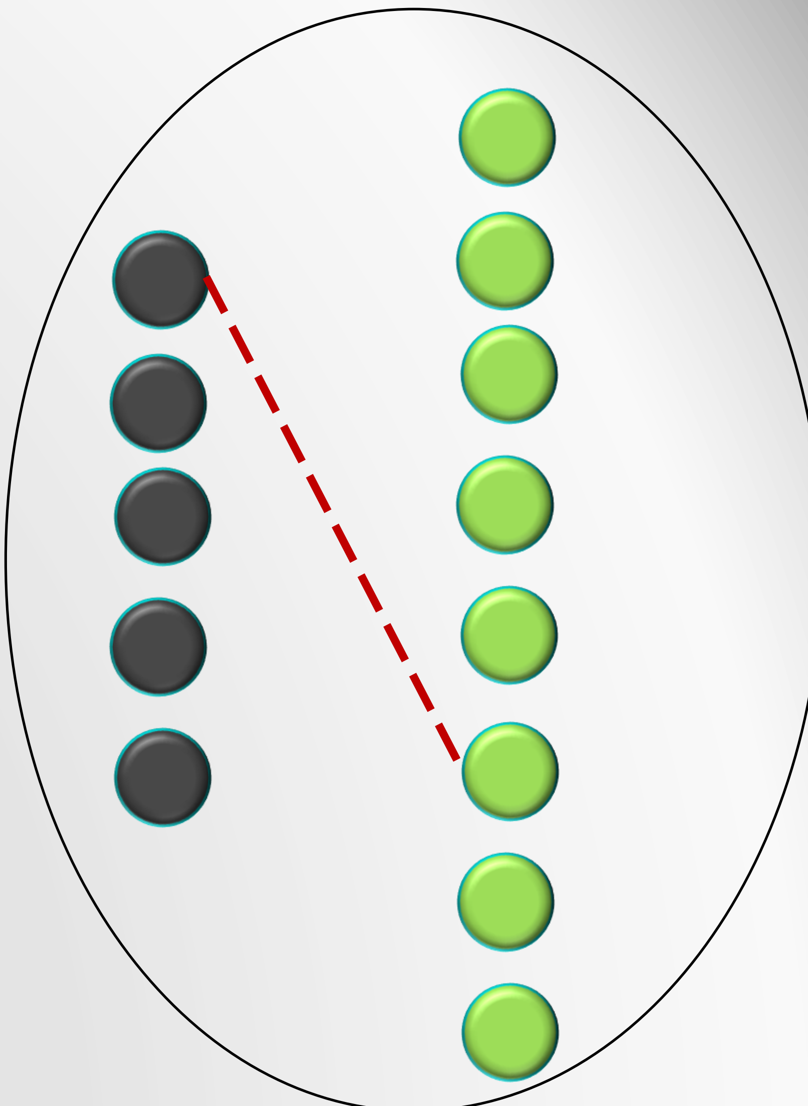
{**H1**, **H2**, 3, **H4**, 5}
{**H1**, **H2**, 3, **H4**, **H5**}
{**H1**, **H2**, **H3**, **H4**, **H5**}
{**H1**, **H2**, **H3**, **H4**, 5}
{**H1**, **H2**, **H3**, 4, **H5**}
{**H1**, 2, **H3**, 4, **H5**}
{**H1**, 2, **H3**, **H4**, **H5**}
{**H1**, 2, **H3**, **H4**, 5}

$$i\,(CC(P_k)) = i\,(CC(P_{k-1})) + i\,(CC(P_{k-2}))$$

# GRAFO INDEPENDENTE

## Grafo Caterpillar completo: $CC(P_k)$



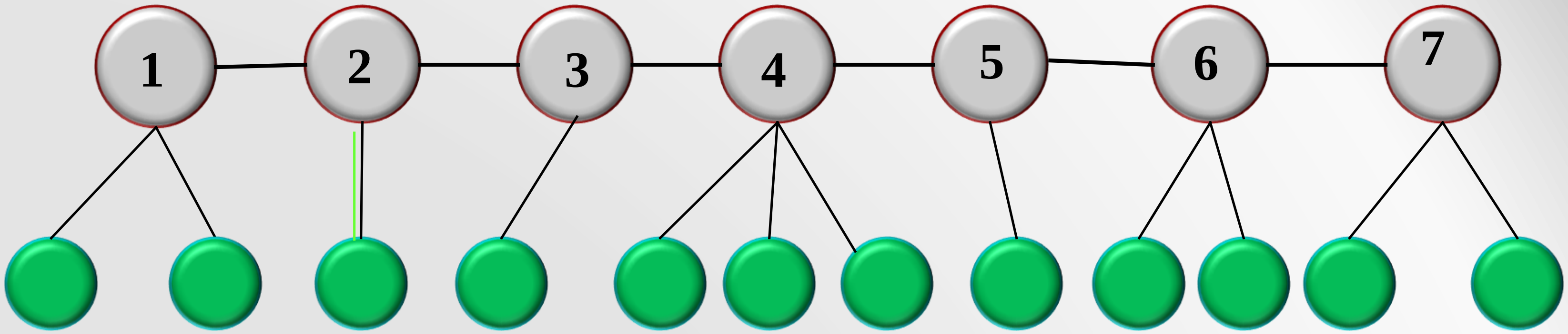



| | |
|---|---|
| {1, H2, 3, H4, 5} ❖ | {H1, H2, 3, H4, 5} |
| {1, H2, 3, H4, H5} | {H1, H2, 3, H4, H5} |
| {1, H2, H3, H4, H5} | {H1, H2, H3, H4, H5} |
| {1, H2, H3, H4, 5} | {H1, H2, H3, H4, 5} |
| {1, H2, H3, 4, H5} | {H1, H2, H3, 4, H5} |
| | {H1, 2, H3, 4, H5} ❖ |
| | {H1, 2, H3, H4, H5} |
| | {H1, 2, H3, H4, 5} |

# GRAFO INDEPENDENTE
## (Ortiz & Villanueva)

| | Grafo | $\mathcal{J}(G)$ |
|---|---|---|
| 2012 | Caterpillar completo | Grafo completo – *e* |
| 2017 | Caminho | Co-bipartido |
| | Ciclo | Co-tripartido |
| | Reticulado com 2 linhas | Co-bipartido |
| 2019 | Sol sem cordas | Contêm um grafo completo |
| | Sol completo | Grafo completo |
| | Sol | Grafo completo – *e* |
| | Pneu (*wheel*) | $\mathcal{J}$ (ciclo) + *u* [*u* universal no Pneu] |
| | Timão (*helm*) | Grafo completo [*n* impar ]<br>Grafo completo – *e* [*n* par] |
| | Lollipop | Co-tripartido |
| | Girino (*tadpole*) | Co-tripartido |

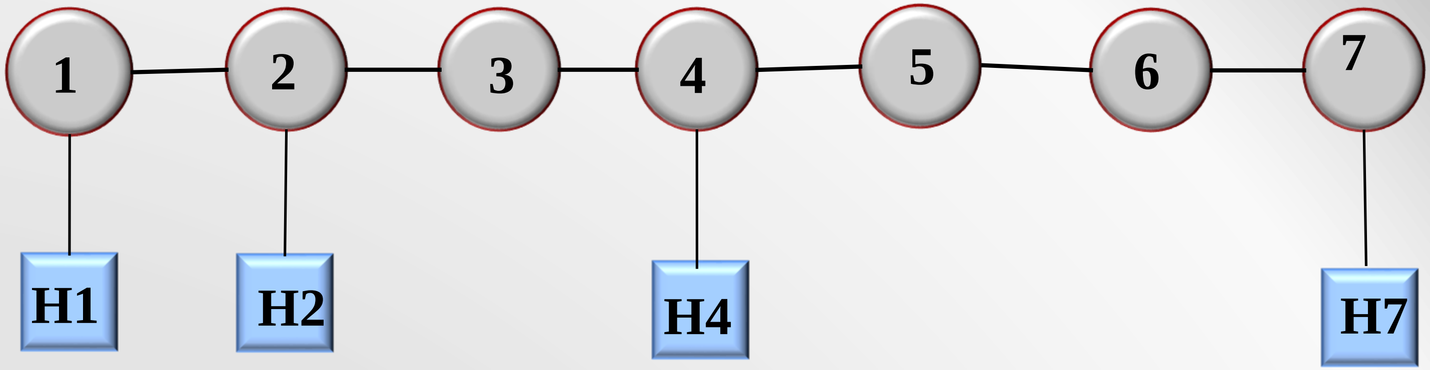
# Grafo caterpillar: $C(P_k)$




$$v \in P_k$$

$$v \in S \;\; \Rightarrow \;\; \forall u \in H(v) \;\; u \notin S$$

$$v \notin S \;\; \Rightarrow \;\; \forall u \in H(v) \;\; u \in S$$




{1, H2, 3, H4, 5, 7}

{1, H2 , 3, H4, 6, H7}

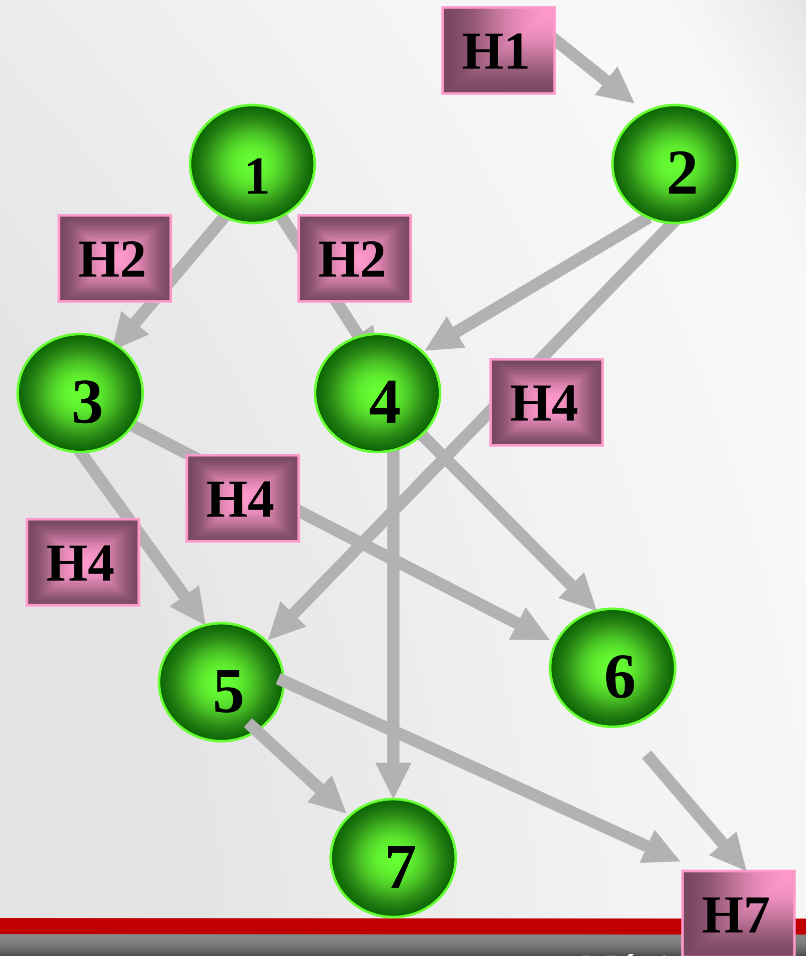
{1, H2 , 4, 6, H7}

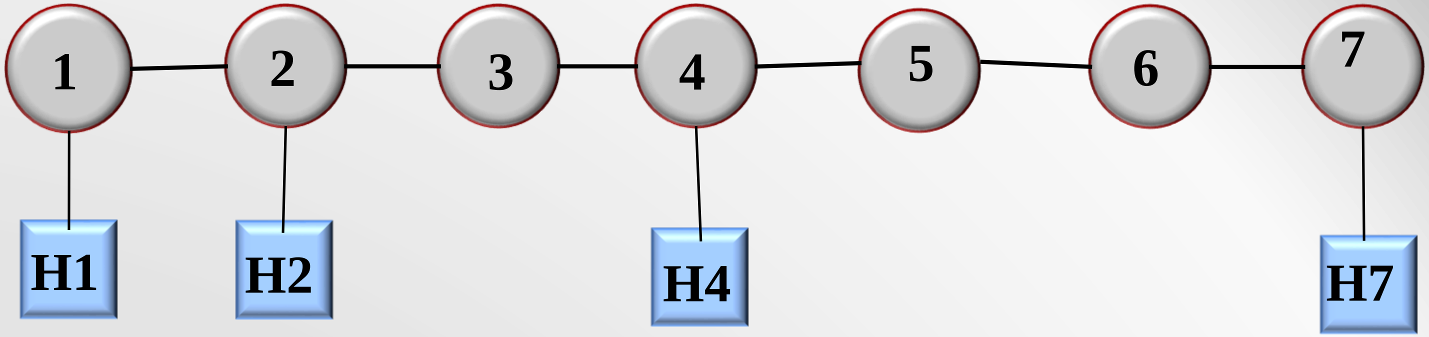
{1, H2 , 4, 7}

{ H1, 2, 4, 6, H7}

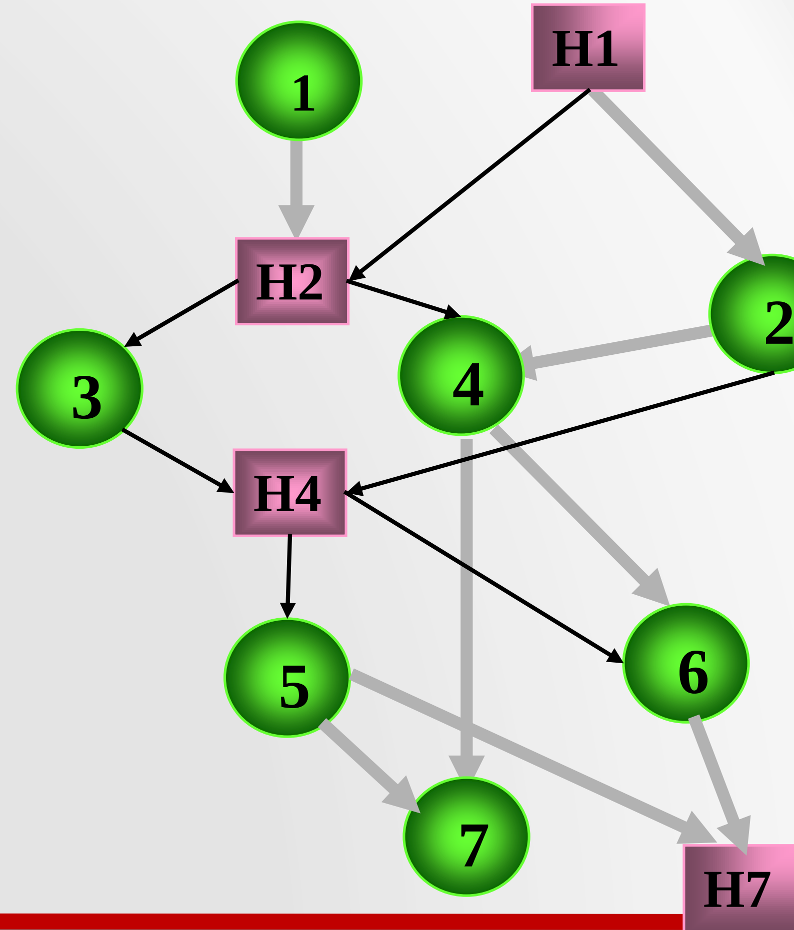
{ H1, 2, 4,7}

{ H1, 2, 5, H4, 7}



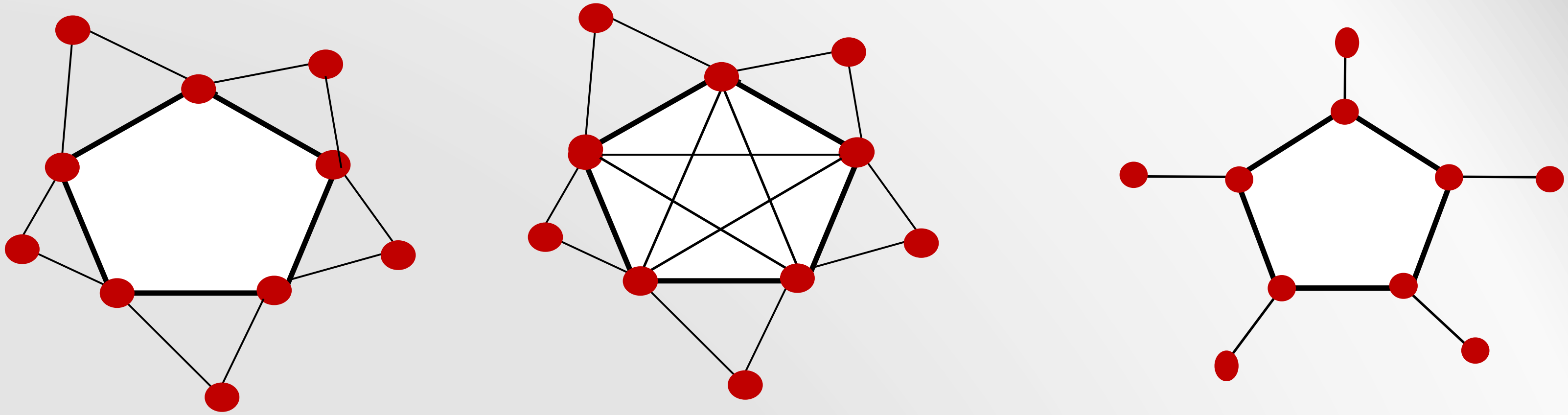



{1, H2, 3, H4, 5, 7}
{1, H2, 3, H4, 5, H7}
{1, H2, 3, H4, 6, H7}
{1, H2, 4, 6, H7}
{1, H2, 4, 7}
{ H1, 2, 4, 6, H7}
{ H1, 2, 4, 7}
{ H1, 2, H4, 5, 7}
{ H1, 2, H4, 5, H7}
{ H1, 2, H4, 6, H7}
{H1, H2, 3, H4, 5, 7}
{H1, H2, 3, H4, 5, H7}
{H1, H2, 3, H4, 6, H7}
{H1, H2, 4, 6, H7}
{H1, H2, 4, 7}

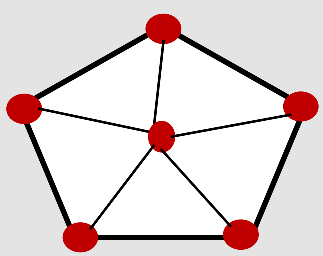


# GRAFOS SIMPLES

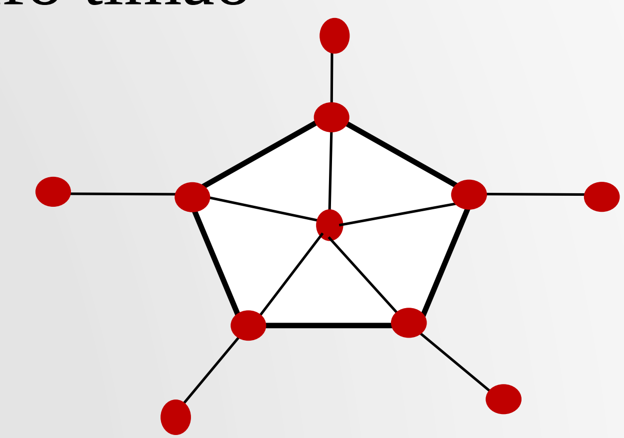
## grafos sol

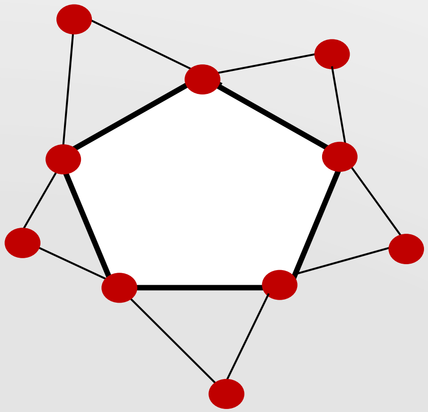


## grafo pneu



## grafo timão





$f(S(C_m)) = 2\,Q_m$

$i(S(C_m)) = L_m$

$\mathcal{J}(S(C_m))$: $V = B_1 \cup B_2$

$\mathcal{K}(S(C_m)) \sim C_m$

$Q_m$: números de Pell-Lucas

$Q_m = 2Q_{m-1} + Q_{m-2}$ , $Q_1 = 1, Q_2 = 3$

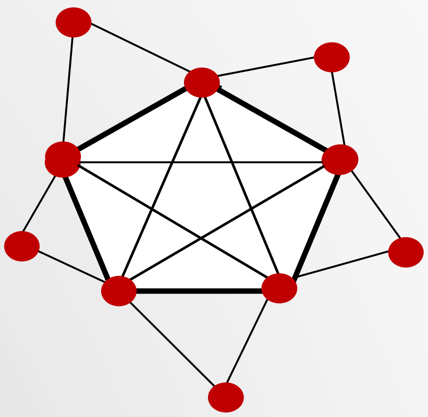
$L_m = L_{m-1} + L_{m-2}$ , $L_0 = 2$, $L_1 = 1$

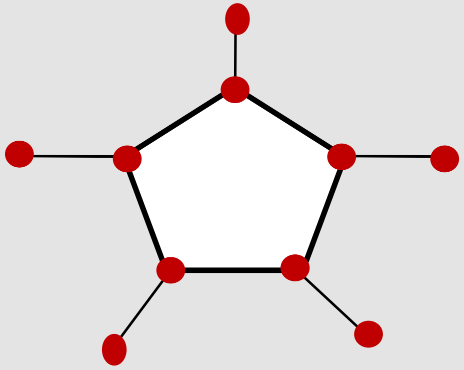
$f(S(K_m)) = 2^m + m2^{m-2}$

$i(S(K_m)) = m + 1$

$\mathcal{J}(S(K_m)) \sim K_{m+1}$

$\mathcal{K}(S(K_m)) \sim W_{m+1}$





$f(SL(C_m)) = 2^{m/2}\, L_{\sqrt{2},m}$

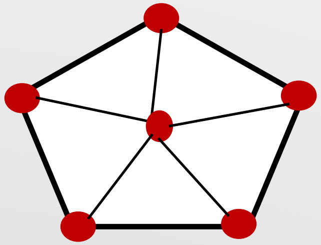
$i(SL(C_m)) = f(C_m) = L_m$

$\mathcal{J}(S(K_m)) \sim K_{L_m}$ - aresta

$\mathcal{K}(S(K_m)) \sim S(C_m)$

$L_{k,m} = k\,L_{k,m-1} + L_{k,m-2}$,

$L_{k,0} = 2$, $L_{k,1} = k$



$f(W_{1,m}) = f(C_m) + 1 = L_m + 1$

$i(W_{1,m}) = i(C_m) + 1$

$\mathcal{J}(W_{1,m}) \sim \mathcal{J}(C_m) \cup \{u\}$

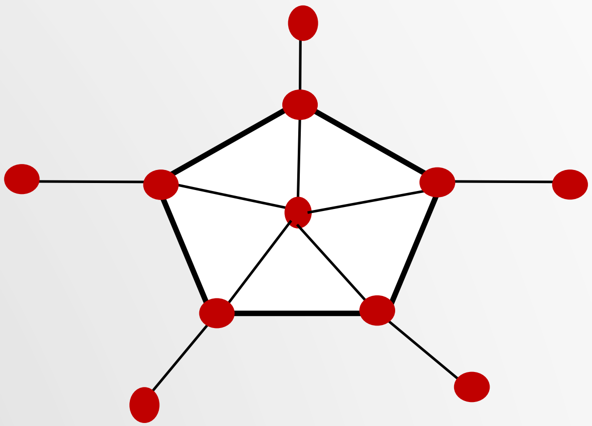
$\mathcal{K}(W_{1,m}) \sim K_m$

$f(H_{1,m}) = 2^m + 2^{m/2} L_{\sqrt{2},m}$

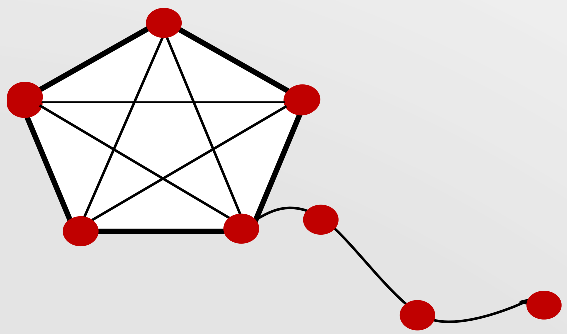
$i(H_{1,m}) = L_m$

$$\mathcal{J}(H_{1,m}) \sim \begin{cases} K_{L_m} & m \text{ impar} \\ K_{L_m} - \text{aresta} & \end{cases}$$

$\mathcal{K}(H_{1,m}) \sim S(K_m)$



# grafo *lollipop*



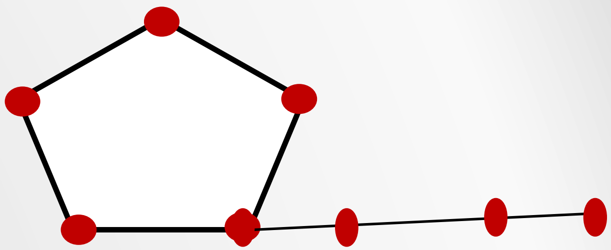
$f(L_{m,n}) = m\, F_{n+1} + F_n$

$i(L_{m,n}) = (m\text{-}1)\; i(P_n) + i(P_{n-1}),\; i(L_{m,1}) = m$

$\mathcal{J}(L_{m,n})$ ~ co-tripartido

$\mathcal{K}(L_{m,n}) \sim P_{n+1}$

# grafo girino (*tadpole*)

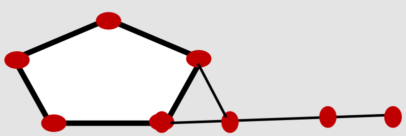


$f(T_{m,n}) = \; L_{m+n} + F_{m-3}\, F_n$ (De Maio & Jacobson, 2014)

$i(L_{m,n}) = i(P_{m-3})\; [\; i(P_{n-1}) + i(P_n)] + i(P_{m-6})\; i(P_n) + i(P_{m-7})\; i(P_{n-2})]$

$\mathcal{J}(L_{m,n})$ ~ co-tripartido

$\mathcal{K}(L_{m,n}) \sim B_{m,2,n-1}$



$B_{5,2,3}$

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# Parabéns Célia