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**LAWCG 22**  
WORKSHOP ON CLIQUES IN GRAPHS

Curitiba, Brazil, October 16-19th, 2022

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# Preface

The Latin American Workshop on Cliques in Graphs (LAWCG) is meant to foster interaction among the Latin American Graph Theory and Combinatorics community, whose research interests include cliques, clique graphs, the behavior of cliques and other topics in Graph Theory.

Twenty years ago, the first edition of LAWCG was held in Rio de Janeiro, in honor of Jayme Swarcfiter's 60th birthday. Now, in 2022, it is our greatest pleasure to celebrate the workshop's 20th anniversary along with Jayme's 80th birthday.

We are grateful to Ana Shirley Ferreira da Silva (UFC, Brazil), Mucuykak Guevara (UNAM, México), and Vinicius Fernandes dos Santos (UFMG, Brazil) for accepting our invitation to come as invited speakers and share their thoughts and experiences with our scientific community.

We sincerely thank all the authors and participants for their effective participation in LAWCG '22. We received 78 submissions by 176 authors from 11 countries: Argentina, Brazil, Canada, France, India, Iran, Mexico, Netherlands, Slovenia, Spain, and The United Kingdom. This book contains the 68 abstracts presented in the workshop.

We hope that these days in Curitiba can be a source of inspiration and increase the collaboration for the development of research in our community.

May our cliques grow larger and larger!

Curitiba, Brazil, October 16th, 2022

André Luiz Pires Guedes  
(General Chair)

# Program

8:00 - 18:00	<b>16/10/2022</b>	
	Reception Cocktail (San Juan Executive Hotel)	

8:00 - 8:20	<b>17/10/2022</b>	
8:20 - 9:10	Opening Ceremony	
9:10 - 9:20	Plenary Talk - Vinicius dos Santos	
9:20 - 9:40	Technical Session 1	Technical Session 2
9:40 - 10:00	Coffee break	
10:00 - 10:20	Technical Session 3	
10:20 - 10:50	Technical Session 4	
10:50 - 11:10	Lunch Break	
11:10 - 11:30	Plenary Talk Ana Shirley Shiva	
11:30 - 11:50	Technical Session 5	Technical Session 6
11:50 - 12:10	Technical Session 7	Technical Session 8
12:10 - 13:50	Coffee break	
13:50 - 14:40	Coffee break	
14:40 - 14:50	Technical Session 8	
14:50 - 15:10	Technical Session 9	
15:10 - 15:30	Technical Session 10	
15:30 - 15:50	Technical Session 11	
15:50 - 16:10	Technical Session 12	
16:10 - 16:40	Technical Session 13	
16:40 - 17:00	Technical Session 14	
17:00 - 17:20	Technical Session 15	
17:20 - 17:40	Technical Session 16	
17:40 - 18:00	Technical Session 17	

8:00 - 8:50	<b>18/10/2022</b>	
8:50 - 9:00	Plenary Talk - MucuyKak Guevara	
9:00 - 9:20	Technical Session 9	Technical Session 10
9:20 - 9:40	Coffee break	
9:40 - 10:00	Technical Session 11	
10:00 - 10:20	Technical Session 12	
10:20 - 10:50	Special Session in Honor of Jayme Swarcffler	
10:50 - 11:10	Lunch Break	
11:10 - 11:30	Conference Trip	
11:30 - 11:50	(Meeting point at San Juan Executive Hotel)	
11:50 - 12:20	(Ending at Venezia Restaurant)	
12:20 - 14:30	Dinner in Honor of Jayme Swarcffler (Venezia Restaurant)	
14:30 - 19:15	19:15 - 23:30	

8:00 - 9:00	<b>19/10/2022</b>	
9:00 - 9:20	Technical Session 13	Technical Session 14
9:20 - 9:40	Coffee break	
9:40 - 10:00	Technical Session 15	
10:00 - 10:20	Technical Session 16	
10:20 - 10:50	Lunch Break	
10:50 - 11:10	Technical Session 17	
11:10 - 11:30	Technical Session 18	
11:30 - 11:50	Closing Session	
11:50 - 12:10	Closing Session	
12:10 - 13:50	Closing Session	
13:50 - 14:10	Closing Session	
14:10 - 14:30	Closing Session	
14:30 - 14:50	Closing Session	
14:50 - 15:10	Closing Session	
15:10 - 15:30	Closing Session	

# Technical Sessions

## Monday - October 17

Monday - October 17		
9:20 - 10:20	<b>Session 1 - Room 1</b> <b>Networks</b> On the hardness of finding arc-disjoint branching flows in $bm(k, \lambda, s)$ -sufficient networks. Cláudio Carvalho, Jonas Silva, Raul Lopes, Ana Karolinna Maia, Nicolas Nisse and Cláudia Sales Positive results for finding arc-disjoint branching flows on $(k, \lambda, s)$ -sufficient networks Cláudio Carvalho, Jonas Costa, Raul Lopes, Ana Karolinna Maia, Nicolas Nisse and Cláudia Linhares Sales Graph properties on routing problems with time intervals Thalissson Clementino, Rosiane de Freitas and Eduardo Uchoa	<b>Session 2 - Room 2</b> <b>Geometry</b> On two-path geometries in digraphs Marisa Gutierrez, Mitre Dourado, Fabio Protti and Sílvia Tondato From word-representable graphs to altered Tverberg-type theorems Deborah Oliveros and Antonio Torres-Hernandez A New Heuristic for the Euclidean Steiner Tree Problem in $n$ Dimensions Nelson Maculan and Renan Pinto
	<b>Session 3 - Room 1</b> <b>Labeling and Coloring</b> Acyclic Coloring of Digraph Products Isnard Costa and Ana Silva Contributions in scheduling theory and special graph colorings with Jayme Rosiane de Freitas Multicolored Ramsey numbers for 4-cycle and stars Lucas da Penha Soares and Emerson Luiz Do Monte Carmelo Two infinite families of Type 1 generalized Petersen graphs Sérgio Fusquino, Mauro Nigro and Diana Sasaki	<b>Session 4 - Room 2</b> <b>Domination and Independence</b> Dominação Romana em Classes de Snarks Guilherme Willian Saraiva da Hora and Atílio Gomes Luiz Domination and Independent Domination Numbers of some Families of Snarks A. A. Pereira and C. N. Campos $k$ -independence in some Cartesian products Márcia Cappelle, Erika Coelho, Otávio Mortosa and Julliano Nascimento Weighted Connected Matchings Guilherme C. M. Gomes, Bruno P. Masquiro, Paulo E. D. Pinto, Vinícius F. dos Santos and Jayme L. Szwarcfiter
14:50 - 16:10	<b>Session 5 - Room 1</b> <b>Labeling and Coloring</b> Equitable total coloring of Semiblowup and Kochol snark families total coloring Isabel F. A. Gonçalves, Simone Dantas and Diana Sasaki Edge coloring of split graphs with even maximum degree Cintia Izabel Cararo, Sheila Morais de Almeida, Cândida Nunes da Silva and Glasielly Demori Proença The $(p, 1)$ -total number of graphs with maximum degree three Mayara Omai, C. N. Campos and Atílio G. Luiz Estudo sobre $(r + 1)$ -atribuição de papéis para prismas complementares, com $r \geq 3$ Diane Castonguay, Elisângela S. Dias, Fernanda N. Mesquita and Julliano R. Nascimento	<b>Session 6 - Room 2</b> <b>Graph Classes</b> K-comportamiento de gráficas cocordales Lesli Hernández-Sayago, Miguel Pizaña and Rafael Villarreal-Flores Hardness of the $f$ -Reversible Process in Directed Graphs Isaac Costa, Carlos Vinícius Lima and Thiago Braga Marclon How to draw a $K(n, 2)$ Kneser graph? Luerbio Faria, Antonio Sousa, Jonas Carneiro and Mario Pabon Fullerene Waves João Pedro Costa and Diego Nicodemos
	<b>Session 7 - Room 1</b> <b>Labeling and Coloring</b> On Total Colouring Bipartite Graphs with at Most Three Bicliques Gustavo Leardini Montanheiro, Leandro Zatesko and Marina Groshaus Local antimagic chromatic number of Bethe trees Francisca Andrea Macedo França, Andre Ebling Brondani and Lara Rodrigues Ventura On non-equitable color class configurations for small Type 1 cubic graphs Matheus Aداuto, Celina Figueiredo and Diana Sasaki Locally irregular decompositions of a class of subcubic graphs Carla Lintzmayer, Guilherme Mota, Lucas Rocha and Maycon Sambinelli	<b>Session 8 - Room 2</b> <b>Graph Operations</b> On tessellations and graph operations: Adding pendant and false twin vertices Alexandre De Abreu, Celina De Figueiredo, Franklin Marquezino and Daniel Posner Reducing the Time Complexity of Computing Square Roots with Girth at Least Six of a Graph Cristopher Carcereri, Alefher Rocha and Renato Carmo On iterated clique graphs with exponential growth Miguel Pizaña and Ismael Robles Critical generators of $K_5$ Gabriela Ravenna and Liliana Alcon

## Tuesday - October 18

		Session 9 - Room 1	Session 10 - Room 2
		Graph Classes	Applications
9:00 - 10:20		Containment among classes of interval graphs with interval count k Livia Medeiros, Fabiano Oliveira and Jayme Szwarcfiter	Monkey Hash Map: a highly performant thread-safe map without locks Judismar Arpini Junior and Vinicius G. Pereira de Sá
		On cycle-free-CPT posets Liliana Alcón, Noemí Amalia Gudíño and Marisa Gutierrez	Clique problems in 3D molecular prediction João Alfredo Holanda Bessa Neto, Clarice Santos, Rosiane de Freitas, Micael Oliveira, Jonathas Nunes and Kelson Mota
		Chordal Thinness Bernardo Amorim, Gabriel Coutinho and Vinicius dos Santos	COVID-19 mortality prediction - Perceptron and Random Forest applications João Pedro Marcelino Terra, Luerbio Faria and Fabiano Oliveira
		On two variants of split graphs Luciano Grippo and Verónica Moyano	Restricted Hamming-Huffman trees Min Lin, Fabiano Oliveira, Paulo Pinto, Moisés Sampaio Jr. and Jayme Szwarcfiter

		Session 11 - Room 1	Session 12 - Room 2
		Flow Graphs	Geometry
10:50 - 11:50		A simple proof of the bijection between Minimal Feedback Arc Sets and Hamiltonian Paths in tournaments Rafael Schneider and Fábio Botler	Spectral properties of threshold k-uniform hypergraphs Lucas Portugal and Renata Del-Vecchio
		Control flow graph, formal verification and constraint programming techniques Jesse Deveza, Lanier Santos, Rosiane de Freitas and Lucas Cordeiro	On a semidefinite relaxation for the maximum k-colourable subgraph problem Marcel K. de Carli Silva, Gabriel Coutinho, Rafael Grandsire and Thiago Oliveira
		FPT algorithm for feedback vertex set in reducible flow hypergraphs Luerbio Faria, André L. P. Guedes and Lilian Markenzon	Positive semidefiniteness of $A\alpha(G)$ on some families of graphs with k cycles Carla Oliveira, André Brondani and Victor Melquades

## Wednesday - October 19

		Session 13 - Room 1	Session 14 - Room 2
		Labeling and Coloring	Computational Complexity
9:00 - 10:20		On total coloring of subcubic graphs Luerbio Faria, Mauro Nigro and Diana Sasaki	Elecciones con Simetrías Claudia De la Cruz and Miguel Pizaña
		Neighbor distinguishing coloring for cacti graphs Vinicius De Souza Carvalho, Maycon Sambinelli and Carla Negri Lintzmayer	NP-Hardness of perfect rainbow polygons David Flores-Peñalcoza and Andrés Fuentes-Hernández
		Edge-Sum Distinguishing game Deise L. de Oliveira, Danilo Artigas, Simone Dantas and Atílio G. Luiz	Parameterized complexity of computing maximum minimal blocking and hitting sets Julio Araujo, Marin Bougeret, Victor Campos and Ignasi Sau
		The $(2,1)$ -total number of powers of paths and powers of cycles M. M. Omai, C. N. Campos and Atílio G. Luiz	Theoretical and empirical analysis of algorithms for the max-npv project scheduling problem Isac M. Lacerda, Rosiane de F. Rodrigues, Eber A. Schmitz and Jayme L. Szwarcfiter

		Session 15 - Room 1	Session 16 - Room 2
		Games	Graph Classes
10:50 - 12:10		Some variations of the Tower of Hanoi and their graph properties Lia Martins, Meng Hsu, Raquel Folz and Rosiane De Freitas	On the Helly Number of trees Moisés Carvalho, Simone Dantas, Mitre Dourado, Daniel Posner and Jayme Szwarcfiter
		The Conflict-Free coloring game and cliques Paola Tatiana Huaynoca, Miguel Palma and Simone Dantas	On the Biclique Graphs of Circular Arc Bigraphs Fabricio Schiavon Kolberg, Marina Groshaus and André L. P. Guedes
		Hardness of general position games Ullas Chandran S.V., Sandi Klavzar, Neethu P. K. and Rudini Sampaio	Tree 3-spanners on prisms of graphs Renzo Gomez, Flavio K. Miyazawa and Yoshiko Wakabayashi
		Notes on graph variations of the NIM game Raquel Folz, Meng Hsu, Lia Martins and Rosiane de Freitas	Extendiendo Gráficas Cuadrado-complementarias Ariadna Juarez-Valencia and Miguel Pizaña

		Session 17 - Room 1	Session 18 - Room 2
		Computational Complexity	Labeling and Coloring
13:50 - 15:10		The Terminal Connection Problem on Rooted Directed Path Graphs is NP-complete Alexander Melo, Celina Figueiredo, Ana Silva and Uéverton Souza	O Número Cromático Total de Grafos Split 2-admissíveis Diego Amaro Costa, Sulamita Klein and Fernanda Couto
		Subdivisions with Parity in Digraphs Marcus Vinicius Martins Melo and Ana Karolinna Mala	Neighbour-distinguishing edge-labelling of powers of paths Luis Gustavo Da Soledade Gonzaga and Christiane Neme Campos
		The absolute oriented clique number problem is NP-complete Erika Moraes Martins Coelho, Hebert Coelho, Luerbio Faria, Mateus de Paula Ferreira and Sulamita Klein	Hunting a conformable fullerene nanodisc that is not 4-total colorable Mariana Cruz, Celina Figueiredo, Diana Sasaki, Marcus Vinicius Tovar Costa and Diego Nicodemos
		Hard instances for the maximum clique problem Rodrigo Nogueira, Victor Campos and Renato Carmo	A New Bound for the Sum of Squares of Degrees in a Class 2 Graph Thiago Cunha and Leandro Zatesko



# Contents

## Invited Talks

A gentle introduction to reconfiguration . . . . .	16
Connectivity Problems on Temporal Graphs . . . . .	17
A tour of kernels in digraphs and their generalizations . . . .	18

## Session 1

On the hardness of finding arc-disjoint branching flows in $(\mathbf{k}, l, \mathbf{s})$ -sufficient networks . . . . .	20
Positive results for finding arc-disjoint branching flows on $(\mathbf{k}, l, \mathbf{s})$ -sufficient networks. . . . .	21
Graph properties on routing problems with time intervals . .	22

## Session 2

On two-path geometries in digraphs . . . . .	24
From word-representable graphs to altered Tverberg-type theorems . . . . .	25
A New Heuristic for the Euclidean Steiner Tree Problem in $n$ Dimensions . . . . .	26



### Session 3

Acyclic Coloring of Digraph Products <sup>†</sup> . . . . .	28
Contributions in scheduling theory and distance graph colorings with Jayme . . . . .	29
Multicolored Ramsey numbers for 4-cycle and stars . . . . .	30
Two infinite families of Type 1 generalized Petersen graphs . . . . .	31

### Session 4

Dominação Romana em Classes de Snarks <sup>†</sup> . . . . .	33
Domination and Independent Domination Numbers of some Families of Snarks <sup>†</sup> . . . . .	34
$k$ -independence in some Cartesian products . . . . .	35
Weighted Connected Matchings . . . . .	36

### Session 5

Equitable total coloring of Semiblowup and Kochol snark families total coloring <sup>†</sup> . . . . .	38
Edge coloring of split graphs with even maximum degree <sup>†</sup> . . . . .	39
The $(p,1)$ -total number of graphs with maximum degree three <sup>†</sup> . . . . .	40
Estudo sobre $(\mathbf{r} + \mathbf{1})$ -atribuição de papéis para prismas complementares, com $\mathbf{r} \geq \mathbf{3}$ . . . . .	41

### Session 6

$\mathbf{K}$ -comportamiento de gráficas cocordales . . . . .	43
Hardness of the $f$ -Reversible Process in Directed Graphs <sup>†</sup> . . . . .	44
How to draw a $K(n,2)$ Kneser graph? <sup>†</sup> . . . . .	45
Fullerene Waves . . . . .	46

## Session 7

On Total Colouring Bipartite Graphs with at Most Three Bicliques <sup>†</sup> . . . . .	48
Local antimagicchromatic number of Bethe trees . . . . .	49
On non-equitable color class configurations for small Type 1 cubic graphs . . . . .	50
Locally irregular decompositions of a class of subcubic graphs . . . . .	51

## Session 8

On tessellations and graph operations: Adding pendant and false twin vertices . . . . .	53
Reducing the Time Complexity of Computing Square Roots with Girth at Least Six of a Graph . . . . .	54
On iterated clique graphs with exponential growth . . . . .	55
Critical generators of $K_5$ . . . . .	56

## Session 9

Containment among classes of interval graphs with interval count $k$ . . . . .	58
On cycle-free-CPT posets . . . . .	59
Chordal Thinness <sup>†</sup> . . . . .	60
On two variants of split graphs . . . . .	61

## Session 10

Monkey Hash Map: a highly performant thread-safe map without locks . . . . .	63
Clique-based problems in the structural prediction of complex molecules . . . . .	64

COVID-19 mortality prediction - Perceptron and Random Forest applications <sup>†</sup> . . . . .	65
Restricted Hamming-Huffman trees . . . . .	66

## Session 11

A simple proof of the bijection between Minimal Feedback Arc Sets and Hamiltonian Paths in tournaments . . . . .	68
Control flow graph, formal verification and constraint programming techniques . . . . .	69
FPT algorithm for feedback vertex set in reducible flow hypergraphs <sup>†</sup> . . . . .	70

## Session 12

Spectral properties of threshold $k$ -uniform hypergraphs . . . . .	72
On a semidefinite relaxation for the maximum $k$ -colourable subgraph problem . . . . .	73
Positive semidefiniteness of $A_a(G)$ on some families of graphs with $k$ cycles . . . . .	74

## Session 13

On total coloring of subcubic graphs . . . . .	76
Neighbor distinguishing coloring for cacti graphs . . . . .	77
Edge-Sum Distinguishing game <sup>††</sup> . . . . .	78
The $(2,1)$ -total number of powers of paths and powers of cycles <sup>†</sup> . . . . .	79

## Session 14

Elecciones con Simetrías . . . . .	81
NP-Hardness of perfect rainbow polygons <sup>†</sup> . . . . .	82

Parameterized complexity of computing maximum minimal blocking and hitting sets . . . . .	83
Theoretical and empirical analysis of algorithms for the <i>max-npv</i> project scheduling problem . . . . .	84

## Session 15

Some variations of the Tower of Hanoi and their graph properties . . . . .	86
The Conflict-Free coloring game and cliques <sup>†</sup> . . . . .	87
Hardness of general position games . . . . .	88
Notes on graph variations of the NIM game . . . . .	89

## Session 16

On the Helly Number of trees <sup>††</sup> . . . . .	91
On the Biclique Graphs of Circular Arc Bigraphs . . . . .	92
Tree <b>3</b> -spanners on prisms of graphs . . . . .	93
Extendiendo Gráficas Cuadrado-Complementarias . . . . .	94

## Session 17

The Terminal Connection Problem on Rooted Directed Path Graphs is <b>NP</b> -complete . . . . .	96
Subdivisions with Parity in Digraphs . . . . .	97
The absolute oriented clique number problem is <b>NP</b> -complete . . . . .	98
Hard instances for the maximum clique problem . . . . .	99

## Session 18

O Número Cromático Total de Grafos Split 2-admissíveis . . . . .	101
Neighbour-distinguishing edge-labelling of powers of paths . . . . .	102

Hunting a conformable fullerene nanodisc that is not 4-total colorable .....	103
A New Bound for the Sum of Squares of Degrees in a Class 2 Graph <sup>†</sup> .....	104
Author Index .....	105



# Invited Talks

A gentle introduction to reconfiguration . . . . .	16
Connectivity Problems on Temporal Graphs . . . . .	17
A tour of kernels in digraphs and their generalizations	18

## A gentle introduction to reconfiguration

Vinicius Fernandes dos Santos

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**Chair: Leandro Zatesko**

Transforming (combinatorial) objects into others is a very general task that can be used to model natural questions. Tasks such as solving certain puzzles, sorting an array or exploring the solution space of a given problem, could sound really different at first glance, but one can find similar elements: there is a set of allowed states, rules for navigating through them and a goal. Combinatorial Reconfiguration is the study of problems having those elements and has been attracting growing attention recently. Common questions in this setting are the connectivity of the reconfiguration graph, the reachability of a target state from an initial state or, equivalently, verifying whether two states belong to the same connected component of the reconfiguration graph), and the shortest path in that graph. In this talk, we will give an overview of this research field, presenting a selection of results of different flavors of reconfiguration in graphs, such as reconfiguration of (labeled) tokens, cliques, separators, among others.



## Connectivity Problems on Temporal Graphs

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**Chair: Cláudia Linhares Sales**

A temporal graph is a graph that changes in time, meaning that, at each timestamp, only a subset of the edges is active. This structure models all sorts of real life situations, from social networks to public transportation, having been used also for contact tracing during the COVID pandemic. Despite its broad applicability, and despite being around for more than two decades, only recently this structure has received more attention from the community. In this talk, we will discuss how to bring some connectivity concepts to the temporal context, and we will learn about the state of the art of complexity results of the related problems. Additionally, we will see various possible adaptations of Menger's Theorem, only a few of which also hold on temporal graphs.

## A tour of kernels in digraphs and their generalizations

Mucuy-kak Guevara

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**Chair: Miguel Pizaña**

A kernel  $N$  of  $D$  is an independent set of vertices such that for every  $w \in V(D) \setminus N$  there exists an arc from  $w$  to  $N$ . A digraph is said to be kernel-perfect if and only if any induced subdigraph has a kernel.

The concept of kernel was introduced by J. Von Neumann and has found many applications, for instance in cooperative  $n$ -person games, Nim-type games, in logic, etc. The main question is: Which structural properties of a digraph  $D$  imply that  $D$  has a kernel?

A classical result obtained by Sands, Sauer and Woodrow asserts that the union of two transitive digraphs is a kernel-perfect digraph. Later it was studied the union of a right-pretransitive and a left-pretransitive digraphs and how it becomes a kernel-perfect digraph. In this talk, will be given sufficient conditions for the union of two kernel-perfect digraphs to become a kernel-perfect digraph.

Let  $F$  be a set of arcs of  $D$  a set  $S \subseteq V(D)$  is called a semikernel of  $D$  modulo  $F$  if  $S$  is an independent set of vertices such that for every  $z \in V(D) \setminus S$  for which there exists an  $Sz$ -arc of  $D \setminus F$ , there also exists a  $zS$ -arc in  $D$ . In this talk, the concept of semikernel modulo  $F$  is used to obtain a new sufficient condition for the existence of kernels in digraph, even in infinite digraphs. As a consequence is obtained a generalization of the result of B. Sands, N. Sauer and R. Woodrow.

Also in this talk, will be presented how a generalization of the line digraph, the partial line digraph, preserves independent sets by directed path,  $(k, l)$ -kernels (that means directed path are used) and some generalization of  $(k, l)$ -kernels. And will be compared the number of kernels and generalizations in  $D$  and in its partial line digraph,  $L(D)$ . Also will be presented a generalization of the Grundy function using neighborhood at distance  $l$ . Furthermore, will be showed that we can say relation between the  $(k, l)$ -kernels and the Grundy functions in  $D$ , as there exists relation between the kernels and the Grundy function. And finally, will be reproduced the previous idea (and results) but now in edge colored digraphs and  $(k, l)$ -kernels by monochromatic directed path.

# Session 1

## *Networks*

Chair: Luerbio Faria

On the hardness of finding arc-disjoint branching flows in $(\mathbf{k}, I, \mathbf{s})$ -sufficient networks . . . . .	20
Positive results for finding arc-disjoint branching flows on $(\mathbf{k}, I, \mathbf{s})$ -sufficient networks. . . . .	21
Graph properties on routing problems with time intervals	22

On the hardness of finding arc-disjoint branching flows in  
 $(k, l, s)$ -sufficient networks

Cláudio Carvalho<sup>1</sup> Jonas Costa<sup>1,†,\*</sup> Raul Lopes<sup>1</sup>

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*Keywords:* network flows, arc-disjoint flows, branchings, branching flows

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An  $s$ -branching is a digraph where every vertex has indegree exactly one, except for  $s$  which has indegree zero. An  $s$ -branching flow in a network  $N = (D, u)$ , where  $u$  is the capacity function, is a flow that leaves from  $s$  with value  $|V(D)| - 1$  such that every other vertex of  $V(D)$  may consume exactly one unit of flow. We say that  $N$  is  $(k, l, s)$ -sufficient if  $d_D^-(X) \geq k|X|/l$  for all  $X \subseteq V(D) - \{s\}$ . The  $(k, l, s)$ -sufficiency is a necessary condition for the existence of  $k$  arc-disjoint branching flows in  $N$ , if  $u(a) = l$  for every arc  $a \in A(D)$ . In [2] the authors asked if it was also a sufficient condition for those networks since it generalises the characterization given by Edmonds in [3] of digraphs with  $k$  arc-disjoint branchings. In [1] it was shown that it is not a sufficient condition in general to the existence of such flows. In this work, we go further in this matter and show that actually it is NP-complete to decide whether a  $(k, l, s)$ -sufficient network admits  $k$  arc-disjoint branching flows.

## References

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<sup>†</sup>Supported by CAPES

Positive results for finding arc-disjoint branching flows on  $(\mathbf{k}, I, \mathbf{s})$ -sufficient networks.

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*Keywords:* network flows, arc-disjoint flows, branchings, branching flows

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An  $s$ -branching is a digraph  $D$  where every vertex has indegree exactly one, except for  $s$  which has indegree zero. An  $s$ -branching flow in a network  $N = (D, u)$ , where  $u$  is the capacity function, is a flow of value  $|V(D)| - 1$  that leaves from  $s$  so that every vertex of  $V(D) \setminus \{s\}$  consumes exactly one unit of flow. Based on a classical result by Edmonds [3], Costa et al. [2] proposed a necessary condition for the existence of branching flows on networks. They stated that if a network  $N = (D, u \equiv I)$  admits  $k$  arc-disjoint branching flows, then every nonempty set  $X \subseteq V(D) \setminus \{s\}$  has indegree at least  $k \lceil |X|/|I| \rceil$ . Carvalho et al. [1] showed that this is not a sufficient condition, in general, to the existence of  $k$  arc-disjoint branching flows. We call the digraphs that meet this condition, and so the networks defined on them, as  $(k, I, s)$ -sufficient. In this work, we show that the condition guarantees the existence of the desired flows in some particular cases: when the underlying simple graph of  $D$  is a spindle; or, for general digraphs, when  $|I| = |V(D)| - 2$ ; or when  $|I|$  or  $k$  is one.

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- [2] Costa, J.; Linhares Sales, C.; Lopes, R.; Maia, A., Um estudo de redes com fluxos ramificados arco-disjuntos, Matemática Contemporânea, (2019), Volume 46, 230–238.
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Graph properties on routing problems with time intervals

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*Keywords: BCP, dynamic routing, integer programming, time intervals.*

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In this work, we deal with vehicle routing problems (Toth, P. and Vigo, D. *Vehicle routing: problems, methods, and applications*, 2014) with two different kinds of time constraints. Given a directed graph  $G = (V, A)$ , where the set of vertices  $V$  is composed by the vertex 0, which represents the depot, and a set of vertices  $N = \{1, \dots, n\}$ , which represents the clients that will be visited.  $A$  is the set of directed edges (or arcs) connecting each pair  $(i, j) \in V \times V$ , where  $i \neq j$ . There is a travel cost  $c_{ij}$  and a travel time  $t_{ij} > 0$  associated with each edge. The travel time  $t_{ij}$  includes the service time at vertex  $i$ . A set of available vehicles is denoted by  $K = \{1, \dots, k\}$ . The vehicles are homogeneous and have the capacity  $Q$ . For each vertex  $i \in V$  is associated with a demand  $q_i > 0$ . Routes must be created and assigned to each vehicle in such a way that the vehicle respects capacity constraints. Also, we are interested in adding two-time constraints: the first one is a traditional time window  $[e_i, l_i]$  associated with each vertex  $i \in V$ , where  $e_i$  and  $l_i$  represent respectively the first and last moments to visit the client or vertex  $i$ ; and, the second one is a release date  $r_i$  associated with the package to be delivered at client  $i \in N$ , representing the time this package is available to leave the depot. The objective is to minimize the total cost of routes such that these routes comply with capacity and time intervals constraints. We will discuss some results obtained with the proposed formulation in a Branch-Cut-and-Price method, exploring properties in graphs.

# Session 2

## *Geometry*

Chair: Marisa Gutierrez

On two-path geometries in digraphs . . . . .	24
From word-representable graphs to altered Tverberg-type theorems . . . . .	25
A New Heuristic for the Euclidean Steiner Tree Problem in n Dimensions . . . . .	26

## On two-path geometries in digraphs

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*Keywords: convex geometry, digraphs*

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Let  $G$  be a simple finite graph, with vertex set  $V(G)$  and  $\mathcal{C}$  a family of subsets of  $V(G)$ . The pair  $(G, \mathcal{C})$  is a graph convexity when the  $\emptyset$  and  $V(G)$  are in  $\mathcal{C}$  and  $\mathcal{C}$  is closed over intersection. The subsets belonging to  $\mathcal{C}$  are called *convex sets*. Given a set  $S \subseteq V(G)$ , the smallest convex set containing  $S$  is called the *convex hull* of  $S$ . An element  $x \in S$ , where  $S \subseteq V(G)$  is a convex set, is called an *extreme vertex* of  $S$  if  $S \setminus \{x\}$  is also convex. A graph equipped with a convexity space is a *convex geometry* if it satisfies the so-called *Minkowski-Krein-Milman* property: *Every convex set is the convex hull of its extreme vertices*. In the last few decades, graph convexity has been studied in many contexts. In particular, some studies are devoted to determine if a graph equipped with a convexity space is a convex geometry. Chordal, Ptolemaic, strongly chordal, interval, and weakly polarizable graphs have been characterized as convex geometries with respect to the monophonic, geodetic, strong, toll, and  $m^3$  convexities, respectively.

In this work we study geometries in digraphs. As a first step we study the geometry that corresponds to directed paths of two edges, called two-path geometry. It is clear that directed cycles are not two-path geometries, as a consequence no digraph with induced directed cycles will be a two-path geometry. We prove that a directed path  $P: x \rightarrow y \rightarrow z \rightarrow t$  and  $P$  with a long chord are not two-path geometries and these two forbidden digraphs characterize the two-path geometries in some classes of digraphs.



From word-representable graphs to  
altered Tverberg-type theorems

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*Keywords:* Tverberg Theorem,  $k$ -word-representable graphs, Nerve Complexes, Erdős-Szekeres Theorems.

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Given a finite collection of points  $S \subset \mathbb{R}^d$  and an  $m$ -partition into  $m$  color classes  $P = S_1, \dots, S_m$  of  $S$ , the nerve of the partition,  $N(P)$  is the simplicial complex with vertex set  $[m] := \{1, 2, \dots, m\}$  whose faces are  $I \subset [m]$  such that  $\bigcap_{i \in I} (\text{conv}(S_i)) \neq \emptyset$ , where  $\text{conv}(S_i)$  is the convex hull of the elements in the color class  $i$ .

Tverberg's theorem, one of the most celebrated and beautiful theorems in discrete geometry, (H. Tverberg (1966)). May be thought as a Ramsey-type theorem, where one studies how every sufficiently large system (set of points) must contain a large well-organized sub-system. "Sufficiently large" means that for every set of points  $S$  with at least  $(d+1)(m-1)+1$  points, there always exists a partition  $P$  into  $m$  color classes such that  $N(P)$ , is a simplex.

De Loera, J., De Loera, J. A., Hogan, T. A., Oliveros, D., and Yang, D. (2021) show that Tverberg's theorem can be seen as a special case of a more general situation, proving that some new Ramsey-Tverberg-type theorem may occur with other nerves different than the simplex. A graph  $G = (V, E)$  is *word-representable* if there exists a word  $W$  over the alphabet  $V$  such that letters  $x$  and  $y$  alternate in  $W$  if and only if  $\{x, y\} \in E$  for each  $x \neq y$ . In particular, if the length of the alternation is  $k$ , the word is called *k-representable*. Not all graphs are word-representable, but we have observed that with a slight generalization and allowing the word  $W$  to contain a  $k$ -alternating subwords, every graph is "*general k-word-representable*" for some  $k$ . In this talk we observe an engaging connection between general  $k$ -word-representable graphs and Ramsey-Tverberg-type results, where nerve structures are shown to arise once we have sufficiently many points.

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## A New Heuristic for the Euclidean Steiner Tree Problem in $n$ Dimensions

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*Keywords: Euclidean Steiner Tree Problem, Combinatorial Optimization, Heuristic, Second-Order Cone Programming*

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Given  $p$  points in  $\mathbb{R}^n$ , called terminal points, the Euclidean Steiner Tree Problem (ESTP) consists of finding the shortest tree connecting them, using or not extra points, called Steiner points. This is a well known NP-hard combinatorial optimization problem. Instances with thousands of points have been solved for  $n = 2$ . However, methods specialized for the ESTP in  $\mathbb{R}^2$  cannot be applied to problems in higher dimensions. Enumeration schemes have been proposed in the literature. Unfortunately, the number of Steiner trees having  $p$  terminal points grows extremely fast with  $p$ , so the enumeration of all trees is only possible for very small values of  $p$ . For  $n \geq 3$ , even small instances with tens of points cannot be solved with exact algorithms in reasonable time. In this work, we present two heuristics for the ESTP. These heuristics differ from most existent ones in the literature in the fact that they do not rely on the minimum spanning tree of the terminal points. Instead, they start with a single extra point connected to all terminal points and new extra points are introduced iteratively according to angle properties for two consecutive edges. The heuristics return the optimal solution in most of the small test instances. For large instances, where the optimum is not known, the heuristics return relatively good solutions, according to their Steiner ratio.

# Session 3

## *Labeling and Coloring*

Chair: David Flores

Acyclic Coloring of Digraph Products <sup>†</sup> . . . . .	28
Contributions in scheduling theory and distance graph colorings with Jayme . . . . .	29
Multicolored Ramsey numbers for 4-cycle and stars . . . . .	30
Two infinite families of Type 1 generalized Petersen graphs . . . . .	31

Acyclic Coloring of Digraph Products<sup>†</sup>Isnard Lopes Costa<sup>1,2</sup> Ana Silva<sup>1,2</sup><sup>2</sup> Departamento de Matemática - Universidade Federal do Ceará  
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*Keywords: acyclic coloring, acyclic chromatic number, digraph products.*

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Given a digraph  $G$ , an *acyclic coloring* of  $G$  is a partition of  $V(G)$  into subsets, each of which induce an acyclic subdigraph of  $G$ ; and the *acyclic chromatic number* of  $G$  is the smallest integer  $c_a(G)$  such that  $G$  admits an acyclic coloring [2]. This is a generalization of the classic chromatic number of graphs. In this work, given digraphs  $G$  and  $H$ , we investigate the acyclic chromatic number of the cartesian ( $G \square H$ ), direct ( $G \times H$ ), strong ( $G \boxplus H$ ), and lexicographic ( $G[H]$ ) products of  $G$  and  $H$ , giving generalizations of some classic results on the chromatic number of products. More specifically, we prove that the following results, whose analogous counterpart are known to hold for the chromatic number of products of graphs, still hold for the acyclic chromatic number of products of digraphs:  $c_a(G \square H) = \max\{c_a(G), c_a(H)\}$ ;  $c_a(G \times H) \leq \min\{c_a(G), c_a(H)\}$ ; and  $c_a(G[H]) = c_a(G[\vec{K}_k])$ , where  $k = c_a(H)$  and  $\vec{K}_k$  denotes the complete digraph on  $k$  vertices. In addition, we investigate the products of directed cycles, giving exact values for  $c_a(\vec{C}_n \times \vec{C}_m)$  and  $c_a(\vec{C}_n \boxplus \vec{C}_m)$  for every  $n, m$ , and for  $c_a(\vec{C}_n[H])$  for every positive integer  $n$ . For more details, we refer to [1].

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Contributions in scheduling theory and distance graph colorings with Jayme

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*Keywords:* algorithms, distance graph coloring, Jayme Szwarcfiter, job-machine constraints, project scheduling, T-coloring, UET scheduling.

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In commemoration of the eightieth birthday celebration of the Brazilian researcher Jayme Szwarcfiter, a reference in graph theory, algorithms, computational complexity, and combinatorics in general, this work summarizes the joint contributions of this author with Jayme, also involving other researchers, in scheduling theory, especially involving UET (Unit Execution Time) scheduling in identical parallel machines (deFreitas *et. al.*, 2011) (Dourado *et. al.*, 2010) (Rodrigues, 2009), scheduling on parallel machines considering job-machine dependency constraints (deFreitas *et. al.*, 2014) (deFreitas *et. al.*, 2012) (Dourado *et. al.*, 2010), project scheduling involving directed graphs (Mendes *et. al.*, 2021) (Mendes *et. al.*, 2020), and also, contributions in distance-constrained vertex coloring problems (deFreitas *et. al.*, 2021) (Dias *et. al.*, 2021) (Dias *et. al.*, 2020) (Dias *et. al.*, 2019).

Multicolored Ramsey numbers for 4-cycle and stars  
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*Keywords: generalized Ramsey number, 4-cycle, stars, upper bound.*

Let  $K_r$  be a complete graph on  $r$  vertices. Given graphs  $G_1, \dots, G_k$ , the *generalized Ramsey number*  $r(G_1, \dots, G_k)$  denotes the smallest positive integer  $r$  such that any  $k$ -coloring of the edges of  $K_r$  contains a monochromatic copy of  $G_i$  in color  $i$  for some  $i$ ,  $1 \leq i \leq k$ .

Let us focus on the classical numbers  $r(C_4, K_{1,n})$ . In 1975, Parsons proved the upper bound  $r(C_4, K_{1,n}) \leq n + \lceil \sqrt{n} \rceil + 1$  and was able to evaluate the exact classes  $r(C_4, K_{1,q^2}) = q^2 + q + 1$  and  $r(C_4, K_{1,q^2+1}) = q^2 + q + 2$  for every prime power  $q$  by using the Erdős-Rényi graph  $ER_q$ , the polarity graph arising from the projective plane of order  $q$ .

It is worth mentioning that exact values of Ramsey numbers are very often highly non-trivial to establish. Indeed, a few exact classes have been evaluated by Parsons (1975-76), Monte Carmelo (2008), Wu, Su and Zhang (2015) and Zhang, Chen and Edwin (2017).

In 2019, Zhang, Chen and Cheng investigated the following generalization

$$r_k(n) = r(\underbrace{C_4, \dots, C_4}_{k \text{ times}}, K_{1,n}).$$

They proved a general upper bound on  $r_k(n)$  and determined the exact class on  $r_2(q^2 - q)$  for a prime power  $q$  by using difference sets.

In this talk, we obtain upper bounds on  $r_k(n)$  on the basis of density arguments on certain related matrices. Our results improve the known general upper bound for certain instances.

Two infinite families of Type 1  
generalized Petersen graphs

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*Keywords: total coloring, generalized petersen graphs, cubic graphs*

The Total Coloring Conjecture states that the total chromatic number of a graph  $G$  is at most  $D(G) + 2$ , where  $D(G)$  is the maximum degree of  $G$ . Clearly, the total chromatic number is at least  $D(G) + 1$ . This conjecture has been proved for cubic graphs, so the total chromatic number of a cubic graph is either 4 (called Type 1) or 5 (called Type 2). It is NP-hard to decide whether a cubic graph is Type 1, even restricted to bipartite cubic graphs.

In this work, we investigate the total coloring of a well-known class of cubic graphs introduced by Watkins in 1969, called generalized Petersen graphs. A generalized Petersen graph  $G(n, k)$  is a cubic graph with  $n \geq 3$ ,  $1 \leq k \leq n-1$ , which has  $V(G(n, k)) = \{u_0, \dots, u_{n-1}, v_0, \dots, v_{n-1}\}$  and  $E(G(n, k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 0 \leq i \leq n-1\}$ , indexes taken modulo  $n$ .

The girth of a graph is the size of the smallest cycle contained in it. Only two generalized Petersen graphs are known to be Type 2 and they have small girth. Besides, it is known that for  $2 \leq k \leq \frac{n}{2}$ , graph  $G(n, k)$  has girth at least 5 whenever  $n \notin \{2k, 3k, 4k\}$  (S. Dantas, C. de Figueiredo, G. Mazzuocolo, M. Preissmann, V. dos Santos, and D. Sasaki, On the total coloring of generalized Petersen graphs, *Discrete Mathematics*, 339 (2016), pp. 1471–1475). So one could hope to find an answer to the following question among this family.

**Question 1** — Brinkmann et al., 2015. Does there exist a Type 2 cubic graph with girth at least 5?

We prove that all members of two infinite families of generalized Petersen graphs are Type 1, by presenting total colorings for them using 4 colors.

**Theorem 1** All generalized Petersen graphs  $G(3j, k)$  and  $G(5j, 5q+2)$ , for  $q \geq 0$ ,  $j \geq 2q+1$  and  $k \not\equiv 0 \pmod{3}$ , are Type 1.

# Session 4

## *Domination and Independence*

Chair: Atilio Gomes Luiz

Dominação Romana em Classes de Snarks <sup>†</sup> . . . . .	33
Domination and Independent Domination Numbers of some Families of Snarks <sup>†</sup> . . . . .	34
$k$ -independence in some Cartesian products . . . . .	35
Weighted Connected Matchings . . . . .	36



Dominação Romana em Classes de Snarks<sup>†</sup>Guilherme Willian Saraiva da Hora<sup>1,\*</sup> Atilio Gomes Luiz<sup>1</sup><sup>1</sup> Universidade Federal do Ceará, Quixadá, Brasil

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*Palavras-chave: dominação em grafos, dominação romana, grafos snarks*

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Dado um grafo  $G = (V(G), E(G))$ , a função  $f: V(G) \rightarrow \{0, 1, 2\}$  é uma *Função de Dominação Romana* (FDR) de  $G$  se todo vértice  $v \in V(G)$  com  $f(v) = 0$  é adjacente a pelo menos um vértice  $u$  com  $f(u) = 2$ . O *peso* de  $f$ , denotado por  $w(f)$ , é definido como  $w(f) = \sum_{v \in V(G)} f(v)$ . O *número de dominação romana* de  $G$  é o menor valor  $w(f)$  dentre todas as FDRs  $f$  de  $G$  e é denotado por  $g_R(G)$ . Este problema foi proposto por Cockayne et al. (Cockayne, Dreyer, Hedetniemi, Hedetniemi, 2004), tendo como motivação um problema de estratégia militar. Dreyer provou que o problema de determinar se um grafo  $G$  possui  $g_R(G) \leq k$  para um inteiro positivo  $k \leq |V(G)|$  é NP-Completo. Dada a complexidade do problema, diversos trabalhos na literatura apresentam limitantes para  $g_R(G)$  ou determinam o valor de  $g_R(G)$  para classes de grafos.

A classe de grafos cúbicos é uma classe de interesse em Teoria dos Grafos pois diversos problemas em grafos são NP-completos para esta classe. Uma subclasse de grafos cúbicos relevante é a dos grafos snarks. Um *snark* é um grafo cúbico que não contém aresta de corte e que não admite uma 3-coloração própria de arestas. O parâmetro  $g_R(G)$  foi determinado para a família dos grafos snarks-flor (Maksimovic, Kratica, Savic, Bogdanovic, 2018), porém permanece não determinado para outras famílias de snarks. Neste trabalho, apresentamos limitantes superior e inferior para  $g_R(G)$  dos Snarks de Loupekine, apresentamos um limitante superior para  $g_R(G)$  dos Snarks de Goldberg e determinamos o valor exato de  $g_R(G)$  dos Snarks de Blanuša generalizados.

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<sup>†</sup> Este trabalho foi apoiado pelo CNPq e PIBIC-UFC.

Domination and Independent Domination Numbers of  
some Families of Snarks<sup>†</sup>

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*Keywords:* Domination number, Independent domination number, Generalized Blanuša Snarks, Loupekine Snarks.

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Let  $G = (V, E)$  be a simple, connected and undirected graph with  $|V| = n$ . A set  $S \subseteq V$  is a *dominating set* of  $G$  if for every  $v \in V$ , either  $v \in S$  or  $v$  is adjacent to some vertex in  $S$ . The *domination number*  $g(G)$  is the minimum cardinality of a dominating set of  $G$ . An *independent dominating set* of  $G$  is both dominating and independent. The minimum cardinality of an independent dominating set of  $G$  is its *independent domination number*  $i(G)$ . Determining  $g(G)$  and  $i(G)$  are NP-hard problems. These problems remain NP-hard even when restricted to cubic graphs.

In 1996, Reed conjectured that every cubic graph has  $g(G) \leq \lceil \frac{n}{3} \rceil$ , which was later proved to be false, even for 2-connected cubic graphs. Nevertheless, finding families of cubic graphs that verify or improve Reed's Conjecture is a hard and interesting problem. It is also challenging to determine the relation between  $g(G)$  and  $i(G)$  since deciding whether  $g(G) = i(G)$  is NP-complete. Even for cubic graphs, the difference  $i(G) - g(G)$  may be unbounded. In this work, we approach these two problems for some classes of *snarks* — connected bridgeless cubic graphs that are not 3-edge-colourable. More specifically, we show that for the two families  $B^k$ ,  $k \in \{1, 2\}$ , of Generalized Blanuša Snarks, graph  $B_i^k \in B^k$ , which has  $n = 8i + 10$  and  $i \geq 1$ , has  $g(B_i^k) = i(B_i^k) = 2i + 4$  when  $k = 1$  and  $i \geq 2$  is odd, and  $g(B_i^k) = i(B_i^k) = 2i + 3$  otherwise. Also, we establish lower and upper bounds for  $g(G)$  and  $i(G)$  when  $G$  belongs to two families of Loupekine Snarks, known as  $LR_0$ -snarks and  $LR_1$ -snarks.

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$k$ -independence in some Cartesian products

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*Keywords:* Cartesian product,  $k$ -independent sets, complexity

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For a positive integer  $k$ , a subset  $S$  of vertices in a graph  $G = (V, E)$  is  $k$ -independent if the maximum degree of the subgraph induced by the vertices of  $S$  is less or equal to  $k - 1$ . The  $k$ -independence number  $a(G)$  is the maximum cardinality of a  $k$ -independent set of  $G$ . Clearly, for  $k = 1$ , the 1-independent sets are the classical independent sets. For more details on the  $k$ -independence, we refer the reader to the survey (Chellali, Favaron, Hansberg, and Volkmann,  $k$ -domination and  $k$ -independence in graphs: A survey. *Graphs Comb.*, 28(1) (2012), 1-55). Given a graph  $G$  and integers  $k$  and  $\ell$ , the  $k$ -INDEPENDENT SET problem consists in deciding whether  $G$  has a  $k$ -independent set with cardinality at least  $\ell$ . This problem is known to be NP-complete on arbitrary graphs (Jacobson and Peters, Complexity questions for  $n$ -domination and related parameters, *Congr. Numer.*, 68 (1989), 7-22). Mao et al. (Mao, Cheng, Wang, and Guo, The  $k$ -independence number of graph products, *The Art of Discrete Appl. Math.*, 1(1) (2018), P1-01) presented bounds on  $k$ -independence in graph products, including Cartesian products.

We consider the Cartesian product of two paths, known as grid graphs, and the Cartesian product of an arbitrary graph  $G$  and the 2-path, known as prism graphs. We present results on  $k$ -independence on grids and we prove that  $k$ -INDEPENDENT SET remains NP-complete even when restricted to prisms, for  $k = 1, 2$ .

## Weighted Connected Matchings

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*Keywords: Matchings, Algorithms, Complexity, Induced Subgraphs*

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Graph matching problems are well-studied computational problems with applications in many areas. A matching  $M$  is a set of pairwise non-adjacent edges of a graph, and is denoted as a  $P$ -matching if the subgraph induced by the endpoints of the edges of  $M$  satisfies the property  $P$ . Some properties  $P$  studied over the years include that the graph is 1-regular, acyclic, connected or disconnected. For most of these properties  $P$ , finding a  $P$ -matching with maximum cardinality is a knowingly NP-hard problem. One exception is connected matchings, whose property  $P$  is that the graph is connected. Matching problems are also studied for edge weighted graphs, having applications like the ASSIGNMENT problem. Motivated by these facts, in addition to recent researches in weighted versions of  $P$ -matchings, we study the problem MAXIMUM WEIGHT CONNECTED MATCHING, where we want to find a connected matching whose sum of the edge weights is maximum. We prove that MAXIMUM WEIGHT CONNECTED MATCHING is NP-hard even for bounded diameter bipartite graphs, starlike graphs, planar bipartite, and subcubic planar graphs, while solvable in linear time for trees and graphs having degree at most two. For graphs having no negative edge weights, the problem turns to be polynomially solvable for chordal graphs, while remains NP-hard for most of the cases where weights can be negative. Our final contributions are on parameterized complexity, presenting a single exponential time algorithm when parameterized by treewidth. Concerning kernelization, we show that the decision problem of MAXIMUM WEIGHT CONNECTED MATCHING does not admit a polynomial kernel when parameterized by vertex cover under standard complexity-theoretical hypotheses.

# Session 5

## *Labeling and Coloring*

Chair: Diana Sasaki Nobrega

Equitable total coloring of Semiblowup and Kochol snark families total coloring<sup>†</sup> . . . . . 38

Edge coloring of split graphs with even maximum degree<sup>†</sup>  
39

The  $(p,1)$ -total number of graphs with maximum degree three<sup>†</sup> . . . . . 40

Estudo sobre  $(\mathbf{r} + \mathbf{1})$ -atribuição de papéis para prismas complementares, com  $\mathbf{r} \geq \mathbf{3}$  . . . . . 41

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Equitable total coloring of Semiblowup and Kochol snark families total coloring<sup>†</sup>

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*Keywords:* equitable total coloring, total coloring, SemiBlowup snarks, superposition.

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The well known *Total Coloring Conjecture (TCC)* states that the total chromatic number of a graph  $G$  is at least  $D + 1$  (graphs called *Type 1*) and at most  $D + 2$  (graphs called *Type 2*), where  $D$  is the maximum degree of  $G$ . The TCC has been settled for some specific graph families, but it remains open for several graph classes for more than fifty years. Similarly, the *Equitable Total Coloring Conjecture (ETCC)* states that the equitable total chromatic number of a graph is at most  $D + 2$ . The ETCC was proved for cubic graphs, and this implies that, if a cubic graph is *Type 2*, both the total chromatic number and the equitable total chromatic number are equal to 5; if a cubic graph is *Type 1*, the equitable total chromatic number can be either 4 or 5. Dantas, De Figueiredo, Mazzuocolo, Preissmann, dos Santos, and Sasaki (2016) proved that it is NP-complete to decide whether the equitable total chromatic number is equal to 4 for a bipartite cubic graph. About the equitable total coloring of cubic graphs with girth at least 5, Dantas et al. (2016) asked whether there exists a *Type 1* cubic graph with girth at least 5 and equitable total chromatic number 5. In 1996, Kochol proposed the *superposition* construction which provides infinite families of large girth snarks, and, in 2016, Hägglund defined two other infinite families of snarks: the *Blowup* and the *SemiBlowup* families. Gonçalves, Dantas and Sasaki (2021) proved that all *Blowup* snarks have equitable total chromatic number 4. In this work, we contribute to Dantas et al. (2016) question, by providing negative evidence. We present equitable 4-total colorings for all members of the *SemiBlowup* and 4-total colorings for all members of two infinite families obtained by the Kochol superposition.

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<sup>†</sup>This work was partially supported by CNPq, CAPES and FAPERJ.

Edge coloring of split graphs with even maximum degree<sup>†</sup>

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*Keywords: chromatic index, edge coloring, split graphs*

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A *proper edge coloring* of a graph  $G$  is an assignment of colors to the edges of  $G$  such that adjacent edges have distinct colors. The minimum number  $k$  of colors needed in a proper edge coloring of  $G$  is called the *chromatic index* of  $G$ , denoted  $c'(G)$ . Since every pair of adjacent edges must have distinct colors,  $c'(G) \geq D(G)$ , where  $D(G)$  is the *maximum degree* of  $G$ . In 1964, Vizing established that  $c'(G) \leq D(G) + 1$  for any simple graph  $G$ . Graphs with  $c'(G) = D(G)$  are said to be *Class 1*, while graphs with  $c'(G) = D(G) + 1$  are said to be *Class 2*. To decide if a given graph is Class 1 is an NP-complete problem, as shown by Holyer in 1981.

A graph  $G = (V(G), E(G))$  is a *split graph* if  $V(G)$  can be partitioned into a clique  $Q$  and a stable set  $S$ . In 1995, Chen, Fu and Ko showed that every split graph with odd maximum degree is Class 1. Their proof is constructive, in the sense that they developed an algorithm that colors any split graph  $G$  with odd maximum degree with  $D(G)$  colors. For split graphs with even maximum degree determining the chromatic index is an open problem. In 2012, Almeida showed that the algorithm of Chen, Fu and Ko which obtains an edge coloring with  $D(G)$  colors can also be used when  $D(G)$  is even given that  $G$  satisfies certain properties. One such property is when  $G$  has a vertex  $u \in S$  such that  $\lceil \frac{|Q|}{2} \rceil \leq d(u) \leq \frac{D(G)}{2}$ . We generalize this condition showing that if  $G$  is a split graph,  $D(G)$  is even, and there is a subset  $S' \subseteq S$  such that  $\lceil \frac{|Q|}{2} \rceil \leq |N(S')| \leq \frac{D(G)}{2}$ , then  $G$  is Class 1.

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<sup>†</sup>Partially supported by UTFPR, CAPES, and CNPq (428941/2016-8, 420079/2021-1).

The  $(p,1)$ -total number of graphs  
with maximum degree three<sup>†</sup>

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*Keywords:* Graph labelling,  $(p,1)$ -total labelling, Snarks, Subcubic graphs.

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For  $G = (V(G), E(G))$  a simple graph, a  $k$ - $(p,1)$ -total labelling of  $G$  is a function  $p: V(G) \cup E(G) \rightarrow \{0, \dots, k\}$  which satisfies the following properties:  $|p(uv) - p(u)| \geq p$  and  $|p(uv) - p(v)| \geq p$  for  $uv \in E(G)$ ;  $p(uv) \neq p(vw)$  for  $uv, vw \in E(G)$ ; and  $p(u) \neq p(v)$  for  $uv \in E(G)$ . The least integer  $k$  for which  $G$  admits a  $k$ - $(p,1)$ -total labelling is denoted  $I_p^t(G)$  and called  $(p,1)$ -total number. This labelling, proposed by Havet and Yu (2002), is a generalization of the well-known  $L(2,1)$ -labellings.

In their seminal work, Havet and Yu (2002) conjectured that  $I_p^t(G) \leq \min\{D(G) + 2p - 1, 2D(G) + p - 1\}$  for any graph  $G$ . They also conjectured that every connected graph  $G \neq K_4$  with  $D(G) \leq 3$  has  $I_2^t(G) \leq 5$ . Since the latter conjecture has been verified for  $G$  having  $D(G) \in \{1, 2\}$ , we focus on the case  $D(G) = 3$ . More specifically, we prove that  $I_2^t(G) = 5$  and  $I_p^t(G) = p + 4$ ,  $p > 2$ , for Goldberg Snarks, Generalized Blanuša Snarks and LP-Snarks. Moreover, for  $G$  with  $D(G) = 3$  and without adjacent vertices of maximum degree, we show that  $I_2^t(G) \leq 5$  and prove that this bound is tight. For such graphs, we also prove that  $I_p^t(G) = p + 3$ ,  $p > 2$ , when every pair of maximum degree vertices is at distance at least four; otherwise,  $I_p^t(G) \leq p + 4$ .

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<sup>†</sup>Partially supported by CNPq (425340/2016-3) and CAPES.



Estudo sobre  $(r + 1)$ -atribuição de papéis para prismas complementares, com  $r \geq 3$

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*Keywords: atribuição de papéis, prisma complementar, rede social.*

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O uso das redes sociais, como Facebook, Twitter e Instagram, foi potencializado pelo distanciamento social e a necessidade de se manter conectados durante a pandemia, devido ao surgimento do novo coronavírus (COVID-19). Com isso, as mesmas tornaram-se fonte de um volume gigantesco de dados, tornando indispensável a modelagem das redes sociais a fim de extrair informações. Deste modo, os grafos constituem uma ferramenta poderosa em que os vértices representam indivíduos e as arestas relações entre eles. Com base em modelos de grafos para redes sociais, Everett e Borgatti (1991) formalizaram a atribuição de papéis sob o nome de *role coloring*. Assim, uma  $r$ -atribuição de papéis de um grafo simples  $G$  é uma atribuição de  $r$  papéis distintos aos vértices de  $G$ , tal que, dois vértices com o mesmo papel têm o mesmo conjunto de papéis nos vértices relacionados. Além disso, uma  $r$ -atribuição de papéis específica define um *grafo de papéis*, no qual os vértices são os  $r$  papéis distintos, e existe uma aresta entre dois papéis sempre que há dois vértices relacionados no grafo  $G$  que correspondem a esses papéis. Consideramos a classe dos prismas complementares, que são os grafos formados a partir da união disjunta do grafo com seu respectivo complemento, adicionando as arestas de um emparelhamento perfeito entre seus vértices correspondentes. Castonguay *et al.* (2019) caracterizam que qualquer prisma complementar de um grafo, que não seja o prisma complementar do caminho com três vértices, tem uma 2-atribuição de papéis. Neste trabalho, consideramos,  $r \geq 3$ , o grafo de papéis  $K'_{1,r}$  que é o grafo bipartido  $K_{1,r}$  com laço no vértice de grau  $r$ . Concluímos que o problema de decidir se um prisma complementar tem uma  $(r + 1)$ -atribuição de papéis, quando o grafo de papéis é  $K'_{1,r}$ , é NP-completo. Conjecturamos que, para  $r \geq 3$ , o problema de decidir se um prisma complementar tem uma  $(r + 1)$ -atribuição de papéis, é NP-completo.

# Session 6

## *Graph Classes*

Chair: Sulamita Klein

<b>K</b> -comportamiento de gráficas cocordales . . . . .	43
Hardness of the $f$ -Reversible Process in Directed Graphs <sup>†</sup> 44	
How to draw a $K(n,2)$ Kneser graph? <sup>†</sup> . . . . .	45
Fullerene Waves . . . . .	46

**K**-comportamiento de gráficas cocordalesLesli Hernández-Sayago<sup>1,\*</sup> Miguel Pizaña<sup>2</sup>Rafael Villarroel-Flores<sup>3</sup><sup>1,2</sup> Universidad Autónoma Metropolitana<sup>3</sup> Universidad Autónoma del Estado de Hidalgo

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*Keywords:* Gráficas de clanes,  $K$ -comportamiento, gráficas cordales, complemento de gráficas cordales

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La *gráfica de clanes*  $K(G)$  de una gráfica  $G$  es la gráfica de intersección de sus clanes (subgráficas completas maximales). Definimos  $K^n(G)$  por  $K^0(G) = G$  y  $K^n(G) = K(K^{n-1}(G))$ . Decimos que una gráfica es  $K$ -convergente si  $K^n(G) = K^m(G)$  para alguna  $n < m$ ; caso contrario decimos que es  $K$ -divergente. Una gráfica  $G$  es cordal si todo ciclo de longitud al menos 4 tiene una cuerda. Si el complemento de  $G$  es cordal, decimos que  $G$  es *cocordal*. Un *octaedro*  $O_n$  es el complemento de  $n$  aristas disjuntas. Un *suboctaedro fuerte* de  $G$  es una subgráfica inducida de  $G$  isomorfa a  $O_n$ , tal que uno de los clanes del octaedro inducido es también un clan de  $G$ .

Experimentos computacionales y resultados preliminares sugieren que el  $K$ -comportamiento de una gráfica cocordal siempre se puede determinar con criterios conocidos: Al parecer, una gráfica cocordal siempre se desmantela a una gráfica clan-Helly (y por tanto es  $K$ -convergente) o se retrae a un octaedro  $O_n$  con  $n \geq 3$  (y por lo tanto es  $K$ -divergente). Sea  $N$  la dimensión máxima de un suboctaedro fuerte  $O_N$  de  $G$ ; defina  $N = 0$  si  $G$  no tiene suboctaedros fuertes. Los casos a considerar son:

$N = 0$ :  $G$  es desmantelable a un vértice ( $K$ -convergente).

$N = 1$ :  $G$  es desmantelable a una gráfica sin aristas ( $K$ -convergente).

$N = 2$ :  $G$  es desmantelable a una gráfica clan-Helly ( $K$ -convergente).

$N \geq 3$ :  $G$  se retrae a un octaedro  $O_N$  ( $K$ -divergente).

Ya hemos logrado demostrar los casos  $N = 0$ ,  $N = 1$  y  $N \geq 3$  usando técnicas de topología combinatoria, mientras que el caso  $N = 2$  es un resultado experimental (con algo de sustento teórico también) que todavía requiere ser probado o desmentido. En esta plática reportaremos nuestros avances en torno a estas investigaciones aún no publicadas.

Hardness of the  $f$ -Reversible Process in Directed Graphs<sup>†</sup>

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*Keywords:* Reversible Process, Infection, Directed Graph, Hitting Set, DAG's

Given a graph  $G$  and a function  $f: V(G) \rightarrow \mathbb{N}$  we study the iterative process on  $G$  such that, given an initial vertex labeling  $c_0: V(G) \rightarrow \{0, 1\}$ , each vertex  $v$  changes its label if and only if  $v$  has at least  $f(v)$  neighbors with the opposite one. We call such processes as  **$f$ -reversible processes** and denote them by  $P_f(G, c_0) = (c_0, c_1, \dots)$ , such that  $c_t: V(G) \rightarrow \{0, 1\}$  is called the **configuration** at time step  $t \in \mathbb{N}$ . Moreover,  $c_t(v)$  denotes the **state** of  $v$ , at  $t \in \mathbb{N}$ . In non oriented graphs (Dourado et al., Reversible iterative graph processes, Theor. Comput. Sci. **460**, (2012), 16–25), a vertex  $v$  takes into account all the states of its neighbors in order to obtain the state in the next time step. We deal with a slightly modification on an orientation  $D$  of a graph  $G = (V, E)$ , that is, a digraph obtained from  $G$  by replacing each edge by exactly one of the two possible arcs with the same end vertices. Now, a vertex takes into account all of its inner neighbors in order to define its next state. We study the problem of determining the minimum cardinality  $r_f(G)$  of a vertex subset initially infected (of state equal to 1) so that all vertices become infected as well. We call such a subset as an  **$f$ -conversion set** of  $G$  and we say that the process **converges**. (Dourado et al., Reversible iterative graph processes, Theor. Comput. Sci. **460**, (2012), 16–25) show that  $r_f(G) = V(G)$  of an undirected graph  $G$  when all thresholds are equal to 1 and that determining whether  $r_f(G) \leq k$  when the thresholds are all equal to 2 is NP-hard, but polynomial for trees. Moreover, it remains open to determining  $r_f(P)$ , for an induced path  $P$  with thresholds in  $\{1, 2\}$ . We prove that the directed version can be solved for acyclic orientations in linear time, for any thresholds, and that it is W[2]-hard to determine whether  $r_f(G) \leq k$ , even if all thresholds are equal to 1, there exists only one cycle in  $G$ , and the process converges into two time steps.

<sup>†</sup>Partially supported by UFCA and CNPq Universal [422912/2021-2] grants.

How to draw a  $K(n, 2)$  Kneser graph?<sup>†</sup>

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*Keywords: Kneser graph  $K(n, k)$ , crossing number, 2-page crossing number*

Take a 2-page drawing  $D(K_{2\lceil \frac{n}{2} \rceil - 1})$  of the complete graph  $K_{2\lceil \frac{n}{2} \rceil - 1}$  from algorithm (de Klerk, Pasechnik and Salazar (2013)) (a), (b) and (c). Replace each vertex of  $K_{2\lceil \frac{n}{2} \rceil - 1}$  by  $q = \lceil \frac{n-1}{2} \rceil$  vertices corresponding to clique  $C_i, i \in \{1, 2, \dots, 2\lceil \frac{n}{2} \rceil - 1\}$  with the order of the Hamiltonian cycle from algorithm (Berge (1973)). Add the edges between the pair of vertices of each 2 cliques according to the geometric position of the  $D(K_{2\lceil \frac{n}{2} \rceil - 1})$  edges. Place the 1-page drawing of  $K_{\lceil \frac{n-1}{2} \rceil}$  from (de Klerk, Pasechnik and Salazar (2013)) for each clique  $C_i$  on the half-plane with the fewest outgoing edges of the vertex  $C_i$  of  $D(K_{2\lceil \frac{n}{2} \rceil - 1})$  (d).

Let  $n(G)$  and  $n_2(G)$  be the minimum number of crossings for a drawing  $D(G)$  of  $G$ , respectively, in the plane, and into a 2-page drawing, we prove that  $\frac{n^8}{2^{13}} - 9\frac{n^7}{2^{13}} - \frac{n^6}{2^{10}} - \frac{n^4}{2^7} - \frac{n^3}{2^9} \leq n(K(n, 2)) \leq n_2(K(n, 2)) \leq \frac{n^8}{2^{10}} - \frac{3n^7}{2^8} + \frac{31n^6}{2^{83}} + \frac{7n^5}{2^6} - \frac{563n^4}{2^73} + \frac{517n^3}{2^53} - \frac{267n^2}{2^5} + \frac{107n}{2^33}$ .

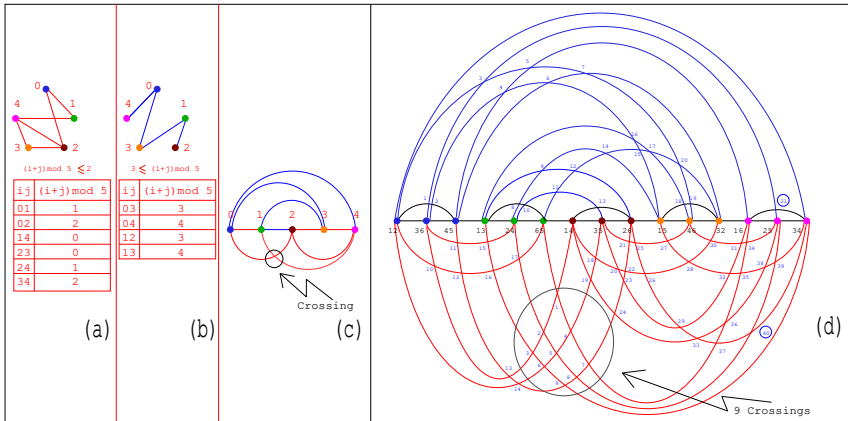


Figure 1 2-page drawing construction of  $K_5$  in (a) and (b), and 2-page drawings of  $K_5$  in (c) and  $K(6, 2)$  in (d).

<sup>†</sup> CAPES 001, CNPq 406036/2021-7, 308654/2018-8, 152340/2021-1, FAPERJ E26/202.902/2018.

## Fullerene Waves

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*Keywords:* Graph. Fullerene. Fullerene Nanodisc. Diameter

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*Fullerene graphs* are 3-connected, cubic and planar graph with only pentagonal (exactly 12) and hexagonal faces. Andova and Škrekovski (2013) [1], after studying Fullerene graphs with full icosahedral symmetry, conjectured that the *diameter* of a Fullerene graph on  $n$  vertices is at least  $\left\lfloor \sqrt{\frac{5n}{3}} \right\rfloor - 1$ .

*Fullerene nanodiscs* are Fullerene graphs formed by grouping the hexagonal faces into two opposite isomorphic discs glued along their boundaries by a strip containing the 12 pentagonal faces. Nicodemos e Stehlík (2016) [2] disproved Andova and Škrekovski's conjecture showing that Fullerene nanodiscs on  $n \geq 300$  vertices have diameter at most  $\sqrt{\frac{4n}{3}}$ . Since then, finding which classes of Fullerene graphs satisfy the conjecture is an open problem. We have found another class of Fullerene graphs that disproves the aforementioned conjecture: *Fullerene 1-nanodiscs*.

A Fullerene 1-nanodiscs are Fullerene graphs with Fullerene nanodiscs similar construction: two opposite isomorphic discs glued along their boundaries by a strip. However, in each disc we have one pentagonal face and the strip now contains just 10 pentagonal faces. We show that the diameter of Fullerene 1-nanodiscs on  $n$  vertices is at most  $\sqrt{\frac{8n}{5}} - 1$ . This gives us indications that nanodiscs may be a boundary class between the Fullerenes that satisfy and those that do not satisfy Andova and Škrekovski's conjecture.

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# Session 7

## *Labeling and Coloring*

Chair: Rosiane de Freitas

On Total Colouring Bipartite Graphs with at Most Three Bicliques <sup>†</sup> . . . . .	48
Local antimagicchromatic number of Bethe trees . . . . .	49
On non-equitable color class configurations for small Type 1 cubic graphs . . . . .	50
Locally irregular decompositions of a class of subcubic graphs . . . . .	51

## On Total Colouring Bipartite Graphs with at Most Three Bicliques<sup>†</sup>

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*Keywords:* Graph total colouring, Bicliques, Bipartite graphs

The problem of determining the total chromatic number has been studied for many graph classes, including graphs with few cliques. For graphs with a universal vertex, it was solved by Hilton in 1990. In 2012, Campos et al. solved the problem for split-indifference graphs, all of which have at most three cliques, proving that such a graph  $G$  is: Type 2 if  $G$  has some  $D$ -subgraph  $H$  (i.e a subgraph with  $D(H) = D(G)$ ) which has a universal vertex and is Type 2; Type 1 otherwise. In 1991, Hilton also solved the problem for bipartite graphs with adjacent *bi-universal* vertices (a vertex in a part of a bipartite graph is *bi-universal* if it is adjacent to all vertices in the other part). We conjecture that a bipartite graph  $G$  with at most three bicliques is: Type 2 if  $G$  has some Type 2  $D$ -subgraph with adjacent bi-universal vertices; Type 1 otherwise. If  $G$  has at most two bicliques, then the conjecture holds by Hilton's result. If  $G$  has three bicliques, we prove that the graph obtained after successively removing twins is either a  $P_3$  (Fig. 1) or an  $A$  (Fig. 2). For the latter case,  $G$  has adjacent bi-universal vertices. For the former case, let  $A, B, C, D, E$  be the sets of twin vertices as in Fig. 1, being  $a, b, c, d, e$  their corresponding cardinalities, with  $a \geq e$ . Then, we prove that our conjecture holds for all the following cases:  $b + d = c + a > ad + \min(a, d)$ ;  $b + d > c + a$ ;  $b + d < c + a$ ;  $\max(d, e) < a$  and  $ad \geq b$ . So, the conjecture remains open when  $b + d = c + a \leq ad + \min(a, d)$  and: either  $d \geq a$ , or  $a = e > d$ .

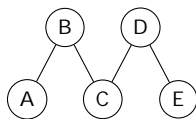


Figure 1

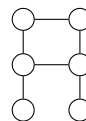


Figure 2

<sup>†</sup>Partially supported by CNPq (428941/2016-8 and 420079/2021-1).



## Local antimagicchromatic number of Bethe trees

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*Keywords:* Local antimagic labeling, Local antimagic chromatic number, Bethe tree

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Let  $G = (V, E)$  be a simple connected graph and let  $f: E \rightarrow \{1, 2, \dots, |E|\}$  be a bijection. For each  $u \in V$ , the *weight* of  $u$  is given by  $w(u) = \sum_{e \in E(u)} f(e)$ , where  $E(u)$  is the set of incident edges in  $u$ . If  $w(u) \neq w(v)$  for any two distinct vertices  $u$  and  $v$  in  $V$ , then  $f$  is called an *antimagic labeling* of  $G$ . A graph  $G$  is called *antimagic* if  $G$  has an antimagic labeling, (Hartsfield and Ringel, 1994). A bijection  $f: E \rightarrow \{1, 2, \dots, |E|\}$  is called a *local antimagic labeling* if  $w(u) \neq w(v)$  for all  $fu, vq \in E$ . A graph  $G$  is *local antimagic* if  $G$  has a local antimagic labeling (Arumugam et al., 2017). The *local antimagic chromatic number*  $c_{la}(G)$  is the minimum number of colors taken over all colorings induced by local antimagic labelings of  $G$ . Hartsfield and Ringel (1994), proved that for any tree  $T$  with  $\ell$  leaves has  $c_{la}(T) \geq \ell + 1$ . In this work, we present some results obtained in the research on local antimagic labeling for some subfamilies of Bethe trees, which are rooted tree with  $k$  levels whose root on level 1 and has degree equal to  $d$ , the vertices of levels from 2 to  $k - 1$  have degrees equal to  $d + 1$  and the vertices on the level  $k$  have degree equal to 1.

On non-equitable color class configurations  
for small Type 1 cubic graphs

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*Keywords: Graph Theory, Cubic Graphs, Equitable Total Coloring*

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A total coloring assigns colors to the vertices and edges of a graph without conflicts. The Total Coloring Conjecture (TCC) establishes that for any simple graph  $G$ , we have  $c''(G) \leq D(G) + 2$ . If  $c''(G) = D(G) + 1$ , then the graph is called Type 1. A total coloring is called equitable if the cardinalities of any two color classes differ from at most 1.

The smallest known Type 1 cubic graph with no equitable 4-total coloring has 20 vertices. Every 4-total coloring must be equitable on all cubic graphs with 6, 8, 10, and 14 vertices. For cubic graphs with 12, 16, and 18 vertices, we characterize the color class configurations that might allow a non-equitable 4-total coloring. We obtained these results by analyzing the configurations of the total independent sets of cubic graphs with less than 20 vertices and determining the possible configurations of the color classes.

We consider the class of circular ladder graphs and generalized Petersen graphs. We provide non-equitable 4-total colorings for  $G(8, 1)$ ,  $G(8, 2)$ , and  $G(8, 3)$  with color class configuration 11, 10, 10, 9; and for  $G(9, 1)$  and  $G(9, 2)$  with color class configuration 12, 12, 10, 10, extending the known non-equitable 4-total colorings for these graphs. We aim to establish for Type 1 cubic graphs: the smallest value of  $n$  for which there is a graph such that all its 4-total colorings are non-equitable, and the largest value of  $n$  which implies that all 4-total colorings are equitable.

## Locally irregular decompositions of a class of subcubic graphs

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*Keywords: locally irregular decompositions, graph decompositions, graph colorings.*

Given an undirected simple graph  $G$ , we are interested in finding the minimum number of colors such that there is a *locally irregular decomposition* of  $G$ , which is a coloring of  $E(G)$  in which the subgraphs induced by each of the colors have no adjacent vertices with the same degree. We denote this number by  $c'_{irr}(G)$  and call it the *irregular chromatic index* of  $G$ . If such edge coloring exists we say  $G$  is *decomposable*. The main conjecture on locally irregular decompositions is given below.

**Conjecture 1** — Baudon, Bensmail, Przybyto, Wo niak (2015). For every decomposable graph  $G$ , we have  $c'_{irr}(G) \leq 3$ .

The conjecture does not hold in its full generality, but so far only one graph  $G$  (with 10 vertices) is known for which  $c'_{irr}(G) = 4$ . We present our results related to Conjecture 1 for subcubic graphs, which are graphs with maximum degree at most 3. These results include an analysis of the structure of a minimal counterexample for the conjecture, in case it exists, and the following intermediate results.

**Lemma 1** If  $G$  is a decomposable subcubic graph with no adjacent vertices of degree 3, then  $c'_{irr}(G) \leq 3$ . Furthermore, if  $u \in V(G)$  and  $d(u) = 3$ , there is a locally irregular decomposition of  $G$  with 3 colors such that every edge incident to  $u$  has the same color.

**Lemma 2** Let  $G$  be a non-decomposable graph and  $v$  be a vertex that is not in  $V(G)$ . If  $u$  is a vertex of  $G$  with  $d(u) = 2$  that is not contained in triangles, then  $c'_{irr}(G') \leq 3$ , where  $G'$  is a graph with  $V(G') = V(G) \cup \{v\}$  and  $E(G') = E(G) \cup \{uv\}$ . Furthermore,  $G'$  admits a locally irregular decomposition with 3 colors where every edge incident to  $u$  has the same color.

# Session 8

## *Graph Operations*

Chair: Diego Nicodemos

On tessellations and graph operations: Adding pendant and false twin vertices . . . . .	53
Reducing the Time Complexity of Computing Square Roots with Girth at Least Six of a Graph . . . . .	54
On iterated clique graphs with exponential growth . . .	55
Critical generators of $K_5$ . . . . .	56

On tessellations and graph operations:  
Adding pendant and false twin vertices

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*Keywords: Tessellations, Graph operations, Quantum computation*

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Portugal *et al.* proposed the staggered quantum walk model, which is more general than the previous models (Portugal, R., Santos, R., Fernandes, T., and Gonçalves, D.. The Staggered Quantum walk Model. *Quantum Inf. Process.* (2016), 15(1):85-101). In the staggered quantum walk, the quantum operators that rule the walk are generated from tessellations on  $G$ . A tessellation  $\mathcal{T}$  is a partition of the vertices of a graph into cliques. We say that an edge belongs to a tessellation  $\mathcal{T}$  if its endpoints belong to the same clique in  $\mathcal{T}$ . We say that  $\mathcal{T} = \{T_1, \dots, T_m\}$  is a tessellation cover if  $\mathcal{T} = \bigcup_{i=1}^m T_i$  covers all edges of  $G$ . Abreu *et al.* proved that  $T(G) \leq \min\{c'(G), c(K(G))\}$  for any graph  $G$ , where  $T(G)$  is the size of a smallest tessellation cover of  $G$ , and they also showed that adding a true twin vertex  $v'$  of  $v$  in  $G$  does not change  $T(G)$  (Abreu, A., Cunha, L., Figueiredo, C., Kowada, L., Marquezino, F., Posner, D., and Portugal, R.. The Graph Tessellation Cover Number: Chromatic Bounds, Efficient Algorithms and Hardness. *Theor. Comput. Sci.* (2020), 801:175-191). In this work, we proved that adding a pendant vertex  $v'$  to  $v$  in  $G$  may increase  $T(G)$  at most in one tessellation, while adding a false twin vertex  $v'$  of  $v$  in  $G$  may increase  $T(G)$  in at most  $c(G^c[N(v)])$  tessellations.

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Reducing the Time Complexity of Computing Square  
Roots with Girth at Least Six of a Graph

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*Keywords: graph powers; graph square roots; cycle detection*

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The square of a graph  $H$  is the graph  $H^2$  given by  $V(H^2) = V(H)$  and  $E(H^2) = \{uv \in E(H) : d_H(u, v) \leq 2\}$ . The graph  $H$  is then called a *square root* of  $H^2$ . The *girth* of a graph  $G$  is the length of a shortest cycle in  $G$ . Farzad, Lau, Le and Tuy provided an  $O(d \cdot n^4)$  algorithm to compute the square root with girth at least six of a graph  $G$  with  $n$  vertices and minimum degree  $d$ , if it exists (Farzad, Lau, Le and Tuy, 2009). The factor  $n^4$  comes from a step to verify that a potential solution does not contain a 4-cycle. We show that the complexity of the algorithm can be reduced to  $O(d \cdot M(n))$  (where  $M(n)$  is the time to multiply two  $n \times n$  matrices) through the use of a more efficient algorithm for 4-cycle detection. We note that, currently,  $M(n) = O(n^{2.373})$  (Alman and Williams, 2021) so that the complexity of the algorithm is reduced to  $O(d \cdot n^{2.373})$ .

On iterated clique graphs with exponential growth

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*Keywords: graph theory, graph dynamics, iterated clique graphs*

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The *clique graph*  $K(G)$  of a graph  $G$  is the intersection graph of the set of all (maximal) cliques of  $G$ . The iterated clique graphs  $K^n(G)$  are defined inductively by  $K^0(G) = G$  and  $K^{n+1}(G) = K(K^n(G))$ .

Let  $|K^n(G)|$  be the order of  $K^n(G)$  and let  $f_G(n) = |K^n(G)|$ , be the clique growth function of  $G$ . Given a function  $g(n)$ , a problem that arises is whether a graph  $G$  exists such that  $f_G(n) = Q(g(n))$ . There are examples of graphs  $G$  where  $f_G(n)$  is linear (Larrión and Nuemann-Lara, 1997), polynomial (Larrión and V. Neumann-Lara, 1999) or super-exponential (Larrión, Neumann-Lara and Pizaña, 2009). Nevertheless, it is not known if there are graphs with exponential growth (i.e.,  $f_G(n) = Q(a^n)$  for some  $a > 1$ ).

In this talk we will show some results of our current investigation on this problem (a work in progress).

Critical generators of  $K_5$ Gabriela Ravenna <sup>1,\*</sup> Liliana Alcón <sup>2</sup><sup>1</sup> UNMDP - CONICET<sup>2</sup> CeMalp - CONICET

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*Keywords: Clique graph, Critical generator, complete graphs*

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Let  $H$  be a graph, the clique graph  $K(H)$  of  $H$  is defined as having the cliques of  $H$  as vertices and two cliques  $C_i$  and  $C_j$  being adjacent in  $K(H)$  if and only if  $C_i \cap C_j \neq \emptyset$ . The graph  $H$  is said to be clique critical if  $K(H) \neq K(H-x)$  for all  $x \in V(H)$ , where  $H-x$  is the graph obtained from  $H$  by removing the vertex  $x$  and all the edges incident on it. This notion of clique-critical graph was introduced by Escalante and Toft (1974). If  $H$  is a clique critical graph then we say that  $H$  is a critical generator of  $K(H)$ .

It is not easy to know when a graph is a clique graph, much less what its generators are. Roberts and Spencer (1971) showed that a graph  $G$  is a clique graph if there exists a family edge cover of  $G$  satisfying the Helly property. We called these families, RS-families. Gutierrez and Meidanis (2006) showed that if there is a separating (the intersection of all the members containing  $v$  is  $v$ ) RS-family then the intersection graph of this family is a generator of  $G$ .

It is known that  $K_3$  has two critical generators ( $K_{1,3}$ ,  $4-fan$ ) and  $K_4$  has seven critical generators.

In this work we present how to obtain the fortythree critical generators of  $K_5$  looking for its minimal RS-separating families.



# Session 9

## *Labeling and Coloring*

Chair: Marina Groshaus

Containment among classes of interval graphs with interval count $k$ . . . . .	58
On cycle-free-CPT posets . . . . .	59
Chordal Thinness <sup>†</sup> . . . . .	60
On two variants of split graphs . . . . .	61

Containment among classes of interval  
graphs with interval count  $\mathbf{k}$

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*Keywords: interval count, interval graphs, interval orders*

A graph  $G$  is an *interval graph* if there exists a bijection  $q$  of  $V(G)$  to a family  $\mathcal{M}$  of intervals on the real line, called a *model*, in which for all  $u, v \in V(G)$  with  $u \neq v$ ,  $(u, v) \in E(G)$  if and only if  $q(u) \cap q(v) \neq \emptyset$ . An *interval order* is a partial order on a family of intervals on the real line in which the precedence relation corresponds to that of the intervals, that is, the interval  $I_a$  precedes the interval  $I_b$  in the order if and only if  $I_a$  is entirely to the left of  $I_b$ .

Ronald Graham suggested the problem of determining a model of a given interval graph having the smallest number of distinct interval lengths, which is called the *interval count* problem. An  $fa, bg$ -model is a model in which each interval has length  $a$  or  $b$ . The class of graphs which admit an  $fa, bg$ -model is denoted by  $\text{LEN}(a, b)$ . Skrien provided a characterization for  $\text{LEN}(0, 1)$ . Rautenbach and Swarcfiter described a characterization and a linear-time algorithm to recognize graphs of  $\text{LEN}(0, 1)$ . Boyadzhiyska, Isaak and Trenk presented a characterization for interval orders which admit a  $f0, 1g$ -model, and partially for those which admit a  $f1, 2g$ -model. But, the question regarding the inclusion relation among these two classes was not considered. In Francis et al. (Francis, M. C., Medeiros, L. S., Oliveira, F. S., and Swarcfiter, J. L., On subclasses of interval count two and on Fishburn's conjecture. *Discrete Appl. Math.*, 2022 (to appear)), it has been shown that  $\text{LEN}(a', b') \subseteq \text{LEN}(a, b)$  for all  $0 < a' < b'$  and  $0 < a < b$  if and only if  $\frac{b'}{a'} \neq \frac{b}{a}$ . The motivation of this work was to investigate the more general inclusion relationship between the classes  $\text{LEN}(a_1, \dots, a_k)$  and  $\text{LEN}(b_1, b_2, \dots, b_k)$ . For all  $0 < a_1 < a_2 \dots < a_k$ , and  $0 < b_1 < b_2 \dots < b_k$ , we prove that  $\text{LEN}(a_1, a_2, \dots, a_k) \subseteq \text{LEN}(b_1, b_2, \dots, b_k)$  if and only if there does not exist constant  $r$  such that  $b_j = ra_j$  for all  $1 \leq j \leq k$ . We also discuss the case for  $a_1 = 0$  or  $b_1 = 0$ .

## On cycle-free-CPT posets

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*Keywords:* Comparability graph, Poset, Cycle-free

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Given a poset  $\mathbf{P} = (X, P)$  we say that a family of sets  $F = (F_x)_{x \in X}$  is a *containment model* of  $\mathbf{P}$  if each element  $x$  can be assigned to a set of  $F_x$  such that  $x < y$  in  $\mathbf{P} \leftrightarrow F_x \subset F_y$ . When the elements of the family  $F$  are paths of a tree we say that  $\mathbf{P}$  is a *CPT* poset [2]. In [1] we prove the following necessary condition for a poset  $\mathbf{P}$  to be *CPT*:

The down-set of each vertex of  $\mathbf{P}$  induces a *CI* subposet. (i)

The *comparability graph* of  $\mathbf{P}$  is the simple graph  $G_{\mathbf{P}} = (X, E)$  where  $xy \in E$  if and only if  $x < y$  in  $\mathbf{P}$  or  $x > y$  in  $\mathbf{P}$ . Two posets are associated if their comparability graphs are isomorphic. If  $\mathbf{P}$  and its dual poset  $\mathbf{P}^d$  are *CPT* we say that  $\mathbf{P}$  is *dually-CPT*. If  $\mathbf{P}$  and every other poset associated with  $\mathbf{P}$  are *CPT* we say  $\mathbf{P}$  is *strongly-CPT*.

We say that a poset  $\mathbf{P} = (X, P)$  is *cycle-free* if its comparability graph  $G_{\mathbf{P}}$  is a chordal graph. In this paper we prove that chordal posets admit a recursive construction process. Moreover, condition (i) is also sufficient for the class of chordal posets. As a consequence we obtain a characterization of *CPT* chordal posets by an infinite family of minimal forbidden subposets. Moreover, we obtain a characterization of the dually-CPT posets.

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Chordal Thinness<sup>†</sup>

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*Keywords: Chordal Graphs, Thinness, Algorithms, Complexity*

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The thinness of a graph is a measure of “how distant” the graph is from an interval graph (Carlo Mannino et al, The stable set problem and the thinness of a graph, Operations Research Letters 35.1 (2007), 1–9), which are exactly the graphs with thinness 1. In this work we introduce an analogous concept, the chordal thinness. A graph  $G$  is said to be  $k$ -chordal thin if there are a  $k$ -partition  $V_1, \dots, V_k$  and an ordering  $v_1, \dots, v_n$  of  $V(G)$  such that for each  $v_i$  and each  $V_j$ , the set  $N(v_i) \cap \{v_{i+1}, \dots, v_n\} \cap V_j$  induces a clique. Note that in the case  $k = 1$  this condition is equivalent to say that the ordering is a perfect elimination order.

In this work, we determine the complexity of some classic problems in graphs of bounded chordal thinness. We show that INDEPENDENT SET remains **NP**-Hard even in graphs of chordal thinness 3, and that CLIQUE is **NP**-Hard for  $k$ -chordal thin graphs when  $k \geq 3$  and in **P** otherwise. Moreover, we investigate problems regarding recognition and show that, given a  $k$ -partition of the vertices, finding an ordering consistent with it is polynomially solvable, while finding a  $k$ -partition given the ordering is **NP**-Hard.

Furthermore, we show some results about how  $k$ -chordal thin graphs relate to other graph classes. In particular, we show that strictly subcubic graphs have bounded chordal thinness and that  $(1, k)$ -graphs, the graphs that can be partitioned into 1 independent set and  $k$  cliques, are  $k$ -thin graphs. We also present a relation between cographs and its chordal thinness, showing also that the chordal thinness of a cograph can be found in polynomial time. On top of that, we show upper bounds relating  $k$ -chordal thinness with other well studied graph properties.

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<sup>†</sup>Partially supported by FAPEMIG and CNPq.

## On two variants of split graphs

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*Keywords:  $k$ -probe-split graph, split-width, 2-unipolar graph*

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A graph is called split if its vertex set can be partitioned into a stable set and a clique. In this talk we present results on two variants of split graphs.

A graph  $G$  is polar if its vertex set can be partitioned into two sets  $A$  and  $B$  such that  $G[A]$  is a complete multipartite graph and  $G[B]$  is a disjoint union of complete graphs. If  $G[B]$  is an independent set then  $G$  is called monopolar. J. L. Szwarcfiter and M. R. Cerioli (1999) characterized the starlike graphs a subclass of monopolar graphs. A 2-unipolar graph is a polar graph  $G$  such that  $G[A]$  is a clique and  $G[B]$  the disjoint union of complete graphs with at most two vertices. We present a characterization for 2-unipolar graphs and show that they can be represented as intersection of substars of a special cactus.

Let  $\mathcal{G}$  be a graph class, the  $\mathcal{G}$ -width of a graph  $G$  is the minimum positive integer  $k$  such that there exist  $k$  independent sets  $N_1, \dots, N_k$  such that a set  $F$  of nonedges of  $G$ , whose endpoints belong to some  $N_i$  with  $i = 1, \dots, k$ , can be added so that the resulting graph  $G'$  belongs to  $\mathcal{G}$ . We said that a graph  $G$  is  $k$ -probe- $\mathcal{G}$  if it has  $\mathcal{G}$ -width at most  $k$  and when  $\mathcal{G}$  is the class of split graphs it is denominated  $k$ -probe-split. The 1-probe split graphs was characterized by V. Bang Le and H. N. de Ridder (2007). We present a characterization by minimal forbidden induced subgraphs for 2-probe-split graphs and a result on the complexity of the problem of deciding if a graph is  $k$ -probe-split graph.

# Session 10

## *Applications*

Chair: Déborah Oliveros Brani

Monkey Hash Map: a highly performant thread-safe map without locks . . . . .	63
Clique-based problems in the structural prediction of complex molecules . . . . .	64
COVID-19 mortality prediction - Perceptron and Random Forest applications <sup>†</sup> . . . . .	65
Restricted Hamming-Huffman trees . . . . .	66

Monkey Hash Map: a highly performant  
thread-safe map without locks

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*Keywords:* concurrent data structures, hashing, wait-free

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Hash tables are arguably the most powerful data structures ever known. A shared data structure is *lock-free* if infinitely often *some* thread completes its task within a finite number of steps. A shared data structure is *wait-free* if *each thread* completes its execution within a finite number of steps.

We exploit multiple-choice hashing to create a high-performance, wait-free hashing scheme with  $O(1)$  worst-case time for lookup, `getValue`, insert, update and remove operations, a hash table that provides thread-safety without requiring any kind of thread synchronization. In short, our *monkey hashing* scheme consists of a single hash table and a family of  $k \geq 1$  hash functions, meaning multiple alternative locations for each key in the same table. Unlike what happens in the well-known cuckoo hashing, elements will never be evicted from where they first landed, so new keys being inserted must always find an unoccupied spot to call their own. The actual counts of hash functions in use are kept track of, making it possible that lookups of absent keys fail before the entire family of hash functions has been exhausted. Dynamic memory allocation—and its inherent pauses, e.g. garbage collection—is avoided via a key-value object pool, and thread-safe is attained via (i) pre-allocation of the underlying array, meaning no rehashing will ever take place, and (ii) the fact that no collision-handling lists are called for, by design.

The proposed scheme works particularly well in scenarios with a single writer and multiple reader threads, dramatically outperforming popular solutions such as `ConcurrentHashMap` (Java) and `Intel TBB concurrent_hash_map` (C++) in heavily concurrent test scenarios. The prices to pay are (i) eventual consistency, which is dealt with well in numerous concurrent settings, and (ii) a non-zero probability that an insertion might fail, which can be made small enough, though, to suit all imaginable applications.

Clique-based problems in the structural  
prediction of complex molecules

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*Keywords: algorithms, complex molecules, distance geometry, k-cliques.*

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In this work we present the application of clique-based problems in algorithmic strategies for predicting the spatial structure of complex molecules. A clique  $C$  is a subset of vertices  $C \subseteq V$  in an undirected graph  $G = (V, E)$  where all vertices are adjacent to each other, and then, if there is a subset of  $k$  vertices that are connected to each other, that graph contains a  $k$ -clique. Moreover, a maximal clique is a subgraph  $H \subseteq G$  isomorphic to a complete graph and there is no vertex  $v \in V(G) \setminus V(H)$  so that  $v$  is adjacent to each vertex of  $H$  (Bondy and Murty, 1976)(Szwarcfiter, 2018). To determine if there is a clique of a given size in a graph is NP-complete, and one of variants of this general problem is to list all the maximal cliques of a graph, these can be found by the Bron-Kerbosh algorithm, a backtracking algorithm of  $O(3^{n/3})$  time complexity. Another variation is to find all cliques of size  $k$ , where the brute-force algorithm for any trivial case has complexity  $O(n^k k^2)$ . We handle these variations of clique search and their intersection and coverage properties on atomic data instances for the Molecular Distance Geometry problem (MDGP). Given a molecule formed by  $n$  atoms  $a_1, a_2, \dots, a_n$  and a set of distances  $d_{ij}$ . The MDGP is to obtain a three-dimensional configuration  $x_1, x_2, \dots, x_n$  for the molecule respecting the set of distances (Lavor *et. al.*, 2012). The MDGP in a complete set of distances can be solved in polynomial time, otherwise the problem is NP-hard for  $\mathbb{R}^2$  or more (Santos *et. al.*, 2021). In this work we determine some maximal and  $k$ -cliques,  $k \leq 20$ , of NMR protein structures (protein partial instances) to investigate the possibility of using a cliques search preprocessing to improve an strategy based on incomplete sets resolution methods(Wu and Wu,2007)(Souza,2021), by performing the insertion of complete subsets of distances ( $K$ -cliques) into the plane, maintaining the single-atom immersion rules of the original algorithm.



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## COVID-19 mortality prediction - Perceptron and Random Forest applications<sup>†</sup>

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*Keywords: Perceptron, COVID-19, Random Forest*

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In Brazil, DATASUS centralizes data on the evolution of COVID-19. Perceptron is a binary classifier based on a linear separator, yielding a hyperplane separating the data into two classes: points in the region above the hyperplane are classified as 1 while below as 0. Random forest can also be used as a classifier, consisting of several decision trees. They are machine learning methods that work based on training and validating turns used for decision taking. Both methods were recently worldwide used for COVID-19 mortality prediction. Moulaei et al. (2022) used random forest for mortality of COVID-19, the accuracy was 95%, using 1500 patients from Iran. Borghi et al. (2021) used single and multiple layer Perceptron to predict to predict the number of deaths and infections over six days. In this paper the DATASUS file provides 440,915 valid patients, which Perceptron successfully predicted 77.7% and the Random Forest 78%. We experimentally found that the COVID-19 database is not separable. Hence, our strategy to use Perceptron on this problem consisted of in each iteration algorithm, selecting the current hyperplane that better separates the training data set.

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<sup>†</sup> CAPES 001, CNPq 406036/2021-7, 308654/2018-8, 152340/2021-1, FAPERJ E26/202.902/2018.

## Restricted Hamming-Huffman trees

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*Keywords:* Hamming-Huffman codes, hypercube graphs, minimum neighborhood

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In information theory, there is a common trade-off that arises in data transmission processes: data compression and preparation for error detection. While data compression shrinks the message as much as possible, data preparation for error detection adds redundancy to messages so that a receiver can detect corrupted bits, and fix them when possible.

Data compression can be achieved using different strategies. One of the most traditional methods is that of Huffman (David A. Huffman, A method for the construction of minimum redundancy codes, Proceedings of the IRE (1951), 40:1098–1101), which uses binary rooted trees, known as Huffman trees, to encode the symbols of a given message.

In 1980, Hamming proposed the union of both compression and error detection features through a structure called Hamming-Huffman tree (Richard W. Hamming, Coding and Information Theory, Prentice-Hall (1986)). This data structure compresses data similarly to Huffman trees with the additional feature of enabling error detection. In contrast to Huffman trees, building optimal Hamming-Huffman trees is still an open problem.

In this work, we define a more restricted version of the problem of building optimal Hamming-Huffman trees. We tackle the problem of building optimal Hamming-Huffman trees in which the symbol leaves lie in exactly  $k$ -distinct levels. For  $k \leq 2$ , we presented a polynomial time algorithm to solve the problem. Moreover, we proved that, for symbols with uniform frequencies, the optimal tree has at most five distinct levels with symbol leaves.

# Session 11

## *Flow Graphs*

Chair: Ana Shirley Ferreira da Silva

A simple proof of the bijection between Minimal Feedback Arc Sets and Hamiltonian Paths in tournaments . . . .	68
Control flow graph, formal verification and constraint programming techniques . . . . .	69
FPT algorithm for feedback vertex set in reducible flow hypergraphs <sup>†</sup> . . . . .	70

# A simple proof of the bijection between Minimal Feedback Arc Sets and Hamiltonian Paths in tournaments

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*Keywords: Minimal feedback arc set, Hamiltonian Paths, Tournament*

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In what follows, fix a *tournament*  $T$ , i.e., an orientation of the arcs of a complete graph. It is known that every tournament has a Hamiltonian path, i.e., a (directed) path that contains all of its vertices. Given a vertex  $v \in V(T)$ , let  $A^+(v) = \{vw \in A(T) : w \in V(T)\}$  and  $A^-(v) = \{uv \in A(T) : u \in V(T)\}$ , and put  $A(v) = A^-(v) \cup A^+(v)$ .

A set of arcs  $F \subseteq A(T)$  is a *feedback arc set* (FAS) if  $T \cap F$  is acyclic; and a FAS  $F$  is *minimal* (MFAS) if  $F - uv$  is not a FAS for every arc  $uv \in F$ . In 1988, it was shown (Bar-Noy, A., & Naor, J. (1990). Sorting, minimal feedback sets, and Hamilton paths in tournaments. SIAM Journal on Discrete Mathematics, 3(1), 7-20] that in any tournament there is a one-to-one relation between its MFASs and its Hamiltonian paths. We present an alternative proof of this result.

Let  $P = v_1 \cdots v_n$  be a Hamiltonian path in  $T$ . We say that  $v_i v_j$  is a *backward arc* (w.r.t.  $P$ ) if  $i > j$ . Note that, for such an arc,  $P \cup v_i v_j$  contains a cycle. Then the set  $F$  of backwards arcs w.r.t.  $P$  is an MFAS of  $T$ . In this case, we say that  $P$  *induces*  $F$ . It is not hard to check that distinct Hamiltonian paths induce distinct MFASs.

We now present an injection from the MFASs to the Hamiltonian paths of  $T$ . Given an MFAS  $F$ , a vertex  $v \in V$  is called an *F-initial vertex* if  $F \cap A(v) = A^-(v)$ . We can prove that there is precisely one  $F$ -initial vertex in  $T$ , say  $u_1$ . Let  $T' = T - u_1$  and let  $F' = F \cap A(T')$ . It is not hard to check that  $F'$  is an MFAS of  $T'$ . Now, let  $u_2$  be the (unique)  $F'$ -initial vertex of  $T'$ . We claim that  $u_1 \rightarrow u_2$ . Indeed, if  $u_2 \rightarrow u_1$ , then, by the minimality of  $F$ , there is a  $u_1 u_2$ -path in  $T \cap F$  (with at least two arcs), which is a contradiction, because  $A^-(u_2) \subseteq F' \subseteq F$ . The result now follows by repeating this step, and uniqueness of  $F$ -initial vertices guarantees that this is an injection.

Control flow graph, formal verification and constraint programming techniques

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*Keywords:* bounded-model checking, constraint programming, directed graph, interval arithmetic, SAT problem.

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Formal verification of a program is an undecidable problem because means checking whether the software conforms to its requirements, which includes a statement indicating the program must produce a determined answer. But, checking whether the program eventually produces a result is equivalent to the famous Halting problem, which is undecidable (Hopcroft and Ullman, 1979). Bounded Model Checking (BMC) is a method that can achieve decidability, by searching for violations of properties of a program up to a bound  $k$  (Cordeiro *et al.*, 2009). BMC reduces the program verification problem to the classic NP-complete Boolean Satisfiability (SAT) problem, and there are several solvers as the ESBMC, an open-source, context-bounded model checker based on satisfiability modulo theories (SMT) for the verification C/C++ programs (Cordeiro *et al.*, 2009). However, it can still lead to an exponential state-space exploration due to the program's large and possibly unbounded loops. In this case, there might be many execution paths to traverse through a program during its symbolic execution (Clarke *et al.*, 2012). Therefore, the control flow or computation during the program's execution, mainly in symbolic execution, can be represented as a directed graph named Control-Flow Graph (CFG). In this work we present the CFG properties and discuss some preliminary results of the application of interval arithmetic, constraint programming techniques and combinatorial methods in steps before and after the main software verification process via BMC.

FPT algorithm for feedback vertex set  
in reducible flow hypergraphs<sup>†</sup>

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*Keywords:* feedback vertex set, reducible flow hypergraph, FPT algorithm

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A *directed hypergraph*  $H = (V, A)$  is a finite set of vertices  $V$  and a set of hyper-arcs  $A$ , where each hyper-arc is an ordered pair of nonempty subsets of vertices. A *flow hypergraph*  $H = (V, A, s)$  is a triple, such that  $(V, A)$  is a directed hypergraph,  $s \in V$  is a distinguished vertex such that  $s$  reaches every vertex of  $V$ . *Reducible flow hypergraphs* are a generalization of Hecht and Ullman's reducible flowgraphs and are recognizable in polynomial time (Guedes, Markenzon, Faria (2011)). The FEEDBACK VERTEX SET (FVS) decision problem has a directed hypergraph  $H$  and an integer  $k \geq 0$  as input and the question is whether there is  $V' \subseteq V, |V'| \leq k$ , such that  $H \setminus V'$  is an acyclic directed hypergraph. It is known that FVS is polynomial time solvable for reducible flowgraphs. For reducible flow hypergraphs it is proved that FVS is NP-complete and there is a polynomial-time  $D$ -approximation for FVS in reducible flow hypergraphs, where  $D$  is the maximum number of hyper-arcs adjacent to a vertex of  $H$  (Faria, Guedes, Markenzon (2021)). Each cycle  $C$  in a reducible flow hypergraph has a single entry vertex,  $v_C$ . Consider an instance  $(H, k)$  of FVS with a positive answer and a minimum cycle  $C$  of  $H$ . Let  $V_C$  be the set of vertices of  $C$  that are entry vertices of other cycles of  $H$ , hence  $|V_C| \leq k$ . So, there is a solution for instance  $(H, k)$  with at least one vertex of these entry vertices, and  $\text{FVS}(H, k) = \min_{v \in V_C} \text{FVS}(H - v, k - 1) \cup \{v\}$ . This leads to a FPT  $O((k+1)^k m)$  time algorithm to FEEDBACK VERTEX SET problem to solve  $(H, k)$ , where  $m = |A(H)|$ .

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<sup>†</sup> CAPES 001, CNPq 406036/2021-7, 308654/2018-8, 152340/2021-1, 420079/2021-1, 428941/2016-8, FAPERJ E26/202.902/2018.

# Session 12

## *Geometry*

Chair: Márcia Rodrigues Cappelle Santana

Spectral properties of threshold  $k$ -uniform hypergraphs 72

On a semidefinite relaxation for the maximum  $k$ -colourable subgraph problem . . . . . 73

Positive semidefiniteness of  $A_a(G)$  on some families of graphs with  $k$  cycles . . . . . 74

Spectral properties of threshold  $k$ -uniform hypergraphsLucas Portugal<sup>1,\*</sup> Renata Del-Vecchio<sup>2</sup><sup>1</sup> Institute of Mathematics and Statistics - UFF, RJ, Brazil.<sup>2</sup> Institute of Mathematics and Statistics - UFF, RJ, Brazil.

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*Keywords:* Threshold hypergraph; Adjacency matrix; Distinct eigenvalues.

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In this work, we define a threshold  $k$ -uniform hypergraph with vertex set  $V = \{1, 2, \dots, n\}$  by a binary sequence  $(b_1, b_2, \dots, b_n)$ , where each  $b_i \in \{0, 1\}$ , as following:

- $b_1 = b_2 = \dots = b_{k-1} = 0$ ;
- for  $i \geq k$ , if  $b_i = 0$  and  $\{x_1, \dots, x_k\}$  is an edge satisfying  $x_1 < x_2 < \dots < x_k$ , then  $x_k \neq i$ ;
- for  $i \geq k$ , if  $b_i = 1$  then  $\{x_1, \dots, x_{k-1}, i\}$  is an edge, for every possible  $(k-1)$ -subset  $\{x_1, \dots, x_{k-1}\}$  of  $V$  satisfying  $x_1 < x_2 < \dots < x_{k-1} < i$ .

This generalizes the well known definition of a threshold graph through the binary sequence of zeros and ones. We study the adjacency matrix and the spectrum of some subclasses of 3-uniform threshold hypergraphs. As in the case of threshold graphs, we obtain classes of 3-uniform threshold hypergraphs with few distinct eigenvalues, more specifically, hypergraphs with only 4 or 5 distinct eigenvalues and an arbitrary number of vertices. We also study what happens with the spectrum, considering a  $k$ -uniform threshold hypergraph with  $k > 3$ . In this way, we bring to the context of hypergraphs an important issue of spectral graph theory, the characterization of graphs with few distinct eigenvalues



On a semidefinite relaxation for the maximum  
 $k$ -colourable subgraph problem

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*Keywords:* semidefinite programming ;  $k$ -colourable subgraph ; Lovász theta function

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Let  $a_k(G)$  denote the size of a largest induced subgraph of  $G$  that can be properly coloured with  $k$  colours. A semidefinite relaxation for this parameter was introduced by Narasimhan and Manber in 1988, in analogy to the well-known Lovász theta number of a graph:

$$J_k(G) = \max \{ \text{tr}(X) : X \succeq 0, \text{tr}(X) = k, X \circ A(G) = 0 \}.$$

Few results were known about this parameter until a recent surge in interest, with three papers appearing in the past two years. In this present work, we have shown that  $J_k(G) \geq j_k(G)$ , where

$$j_k(G) = \max \left\{ \sum_{i=1}^k \lambda_i(B) : B \preceq 0, \text{diag}(B) = \mathbf{1}, B \circ A(G) = 0 \right\},$$

where  $\lambda_i(B)$  stands for the  $i$ th largest eigenvalue of  $B$ . The case  $k = 1$  was due to Lovász, who actually showed the equality, but for larger  $k$  the proof is slightly more contrived. In fact, it does not seem that equality holds, and we pose this as an open question. The importance of our work is that the equality for the case  $k = 1$  is a key fact in the celebrated theory that associates the polyhedra STAB and QSTAB with the convex corner known as the theta body and that leads, among other things, to the polynomial algorithm to compute  $a(G)$  and  $c(G)$  for perfect graphs. The inequality we showed for  $J_k(G)$  might lead to partial results towards the construction of an analogous theory for  $k$ -colourable subgraphs. Narasimhan and Manber also showed that  $J_k(G) \leq c_k(\overline{G})$ . In our work we also discuss a sensible definition for the fractional version of  $c_k$  that maintains the inequality.

Positive semidefiniteness of  $A_a(G)$  on  
some families of graphs with  $k$  cycles

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*Keywords: Eigenvalues,  $A_a$ -matrix, cycles*

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Let  $G = (V(G), E(G))$  be a simple graph of order  $n$ . The signless Laplacian matrix of  $G$  is defined by  $Q(G) = D(G) + A(G)$ , where  $D(G)$  is the diagonal matrix of the degrees and  $A(G)$  is the adjacency matrix of  $G$ . Nikiforov (2017) defined for any real  $a \in [0, 1]$ , the convex linear combinations  $A_a(G)$  of  $A(G)$  and  $D(G)$  the following way  $A_a(G) = aD(G) + (1 - a)A(G)$ . It is easy to see that  $A(G) = A_0(G)$ ,  $D(G) = A_1(G)$  and  $Q(G) = 2A_{\frac{1}{2}}(G)$ . The matrix  $Q(G)$  is positive semidefinite, but is not true for  $A_a(G)$  if  $a$  is sufficiently small. Nikiforov (2017) proved that if  $a \geq \frac{1}{2}$  then  $A_a(G)$  is positive semidefinite and if  $a > \frac{1}{2}$  and  $G$  has no isolated vertices then  $A_a(G)$  is positive definite. Nikiforov and Rojo (2017) defined  $a_0(G)$  as the smallest value in the interval  $[0, 1]$  such that the minimum eigenvalues of  $A_a(G)$  is non negative. In the same paper, this problem was solved when  $G$  is  $d$ -regular,  $r$ -colorable and when  $G$  contains bipartite components. In this work, this problem is solved for some families of graphs defined the following way: (1)  $F_{a_1, a_2, \dots, a_k}$  is the family that consist of  $k$  cycles of length  $a_1, a_2, \dots, a_k$ , respectively which share a vertex, where  $k \geq 2$  and  $a_1 \geq a_2 \geq \dots \geq a_k \geq 3$ ; (2) let  $n \geq 3$  and  $k \geq 2$ . The graph  $G_k(C_n)$  consists of  $k$  cycles of the same size  $n$ , where each cycle has one vertex incidents to one extra vertex,  $s$ , that is,  $V(G_k(C_n)) = (\bigcup_{i=1}^k V(C_n)) \cup \{s\}$  and  $E(G_k(C_n)) = (\bigcup_{i=1}^k E(C_n)) \cup (\bigcup_{i=1}^k \{u_{i,n}s\})$ , where  $u_{i,n}$  belongs to  $i$ -th copy of  $V(C_n)$  for each  $i, 1 \leq i \leq k$ .

# Session 13

## *Labeling and Coloring*

Chair: Sheila Morais de Almeida

On total coloring of subcubic graphs . . . . .	76
Neighbor distinguishing coloring for cacti graphs . . .	77
Edge-Sum Distinguishing game <sup>†‡</sup> . . . . .	78
The (2,1)-total number of powers of paths and powers of cycles <sup>†</sup> . . . . .	79

## On total coloring of subcubic graphs

Luerbio Faria<sup>1</sup> Mauro Nigro<sup>1,\*</sup> Diana Sasaki<sup>1</sup><sup>1</sup> Rio de Janeiro State University, Rio de Janeiro, Brazil*Keywords: total coloring, subcubic graph, non-conformable graphs*

A  $k$ -total coloring of  $G$  connected is an assignment of  $k$  colors to the vertices and edges of  $G$  so that adjacent elements have different colors. The total chromatic number  $c''(G)$  is the smallest  $k$  for which  $G$  has a  $k$ -total coloring. Graphs with  $c''(G) = D(G) + 1$  are called *Type 1* and with  $c''(G) = D(G) + 2$  called *Type 2*. The *deficiency* of  $G$  be  $def(G) = \sum_{v \in V(G)} (D(G) - d(v))$ . A  $(D(G) + 1)$ -vertex coloring is called *conformable* if the number of color classes of parity different from that of  $|V(G)|$  is at most  $def(G)$  (Chetwynd and Hilton, Some refinements of the total chromatic number conjecture, Congr. Numer., (1988), pp. 195–216). A graph is *conformable* if it has a conformable vertex coloring. Hilton and Hind (2002) showed that if  $G$  is non-conformable, then  $def(G) \leq D(G) - 2$ . It is well known that if  $G$  is non-conformable, then  $G$  is Type 2. A natural question is: Is there a function  $f$  such that if  $G$  is Type 2, then  $def(G) \leq f(D(G))$ ? We know that it holds for  $D(G) = 2$  and we answer negatively this statement for subcubic graphs (Figure 2).

**Theorem 2** For each integer  $k > 0$  there is a subcubic graph  $G$  with  $def(G) = k$  which is Type 2.

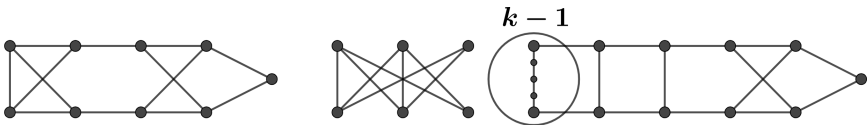


Figure 2 Type 2 graphs used in the proof of Theorem 1.

## Neighbor distinguishing coloring for cacti graphs

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*Keywords: coloring, neighbor distinguishing coloring, cactus graph*

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Given a vertex  $u$  and a proper edge coloring  $c$ , let  $S(u)$  be the set of colors in the edges incident to  $u$ , i.e.,  $S(u) = \{c(uv) : uv \in E(G)\}$ . A proper edge coloring  $c$  of a graph  $G$  is *neighbor distinguishing* if, for every edge  $uv \in E(G)$ , we have  $S(u) \neq S(v)$ . The minimum integer  $k$  such that there exists a neighbor distinguishing edge coloring using  $k$  colors for a graph  $G$  is the *neighbor distinguishing index* of  $G$  and it is denoted as  $c'_a(G)$ .

Zhang *et al.* (2002) conjectured in 2002 that if  $G$  is a graph of order at least 3 and different from  $C_5$ , then  $c'_a(G) \leq D(G) + 2$ . This conjecture was verified for bipartite graphs, graphs with maximum degree 3, almost all 4-regular graphs, and planar graphs with maximum degree at least 12. The upper bound for the last case was in fact later improved to  $D(G) + 1$ . For a general graph  $G$ , it is known that  $c'_a(G) \leq 3D(G)$ , and  $c'_a(G) \leq D(G) + O(\log c(G))$ . If  $G$  has no isolated edge and  $D(G) \geq 10^{20}$ , then we know that  $c'_a(G) \leq D(G) + 300$  (Bonamy and Przybylo, 2017), which was recently improved to show that  $c'_a(G) \leq D(G) + 19$ .

A *cactus* graph is a connected graph in which every two cycles have at most one vertex in common. In this work we prove that cacti graphs with order at least 3 and different from  $C_5$  do have a neighbor distinguishing coloring using  $D(G) + 2$  colors, which means that the conjecture is valid for cacti graphs.

Edge-Sum Distinguishing game<sup>†‡</sup>Deise L. de Oliveira<sup>1,\*</sup> Danilo Artigas<sup>1</sup>Simone Dantas<sup>1</sup> Atílio G. Luiz<sup>2</sup><sup>1</sup> Fluminense Federal University, Brazil.<sup>2</sup> Federal University of Ceará, Brazil.

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*Keywords: graph labeling, labeling game, maker breaker game, edge-sum distinguishing game, combinatorial game.*

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In 2017, Tuza surveyed the area of Graph labeling games, and proposed the *Edge-Sum Distinguishing game* (ESD game) which is a type of *maker-breaker* graph labeling game where the players, traditionally called Alice and Bob, alternately assign an unused label  $f(v) \in L = \{1, \dots, s\}$  to an unlabeled vertex  $v$  of a given graph  $G$ . If both ends of an edge  $vw \in E(G)$  are already labeled, then the (induced) label  $f(vw)$  of the edge  $vw$  is defined as  $f(vw) = f(v) + f(w)$ . A move is *legal* if after it all edge labels are distinct. Alice (the *maker*) wins if the graph  $G$  has an injective vertex labeling of all vertices of  $G$  that induces distinct edge labels, and Bob (the *breaker*) wins if he can prevent this.

Tuza also posed the following question about the ESD game: given a simple graph  $G$ , for which values of  $s$  can Alice win the ESD game? And if Alice wins the ESD game with the set of labels  $\{1, \dots, s\}$ , can she also win with  $\{1, \dots, s+1\}$ ? In this work, we present the first results in the literature about this game. We present computational and theoretical results investigating winning strategies for Alice and Bob on the ESD game for classical families of graphs. Furthermore, we partially answer Tuza's questions by presenting an upper bound on the least number of consecutive non-negative integer labels necessary for Alice to win the ESD game on a graph  $G$ .

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<sup>†</sup>We dedicate this paper to Professor Jayme Szwarcfiter's 80th birthday.

<sup>‡</sup>This work was partially supported by CNPq, CAPES and FAPERJ.

The  $(2,1)$ -total number of powers of paths and powers of cycles<sup>†</sup>

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*Keywords:* graph labelling,  $(2,1)$ -total labelling, powers of paths, powers of cycles.

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For a simple graph  $G = (V(G), E(G))$ , a  $k$ - $(2,1)$ -total labelling of  $G$  is a function  $p: V(G) \cup E(G) \rightarrow \{0, \dots, k\}$  such that:  $p(u) \neq p(v)$  for  $uv \in E(G)$ ;  $p(uv) \neq p(vw)$  for  $uv, vw \in E(G)$ ; and  $|p(uv) - p(u)| \geq 2$  and  $|p(uv) - p(v)| \geq 2$  for  $uv \in E(G)$ . The least integer  $k$  for which  $G$  admits a  $k$ - $(2,1)$ -total labelling is denoted  $I_2^t(G)$  and called  $(2,1)$ -total number. This labelling, proposed by Havet and Yu (2002), is a variant of the well-known  $L(2,1)$ -labellings, which have been extensively studied in the literature.

Havet and Yu (2002) conjectured that  $I_2^t(G) \leq D(G) + 3$  for  $G$  having  $D(G) > 2$ . This conjecture was verified for complete graphs  $K_n$ , by showing that:  $I_2^t(K_n) = n + 2$  if  $n$  is even and  $n \notin \{2, 6, 8\}$ ; otherwise  $I_2^t(K_n) = n + 1$  (Chia et al., 2013, Havet and Yu, 2002). The  $(2,1)$ -total number has also been determined for other classes of graphs, such as paths, cycles, near-ladders, caterpillars, lobsters, and complete bipartite graphs.

In this work, we focus on the  $(2,1)$ -total number of powers of paths and powers of cycles. The  $k$ -th power of a graph  $G$  is the simple graph obtained from  $G$  by adding edges connecting every pair of vertices at distance at most  $k$  in  $G$ . The  $k$ -th power of a path  $P_n$  is denoted by  $P_n^k$  and the  $k$ -th power of a cycle  $C_n$  is denoted by  $C_n^k$ . We show that  $I_2^t(P_n^k) = D(P_n^k) + 1$  when  $k \in \{(n-1)/2, (n-2)/2\}$  and  $I_2^t(P_n^k) = D(P_n^k) + 2$  when  $2 \leq k \leq (n-3)/2$ . Moreover, we show that  $I_2^t(C_n^k) = D(C_n^k) + 2$  when  $n \equiv 0 \pmod{2k+1}$ .

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# Session 14

## *Computational Complexity*

Chair: Guilherme Motta

Elecciones con Simetrías . . . . .	81
NP-Hardness of perfect rainbow polygons <sup>†</sup> . . . . .	82
Parameterized complexity of computing maximum minimal blocking and hitting sets . . . . .	83
Theoretical and empirical analysis of algorithms for the <i>max-npv</i> project scheduling problem . . . . .	84



## Elecciones con Simetrías

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*Keywords: Generación exhaustiva de gráficas, grupos de permutaciones.*

Sean  $X = \{1, 2, \dots, n\}$  y  $A, B \subseteq X$ . Si consideramos a  $A$  y  $B$  como sucesiones de elementos ordenados, debería ser claro cuándo  $A$  es un prefijo de  $B$ , denotado por  $A \prec B$ , y cuándo que  $A$  es (lexicográficamente) menor que  $B$ , denotado por  $A < B$ . Si  $G$  es un grupo de permutaciones de  $X$ , usamos notación exponencial para denotar la acción del grupo sobre los elementos de  $X$ , es decir, para  $x \in X$  y  $g \in G$ ,  $x^g$  es la imagen de  $x$  bajo la permutación  $g$ . De manera natural,  $G$  también actúa en los subconjuntos de  $X$ , es decir,  $A^g = \{a^g : a \in A\}$ . Decimos que  $A$  y  $B$  son *equivalentes* (bajo la acción del grupo) si  $A^g = B$  para alguna  $g \in G$  y lo denotamos como  $A \sim B$ .

Considere el problema de generar todos los subconjuntos de  $k$  elementos de  $X$  hasta equivalencia, es decir, generar  $\binom{X}{k}_G := \binom{X}{k} / \sim$ . En general es impráctico generar primero todos los  $k$ -subconjuntos y luego descartar los equivalentes, pues  $\binom{X}{k}$  puede ser gigantesco. En vez de eso lo que hacemos es ir construyendo  $\binom{X}{k}_G$  y cada uno de sus elementos de manera incremental y lexicográfica (usando ramificación y poda o *backtracking*) y podamos todas las ramas de construcción cada vez que encontramos un prefijo que ya había sido considerado previamente (hasta equivalencia). Esto se puede hacer gracias al siguiente lema:

**Lema 1** Si  $A \prec B$ , y  $A^g < A$  para alguna  $g \in G$ , entonces  $B^g < B$ .

Este problema en particular tiene muchísimas aplicaciones en combinatoria computacional, incluyendo búsquedas exhaustivas en espacios combinatorios, generación exhaustiva de objetos combinatorios (como las gráficas), y demostraciones asistidas por computadoras. En esta plática, reportaremos los avances que hemos realizado en el estudio del estado del arte del problema y las aplicaciones que estamos considerando. Como ejemplo, presentaremos la generación automática de jaulas (*cages*) y la demostración semiautomática de la unicidad de algunas de ellas, con las herramientas que estamos desarrollando y que podrán usar los programadores no expertos en la generación exhaustiva y automática de gráficas sujetas a propiedades preestablecidas.

NP-Hardness of perfect rainbow polygons<sup>†</sup>David Flores-Peñaloza<sup>1,\*</sup> Andrés Fuentes-Hernández<sup>1</sup><sup>1</sup> Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México.

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*Keywords: Rainbow Polygon, NP-Hardness, Colored Point Set, Covering Tree, Combinatorial Geometry.*

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Let  $S$  be a  $k$ -colored point set in the plane in general position (no three points on the same line). A *perfect rainbow polygon* on  $S$  is a simple polygon that contains exactly one point of  $S$  of each of the  $k$  colors, either on its interior or on its boundary.

In Rainbow polygons for colored point sets in the plane (D. Flores-Peñaloza, M. Kano, L. Martínez-Sandoval, D. Orden, J. Tejel, C. D. Tóth, J. Urrutia, and B. Vogtenhuber, 2021), the authors define and study the combinatorial problem of determining the number  $\text{rb-index}(k)$ : the smallest integer such that every  $k$  colored point set in general position has a perfect rainbow polygon with at most  $\text{rb-index}(k)$  vertices. They conjectured that the following related problem is NP-Hard:

Given a  $k$ -colored point set  $S$  in general position, and a positive integer  $\nu$ , decide whether there exists a perfect rainbow polygon on  $S$  with at most  $\nu$  vertices.

We prove this conjecture is true, reducing from **one-in-three 3-SAT**.

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<sup>†</sup>Research by David Flores and Andrés Fuentes was supported by the grant UNAM PA-PIIT, Mexico IN120520. A.F was also supported by a M.Sc. scholarship from CONACYT, México.

Parameterized complexity of computing maximum minimal blocking and hitting sets

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*Keywords:* maximum minimal blocking set, maximum minimal hitting set, parameterized complexity, treewidth, kernelization, vertex cover, upper domination.

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A *blocking set* in a graph  $G$  is a subset of vertices that intersects every maximum independent set of  $G$ . Let  $\text{mmbs}(G)$  be the size of a maximum (inclusion-wise) minimal blocking set of  $G$ .

This parameter has recently played an important role in the kernelization of VERTEX COVER with structural parameterizations. Briefly, Bougeret et al. proved that VERTEX COVER parameterized by *distance to a family*  $F$ , for a minor-closed family  $F$ , admits a polynomial kernel if, and only if, there exists  $k \in \mathbb{N}$  such that  $\text{mmbs}(G) \leq k$ , for each  $G \in F$ .

Given a graph  $G$  and  $k \in \mathbb{N}$ , we provide a panorama of the complexity of deciding whether  $\text{mmbs}(G) \leq k$  parameterized by  $k$ , or by the independence number  $\alpha(G)$ , or both.

Note that, for a given graph  $G$ , a blocking set of  $G$  is indeed a hitting set of the hypergraph  $H$  have the same vertices as  $G$  and whose hyperedges correspond to the maximum independent sets of  $G$ . With this motivation, we also consider the closely related parameter  $\text{mmhs}(H)$ , which is the size of a maximum minimal hitting set of a hypergraph  $H$ . We also study the complexity of determining  $\text{mmhs}(H)$ , under different parameterizations.

Finally, we consider the problem of computing  $\text{mmbs}(G)$  parameterized by the treewidth of  $G$ , especially relevant in the context of kernelization. Given the “counting” nature of  $\text{mmbs}$ , it does not seem to be expressible in monadic second-order logic, hence its tractability does not follow from Courcelle’s theorem. Our main technical contribution is a fixed-parameter tractable algorithm for this problem.

Theoretical and empirical analysis of algorithms for the  
***max-npv*** project scheduling problem

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*Keywords:* Algorithms, computational complexity, directed graphs, factor analysis, net present value, project scheduling.

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The scheduling problem considered in this work refers to maximizing the Net Present Value (NPV) of projects with precedence constraints between activities and unrestricted resources. In terms of the three-field notation, the problem can be presented as  $0 \mid j \mid cpm, d_n, c_j \mid \max\text{-npv}$ . So,  $0$  means unrestricted resources,  $cpm$  means precedence between *finish-start* activities with zero *lag*,  $d_n$  is the *deadline* for the project,  $c_j$  is a cash flow for each activity, and *max-npv* is the objective function (maximization of the net present value). Thus, the objective function can be given as  $\max \sum_{i=2}^{n-1} c_i \cdot e^{-a(s_i+d_i)}$ . Such function is subject to 1)  $s_j + d_j \leq s_i \forall (i, j) \in E$ , meaning that all activity  $i$  have finish date less or equal any successor activity  $j$  (where  $s_i$  and  $d_i$  are the initial date and duration time of activity  $i$ ), 2)  $s_1 = 0$ , meaning the first activity has initial date in time zero, 3)  $s_n \leq d_n$ , meaning initial date of the last activity (with duration zero) is less or equal to the *deadline*, and 4)  $s_i \in \mathbb{N}; i = 2, 3, 4, \dots, n$ , meaning that all initial date of activities belong the natural numbers. The first and last activities are called initial *dummy* ( $s_1$ ) and final *dummy* ( $s_n$ ), respectively demarcating the beginning and end of the project (both with zero duration time). For this problem the three most important algorithms are *Recursive Search* (RS), *Steepest Ascent Approach* (SAA) and *Hybrid Search* (HS). In this work we present theoretical and empirical arguments to answer the open question about the time complexity of these algorithms, related to the number of times the algorithms perform the search in the generated spanning tree to find sets of vertices to scheduling.

# Session 15

## *Games*

Chair: Murilo Vicente Gonçalves da Silva

Some variations of the Tower of Hanoi and their graph properties . . . . .	86
The Conflict-Free coloring game and cliques <sup>†</sup> . . . . .	87
Hardness of general position games . . . . .	88
Notes on graph variations of the NIM game . . . . .	89

Some variations of the Tower of  
Hanoi and their graph properties

Lia Martins<sup>1\*</sup> Meng Hsu<sup>1</sup> Raquel Folz<sup>1</sup> Rosiane de Freitas<sup>1</sup>  
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*Keywords: algorithms; combinatorial games; Sierpinski triangle.*

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The Tower of Hanoi (ToH) game is a game with rich computational mathematical properties and an abstraction of important applications in logistic problems. The ToH consists of three vertical pegs, labeled 0, 1, and 2, then we define the set of towers  $T = \{0, 1, 2\}^n$ , with  $n$  disks with different radii  $1, \dots, n$ . ToH is a computationally intractable problem with lower bound  $\Omega(2^n - 1)$  and upper bound  $O(2^n - 1)$  due to recursive and interactive algorithms. Any distribution of the  $n$  disks over the three sticks, with no larger disk over the smaller one, is a regular state of the game, and a perfect state is when the  $n$  disks are arranged on just one stick. Each state is uniquely represented by an element  $s = s_n \dots s_1 \in T^n$ , where  $s_d$  is the pin on which the disk  $d$  is living. A tower game can be reproduced by a graph  $G = (V, E)$ , where:  $V$  are the vertex set and each vertex correspond to a regular state of the game and  $E$  are the edges and are equivalent to the transitions of a possible movement. Two vertices are adjacent if they are obtained from each other by a legal movement of a disk. ToH is one of those games and can be represented by a connected, flat and simple graph as an approximation of the Sierpinski triangle: the Hanoi graph  $H_n$  (or  $H_k^n$ , with  $k$  pins and  $n$  disks) (Poole, D. G., The towers and triangles of Professor Claus (or, Pascal knows Hanoi), Mathematics Magazine. (1994), 67(5): 323–344). There are several variations of the ToH regarding the number and size/capacity of pins, rules and gameplay, such as: the Tower of Oxford (ToO), the Tower of London (ToL) and the Tower of Bucharest (ToB). We are interested in exploring its algorithmic strategies and properties in graphs. Furthermore, we present the Tower of Hanoi-London as an ongoing research.

The Conflict-Free coloring game and cliques<sup>†</sup>P. T. P. Huaynoca<sup>1,\*</sup> Simone Dantas<sup>1</sup> Miguel A.D.R. Palma<sup>2</sup><sup>1</sup> Fluminense Federal University, Brazil.<sup>2</sup> Federal University of Piauí, Brazil.

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*Keywords: vertex coloring, coloring game, combinatorial game.*

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In 2003, the frequency assignment problem in cellular networks motivated Even, Lotker, Ron and Smorodinsky to introduce a new coloring problem: the *Conflict-Free Closed Neighborhood (CFCN) coloring*. The goal is to assign the minimum number of colors to the vertices  $v$  of  $G$  such that there exists at least one color appearing exactly once in each closed neighborhood  $N[v]$ . Inspired by this coloring problem, coloring games, and in the search for upper bounds to the CFCN coloring, in 2021, Huaynoca, Chimelli, Dantas and Marinho presented the *CFCN  $k$ -coloring game* on classic graph classes. The *CFCN  $k$ -coloring game* is a maker-breaker combinatorial game in which two players, Alice and Bob, alternately take turns assigning one of the  $k$  colors to each vertex of a graph  $G$  (mapping  $c: V \rightarrow \{1, 2, \dots, k\}$ ) such that for every  $v \in V$ , if  $N[v]$  is fully colored, then there exists  $u \in N[v]$  such that  $c(u) \neq c(w)$  for all  $w \in N[v] \setminus \{u\}$ . A coloring of a vertex  $v$  is said to be *legal* if, after it, in every fully colored neighborhood in which  $v$  belongs, there exists a color that appears exactly once. Alice wins if she obtains a CFCN  $k$ -coloring of  $G$ , otherwise Bob wins if he prevents it from happening. Both players are enabled to start the game, they are constrained to use only legal coloring in each turn, and they play optimally. By optimally, it means that the players try to win with the fewest possible turns or, in case of knowing that is not possible to win the game, delay the victory of the opponent. In this work, we extend the results obtained in 2021 for the CFCN  $k$ -coloring game on complete graphs studying the game played on a graph composed by the disjoint union of cliques joined with a single vertex (universal vertex), obtaining, as a particular case, results on Windmill graphs. Finally, we analyze the game played on other graph classes.

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<sup>†</sup>This work was partially supported by CNPq, CAPES and FAPERJ.

## Hardness of general position games

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*Keywords:* general position sets; general position number; achievement game; avoidance game; graph product; PSPACE-hardness

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Given a graph  $G$ , a set  $S$  of vertices in  $G$  is a general position set if no triple of vertices from  $S$  lie on a common shortest path in  $G$ . The gp-achievement and the gp-avoidance games were introduced by Chandran, Klavžar and Neethu P.K. (2021) and are played on a graph  $G$  by players A and B who alternately select vertices of  $G$ . A selection of a vertex by a player is a legal move if it has not been selected before and the set of selected vertices so far forms a general position set of  $G$ . The player who picks the last vertex is the winner in the gp-achievement game and is the loser in the gp-avoidance game. In this paper, we prove that the gp-achievement and the gp-avoidance games are PSPACE-complete even on graphs with diameter at most 4. For this, we prove that the *misère* play of the classical Node Kayles game is also PSPACE-complete. As a positive result, we obtain polynomial time algorithms to decide the winning player of the general position avoidance game in rook's graphs, grids, cylinders, and lexicographic products with complete second factors.



## Notes on graph variations of the NIM game

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*Keywords: combinatorial game; graph theory; impartial game; Nim game; Sprague-Grundy theorem.*

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We deal with variations in graphs of the impartial two-player and finite combinatorial game known as NIM. In its most known version, two players take turns choosing one of  $k, k > 0$ , heaps of stones and taking at least one stone from it. Following the normal play convention, the player who can't make a move loses. There are many variations of NIM, each with theoretical challenges and practical applications (Nowakowski, 1998). In particular, versions in graphs of NIM in the literature change the rules of from which heaps and how many stones a player can take based on an underlying graph. One such example is Edge NimG (Fukuyama, 2003), in which heaps of stones are placed on the edges of a multigraph and a token is placed in one of its vertices. Each player must move the token to another vertex following an edge and take at least one stone from that edge. Another variation is Graph NIM, where the players must remove edges incident in a vertex of a multigraph. Graph variations of NIM can be solved through the use of the Sprague-Grundy theorem. However, the computation of Grundy numbers for each state of the game can be challenging even for small graphs. As such, it is necessary to make use of optimizations through the study of how the games behave when played on specific types of graphs, as we discussed in this work. As ongoing work, we also present a proposal to generalize Edge NimG with multiple tokens.

# Session 16

## *Graph Classes*

Chair: Erika Morais Martins Coelho

On the Helly Number of trees <sup>†‡</sup> . . . . .	91
On the Biclique Graphs of Circular Arc Bigraphs . . . . .	92
Tree <b>3</b> -spanners on prisms of graphs . . . . .	93
Extendiendo Gráficas Cuadrado-Complementarias . . . . .	94

On the Helly Number of trees<sup>†‡</sup>

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*Keywords:*  $P_3$  convexity, Helly property, trees.

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Consider a graph  $G = (V, E)$  and  $C \subseteq V$ . We obtain the  $P_3$ -convex hull (resp.  $P_3^*$ -convex hull) of  $C$  by iteratively adding vertices with at least two neighbors in  $C$  (resp. two non-adjacent neighbors in  $C$ ). We say that  $S \subseteq V$  is  $P_3$ -Helly-independent (resp.  $P_3^*$ -Helly-independent) when the intersection of the  $P_3$ -convex hulls (resp.  $P_3^*$ -convex hulls) of  $S \cap N(v)$  (for all  $v \in S$ ) is empty.

The  $P_3$ -Helly number (resp.  $P_3^*$ -Helly number) is the size of a maximum  $P_3$ -Helly-independent (resp.  $P_3^*$ -Helly-independent). The edge counterparts of  $P_3$ -Helly-independent and  $P_3^*$ -Helly-independent of a graph follow the same restrictions applied to its edges instead of its vertices. The VP3HI, VSP3HI, and EP3HI problems aim to determine the  $P_3$ -Helly number,  $P_3^*$ -Helly number, and edge  $P_3$ -Helly number of a graph, respectively.

In 2019, Carvalho, Dantas, Dourado, Posner, and Szwarcfiter proved that these problems are NP-hard for bipartite graphs. In this work, we show dynamic programming polynomial time algorithms to solve VP3HI, VSP3HI, and EP3HI on trees (acyclic connected graph).

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<sup>†</sup>We are sincerely grateful to professor Jayme Szwarcfiter for this collaboration, and we dedicate this work to his 80th anniversary.

<sup>‡</sup>This work was partially supported by CNPq, CAPES and FAPERJ.

## On the Biclique Graphs of Circular Arc Bigraphs

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*Keywords: circular arc, bipartite graphs, biclique graphs*

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A *biclique* is a maximal bipartite-complete set of vertices within a graph. Bicliques are a subject closely related to cliques, as a biclique in a graph  $G$  is a clique in  $G^2$  (the square graph of  $G$ ).

A *biclique graph* is the intersection graph of all the bicliques in a graph. The study of biclique graphs provides important insights into the structural properties of different graph classes, especially bipartite graphs.

In our work, we study the biclique graphs of *circular arc bigraphs*. A bipartite graph is a circular arc bigraph if there exists a bijection between its vertices and a family of arcs on a circle such that two vertices of opposing partite sets are neighbors if and only if their corresponding arcs intersect. We show that the biclique graphs of the *Helly* subclass of circular arc bigraphs are proper circular arc graphs, and provide a structural characterization of the biclique graphs of *non-bichordal Helly* circular arc bigraphs.

The biclique graph of a given triangle-free graph  $G$  is the square of its *mutually included biclique graph*. The concept of mutually included biclique graphs, introduced by Groshaus and Guedes, is applied in the characterization of the biclique graphs of triangle-free graphs.

We show a structural characterization for the mutually included biclique graphs of non-bichordal Helly circular arc bigraphs, and prove that the mutually included biclique graphs of *normal-proper-Helly* circular arc bigraphs are proper circular arc graphs and, therefore, that their biclique graphs are the squares of proper circular arc graphs.

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## Tree 3-spanners on prisms of graphs

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*Keywords: tree spanner, 3-spanner, prisms of graphs, linear-time algorithm*

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A *tree  $t$ -spanner* of a graph  $G$  is a spanning tree of  $G$  in which the distance between any pair of vertices is at most  $t$  times their distance in  $G$ . The TREE  $t$ -SPANNER PROBLEM ( $\text{TREES}_t$ ) asks whether a given graph admits a tree  $t$ -spanner. Cai and Corneil (1995) showed that  $\text{TREES}_t$  can be solved in linear time when  $t = 2$ , and is NP-complete when  $t \geq 4$ . The case  $t = 3$  remains open. It is known that  $\text{TREES}_3$  can be solved in polynomial time for some classes of graphs, such as planar, convex, interval, and split graphs. Fomin, Golovach and van Leeuwen (2011) proved that  $\text{TREES}_t$  can be solved in polynomial time on bounded-degree graphs (using results on treewidth).

We study  $\text{TREES}_3$  on the class of prisms of graphs, and characterize those that admit a 3-spanner. The *prism of a graph  $G$*  is the graph obtained by considering two copies of  $G$ , and by linking its corresponding vertices by an edge (also defined as the Cartesian product  $G \times K_2$ ). Couto and Cunha (2021) showed that  $\text{TREES}_t$  is NP-complete for  $t \geq 5$ , even on the class of prisms of graphs. The characterization we show is based on simple properties of the nontrivial blocks of a block tree decomposition of  $G$ . As a result, we obtain a linear-time algorithm for  $\text{TREES}_3$  (and the corresponding search problem) on this class of graphs.

## Extendiendo Gráficas Cuadrado-Complementarias

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*Keywords: Gráficas cuadrado-complementarias, grupos de permutaciones, ecuaciones de gráficas.*

El *complemento*  $\overline{G}$  de una gráfica  $G$  es la gráfica obtenida de  $G$  al eliminar todas sus aristas y agregar como aristas a todos los pares de no-aristas de  $G$ . El *cuadrado*  $G^2$  de  $G$  es la gráfica que se obtiene de  $G$  al conectar con aristas a todos los vértices que estén a distancia a lo más 2 en  $G$ . Una gráfica es *cuadrado-complementaria* (squco por abreviar) si  $G = \overline{G^2}$ . El estudio de gráficas squco fue iniciado por Schuster (1981) y luego continuado por Akiyama et al. (Disc. Math. **34** (1981), 209–218), Baltić, et al. (1994), Milanič, et al. (Disc. Math. **327** (2014), 62–75) entre varios otros.

Las gráficas squco, se estudian en el contexto de la *Dinámica de Gráficas* (Prisner, Chapman & Hall/CRC Research Notes in Mathematics Series, 1995) y en particular en el subtema de *Ecuaciones Gráficas* en donde las ecuaciones que son consideradas más interesantes son aquellas en las que sus soluciones no han logrado ser caracterizadas, como es el caso de la ecuación que define a las gráficas squco:  $G = \overline{G^2}$ .

Decimos que una gráfica squco  $H$  extiende a otra gráfica squco  $G$  si  $G$  es una subgráfica inducida de  $H$ . Solamente hay dos métodos conocidos para extender gráficas squco, ambos reportados por Milanič et al. en 2014, como Lemma 2.1 y Proposition 2.5 (el primer resultado fue publicado originalmente por Baltić, et al. en 1994).

En esta plática presentaremos un nuevo método de extensión de gráficas squco que unifica y generaliza estos dos resultados.

# Session 17

## *Computational Complexity*

Chair: Vinicius Fernandes dos Santos

The Terminal Connection Problem on Rooted Directed Path Graphs is <b>NP</b> -complete . . . . .	96
Subdivisions with Parity in Digraphs . . . . .	97
The absolute oriented clique number problem is <i>NP</i> -complete . . . . .	98
Hard instances for the maximum clique problem . . . . .	99

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The Terminal Connection Problem on Rooted Directed Path Graphs is NP-complete

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*Keywords:* Terminal connection; Steiner tree; Rooted directed path

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A *connection tree* of a graph  $G$  for a *terminal set*  $W \subseteq V(G)$  is a tree subgraph  $T$  of  $G$  such that  $\text{leaves}(T) \subseteq W \subseteq V(T)$ . STEINER TREE is a fundamental network design problem, whose input consists of a graph  $G$ , a terminal set  $W$ , and a non-negative integer  $k$ , aiming at deciding whether there is a connection tree of  $G$  for  $W$  with at most  $k$  non-terminal vertices.

In this work, we analyze the variant of the STEINER TREE problem called TERMINAL CONNECTION problem (TCP), which imposes further constraints on the non-terminal vertices of the sought connection tree  $T$ . A non-terminal vertex is called *linker* if its degree in  $T$  is exactly 2, and it is called *router* if its degree in  $T$  is at least 3. Given a graph  $G$ , a terminal set  $W$  and two non-negative integers  $\ell$  and  $r$ , TCP asks whether there is a connection tree of  $G$  for  $W$  that contains at most  $\ell$  linkers and at most  $r$  routers.

TCP was shown to be NP-complete on strongly chordal graphs (Melo, Figueiredo, and Souza, On the Terminal Connection Problem, LNCS 12607 (2021), 278–292), contrasting with the complexity of STEINER TREE, which is known to be polynomial-time solvable on such graphs (White, Farber, and Pulleyblank, Steiner trees, Connected Domination and Strongly Chordal Graphs, *Networks* 15 (1985), 109–124). We extend this hardness result for TCP by proving that the problem remains NP-complete on rooted directed path graphs, a proper subclass of strongly chordal, that comprises the vertex intersection graphs of directed paths in a rooted directed tree.



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## Subdivisions with Parity in Digraphs

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*Keywords: Subdivision, Parity, Digraphs*

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Let  $D = (V, A)$  be a digraph. The problem of when a given digraph  $D$  contains a subdivision of a fixed digraph  $F$  with parity in each arc is considered. A subdivision of a digraph  $F$ , also called an  $F$ -subdivision, is a digraph obtained from  $F$  by replacing each arc  $ab$  of  $F$  by a directed  $(a, b)$ -path. The parity over each arc  $ab$  is binary  $(0, 1)$ , 0 for even and 1 for odd. The parity indicates whether the length of the path that is generated by subdivisions in each arc is even or odd. In this paper, we studied the complexity of finding a subdivision with given parity for particular cases of  $F$ . For directed paths, we generalize existent results to show that it is NP-complete to find an odd path  $P_k$  between two prescribed vertices [3] and polynomial-time solvable if the vertices are not given, for fixed  $k$  [1]. For directed spiders, we use the same approach as in directed paths. We also consider the case of  $F$  being a directed cycle. We study that the problem is polynomial-time solvable for the special case where  $F$  is a  $C_3$  [2]. The general case of  $F$  being a  $C_k$ , for an even  $k$ , remains an open case.

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The absolute oriented clique number problem is *NP*-complete

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*Keywords: Oriented Clique Number, Oriented Chromatic Number*

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Let  $\vec{G} = (V, A)$  be an oriented graph. An *oriented  $k$ -coloring* of  $\vec{G}$  is a partition of  $V$  into  $k$  color classes such that there is no pair of adjacent vertices belonging to the same class and all the arcs between a pair of color classes have the same orientation. The smallest  $k$  such that  $\vec{G}$  admits an oriented  $k$ -coloring is the *oriented chromatic number*  $c_o(\vec{G})$  of  $\vec{G}$ . We say that  $R \subseteq V(\vec{G})$  is an *absolute oriented clique* (Klostermeyer and Macgillivray (2004)) whether  $c_o(R) = |R|$ . The *absolute oriented clique number*  $w_{ao}(\vec{G})$  is the size of a maximum absolute oriented clique of  $\vec{G}$ . Given  $x, y \in V(\vec{G})$  the *directed distance*  $\vec{d}_{\vec{G}}(x, y) = \min\{k, \infty\}$ , where  $k$  is the number of arcs in a shortest path from  $x$  to  $y$ . The *weak directed distance*  $\bar{d}_{\vec{G}}(x, y) = \min\{\vec{d}_{\vec{G}}(x, y), \vec{d}_{\vec{G}}(y, x)\}$ . A set  $R \subseteq V(\vec{G})$  is a *relative oriented clique* (Nandy et al. (2016)) if for each pair  $x, y \in R$ ,  $\bar{d}_{\vec{G}}(x, y) \leq 2$ . The *relative oriented clique number*  $w_{ro}(\vec{G}) = \max\{|R|\}$ , where  $R$  is a relative oriented clique of  $\vec{G}$ . Given an integer  $k \geq 0$ , deciding if  $w_{ro}(\vec{G}) \geq k$  is *NP*-complete (Das et al. (2018)). In this work we show that deciding if  $w_{ao}(\vec{G}) \geq k$  is *NP*-complete and that unless  $P = NP$ , there is no polynomial time  $e$ -approximation for  $w_{ao}$  within a factor of  $n^e$  for some  $e > 0$ .

Hard instances for the maximum clique problem

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*Keywords: Maximum clique, Branch and Bound, Hard Instances*

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The Maximum Clique problem (MC) is a fundamental NP-hard problem which is inapproximable to within a factor of  $n^{1-e}$ , for any  $e > 0$  unless  $P = NP$ . Its decision version is also W[1]-complete under the natural parameterization. Despite its difficulty, there are reports of exact algorithms being able to solve instances of MC of practical interest and considerable size in several application domains quite satisfactorily. Züge and Carmo (2018) approached this theoretical-experimental dichotomy and conclude that, an instance of MC on a graph  $G$  can be solved by tackling  $O(C(G))$  subinstances, where  $C(G)$  is the number of cliques in  $G$ . We note that, when  $G \sim G(n, p)$ , we have  $C(G) = n^{O(\log n)}$  with high probability. It follows that with high probability, an instance of MC whose graph  $G$  is based on the  $G(n, p)$  model can be solved in quasi-polynomial time.

Aiming to find instances which are harder than the random graph model, we study and benchmark a transformation based on a result of Cornaz and Jost (2008) that maps a graph  $G$  into a graph  $G^*$  where each proper coloring of  $G$  is associated with a clique in  $G^*$ . We prove that beginning with a random graph  $G$ , one can build  $G^*$  which has  $N^{O(\sqrt{N})}$  expected number of cliques, where  $N = |V(G^*)|$ . Besides providing a family of harder instances for MC for enumeration algorithms, it gives an intuition that solving the Vertex Coloring problem should be harder than solving MC on  $G(n, p)$ .

Moreover, we add to Lavnikovich's (2013) result which presents a family of graphs that demands exponential time to be solved by B&B methods whose sole upper bound is the chromatic number, which are amongst the best algorithms for the MC. We show a simple but powerful heuristic which solves these instances in polynomial time and provide a randomized construction that protects these instances against this heuristic.

# Session 18

## *Graph Classes*

Chair: André Luiz Pires Guedes

O Número Cromático Total de Grafos Split 2-admissíveis  
101

Neighbour-distinguishing edge-labelling of powers of paths  
102

Hunting a conformable fullerene nanodisc that is not  
4-total colorable . . . . . 103

A New Bound for the Sum of Squares of Degrees in a  
Class 2 Graph<sup>†</sup> . . . . . 104

Author Index . . . . . 105

## O Número Cromático Total de Grafos Split 2-admissíveis

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*Keywords: total coloring, split graphs, 2-admissible split graphs*

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Um grafo  $G$  é  $k$ -total colorível se existe uma atribuição de  $k$  cores aos vértices e arestas do grafo de modo que elementos adjacentes ou incidentes recebam cores distintas. O menor valor de  $k$  tal que  $G$  é  $k$ -total colorível é chamado de *número cromático total* e é denotado por  $c''(G)$ . A *Conjectura da Coloração Total (TCC)* pressupõe que  $D + 1 \leq c''(G) \leq D + 2$ , para todo grafo  $G$ , classificando os grafos em *Tipo 1*,  $c''(G) = D + 1$  ou *Tipo 2*, caso contrário. Em geral, determinar  $c''(G)$  é NP-difícil (Sánchez-Arroyo, 1989).  $G$  é *split* quando  $V(G)$  é particionável em uma clique e um conjunto independente. Os grafos split satisfazem a TCC (Chen, Fu and Ko, 1995), mas determinar  $c''$  para a classe é um problema em aberto, solucionado apenas para algumas subclasses de split (Chen, Fu and Ko, 1995; Yap, 1989).  $G$  é  $t$ -admissível se admite uma árvore geradora na qual a maior distância entre vértices adjacentes de  $G$  é  $t$  (Cai and Cornell, 1995). O menor valor de  $t$  tal que  $G$  é  $t$ -admissível é  $s(G)$ . Grafos split são 3-admissíveis (Panda and Das, 2010), o que particiona a classe em: split com  $s(G) = 1$  (árvores),  $s(G) = 2$  ou 3. Neste trabalho, mostramos que um grafo  $G$  split com  $s(G) = 2$  é Tipo 2 sse existe  $H = G[N[v]]$ , para algum  $v \in V(G)$ ,  $d(v) = D$  tal que  $|E(H)| \geq \lfloor \frac{|V(H)|}{2} \rfloor D$ . Além disso, apresentamos um algoritmo eficiente para colorir grafos split com  $s(G) = 2$  que são Tipo 1.

## Neighbour-distinguishing edge-labelling of powers of paths

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For a simple graph  $G$  with vertex set  $V(G)$  and edge set  $E(G)$ , a pair  $(p, C_p)$  is a *neighbour-distinguishing  $f_1, \dots, k$ -edge-labelling* if  $p: E(G) \rightarrow f_1, \dots, k$  such that, for every  $v \in V(G)$ ,  $C_p(v) = \sum_{u \in N(v)} p(uv)$  and  $C_p(x) \neq C_p(y)$  for every edge  $xy \in E(G)$ . The least  $k$  for which it has been shown that every graph admits a neighbour-distinguishing  $f_1, 2, \dots, k$ -edge-labelling is five. The 1, 2, 3-Conjecture, proposed in 2004 by Karoński et al., states that every graph has a neighbour-distinguishing  $f_1, 2, 3$ -edge-labelling. This conjecture has been verified for a few classes of graphs, such as 3-chromatic graphs (Karoński et al. 2014), graphs in which every cycle is divisible by four (Khatirinejad, et al. 2011), complete multipartite graphs, and split graphs (Luiz et al. 2018).

Observing that there exist graphs that admit a neighbour-distinguishing  $f_1, 2$ -edge-labelling, another interesting problem, also proposed by Karoński et al., is the characterization of such graphs. In 2011, Dudek and Wajk proved that deciding whether an arbitrary graph  $G$  admits a neighbour-distinguishing  $f_1, 2$ -edge-labelling is NP-complete. Indeed, this problem proved to be challenging even for bipartite graphs. In 2016, Thomassen, Wu and Zhang proved that the only bipartite graphs that need labels 1, 2, 3 are precisely odd multi-cacti. In 2017, Luiz and Campos verified the 1, 2, 3-Conjecture for powers of paths and conjectured that a neighbour-distinguishing  $f_1, 2$ -edge-labelling could be built for powers of paths not isomorphic to complete graphs. In this work, we prove this conjecture.

Hunting a conformable fullerene nanodisc that is not  
4-total colorable

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*Keywords: Total Coloring, Fullerene Nanodiscs, Conformable Graphs*

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Fullerene nanodiscs  $D_r$  are mathematical models for carbon-based molecules experimentally found in the early eighties, which are cubic, 3-connected, planar graphs with pentagonal and hexagonal faces. The planar embedding of  $D_r$  has its faces arranged into layers, one layer next to the nearest previous layer starting from a hexagonal cover until we reach the other hexagonal cover. The distance between the inner (outer) layer and the central layer, where lie 12 pentagonal faces, is given by the radius parameter  $r \geq 2$ .

A total coloring of a graph assigns colors to the vertices and edges such that adjacent or incident elements have different colors. The long-standing Total Coloring Conjecture is settled for cubic graphs, implying that every cubic graph admits a 5-total coloring. However, the recognition of 4-total colorable cubic graphs is a challenging problem. Since every known cubic graph that is not 4-total colorable has girth at most 4, it has been conjectured that every cubic graph with girth at least 5 is 4-total colorable. A necessary step toward proving that a cubic graph admits a 4-total coloring is to define a conformable coloring, which is a 4-vertex coloring where the cardinality of each vertex color class has the same parity as the cardinality of the entire vertex set. A graph that admits a conformable coloring is called conformable.

We prove that every fullerene nanodisc is conformable. Although every conformable coloring we were able to exhibit does extend to a 5-total coloring, we are still looking for a suitable conformable coloring that might extend to the desired 4-total coloring. In parallel with these investigations, we present combinatorial contributions referring to the structure of fullerene nanodiscs.

# A New Bound for the Sum of Squares of Degrees in a Class 2 Graph<sup>†</sup>

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*Keywords:* Coloring of graphs and hypergraphs (MSC 05C15), Edge subsets with special properties (MSC 05C70), Vertex degrees (MSC 05C07)

Let  $G$  be a simple graph. For  $v \in V(G)$ , the degree of  $v$  and the set of neighbors of  $v$  are denoted  $d(v)$  and  $N(v)$ , respectively. The maximum degree of  $G$  is denoted  $D$ . The *majors* of  $G$  are its vertices with degree  $D$ . The majors  $u$  with  $\sum_{v \in N(u)} d(v) \geq D^2 - D + 2$  are the *proper* majors of  $G$ . If this sum is equal to (less than)  $D^2 - D + 1$ , then  $u$  is said to be *tightly (strictly) non-proper*. The *hard core* of  $G$  is the subgraph of  $G$  induced by all its proper and tightly non-proper majors. By Vizing (1964), the chromatic index of  $G$  is  $D$  or  $D + 1$ , in which case  $G$  is *Class 1* or *Class 2*, respectively. A critical graph is a connected Class 2 graph that becomes Class 1 by the removal of any edge. Vizing's recoloring procedure yields a condition, known as Vizing's Adjacency Lemma (VAL), about the minimum number of majors adjacent to every vertex of a critical graph. Zatesko et al. (2020) introduced an extended recoloring procedure, with which we have obtained an Extended Adjacency Lemma (EAL), establishing a condition on the minimum number of majors of the hard core adjacent to every vertex. As VAL also yields a lower bound for the the sum of degrees of a Class 2 graph  $G$ , with EAL we obtain a new lower bound for the sum of squares of degrees of  $G$ . For all  $D \geq 6$ , our bound is better from the bound achieved from the literature by simply combining Vizing's result and the lower bound for the sum of squares of degrees by Edwards (1977). For instance,

$$\sum_{u \in V} d^2(u) \geq \frac{46D^3 + 144D^2 - (4D^2 + 24D - 9)^{\frac{3}{2}}}{54} - (D - 1) \quad (\text{our bound})$$

$$\stackrel{(\text{for } D \geq 6)}{\geq} \frac{4((3D^2 + 6D - 1)/8)^2}{D + 1}. \quad (\text{literature})$$

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# Author Index

- Abreu, Alexandre, 53  
Adauto, Matheus, 50  
Alcón, Liliana, 56, 59  
Almeida, Sheila Morais, 39  
Amorim, Bernardo, 60  
Araújo, J., 83  
Arpini Junior, Judismar, 63  
Artigas, Danilo, 78
- Bessa, João Alfredo, 64  
Botler, Fábio, 68  
Bougeret, M., 83  
Brondani, André E., 49, 74
- Campos, C. N., 34, 40, 79, 102  
Campos, Victor, 83, 99  
Cappelle, Márcia R., 35  
Cararo, Cintia Izabel, 39  
Carcereri, Cristopher, 54  
Carli Silva, Marcel K., 73  
Carmo, Renato, 54, 99  
Carneiro, J. C., 45  
Carvalho, Cláudio, 20, 21  
Carvalho, Moisés T., 91  
Carvalho, Vinícius de Souza, 77  
Castonguay, Diane, 41  
Chandran, Ullas, 88  
Clementino, Thailsson, 22  
Coelho, Erika M. M., 35, 98  
Coelho, H., 98  
Cordeiro, Lucas, 69  
Costa, Diego Amaro, 101
- Costa, Isac, 44  
Costa, Isnard Lopes, 28  
Costa, Jonas, 20, 21  
Costa, João Pedro de S. Gomes, 46  
Coutinho, Gabriel, 60, 73  
Couto, Fernanda, 101  
Cruz, Claudia De la , 81  
Cruz, M., 103  
Cunha, T. H. F. M., 104
- Dantas, Simone, 38, 78, 87, 91  
Del-Vechio, Renata, 72  
Deveza, Jesse, 69  
Dias, Elisângela Silva, 41  
Dourado, Mitre C., 24, 91
- Faria, Luerbio, 45, 65, 70, 76, 98  
Ferreira, M., 98  
Figueiredo, C. M. H., 50, 53, 96,  
103  
Fisquino, Sérgio, 31  
Flores-Peñaloza, David, 82  
Folz, Raquel, 86, 89  
França, Francisca A. M., 49  
Freitas, Rosiane, 22, 29, 64, 69, 84,  
86, 89  
Fuentes-Hernández, Andrés, 82
- Gomes, Guilherme C. M., 36  
Gonzaga, L. G. S., 102  
Gonçalves, Isabel F. A., 38  
Grandsire, Rafael, 73

- Grippe, Luciano N., 61  
Groskaus, Marina, 48, 92  
Gudiño, Noemí Amalia, 59  
Guedes, André L. P., 70, 92  
Guevara, Mucuy-kak, 18  
Gutierrez, Marisa, 24, 59  
Gómez, Renzo, 93
- Hernández-Sayago, Lesli, 43  
Hora, Guilherme Willian S., 33  
Hsu, Meng, 86, 89  
Huaynoca, P. T. P., 87
- Juárez-Valencia, Ariadna, 94
- Klavžar, Sandi, 88  
Klein, Sulamita, 98, 101  
Kolberg, Fabricio Schiavon, 92
- Lacerda, Isac M., 84  
Lima, Carlos V. G. C., 44  
Lin, Min C., 66  
Lintzmayer, Carla N., 51, 77  
Lopes, Raul, 20, 21  
Luiz, Atilio G., 33, 40, 78, 79
- Maculan, Nelson, 26  
Maia, Ana Karolinn, 20, 21, 97  
Marcilon, Thiago, 44  
Markezon, Lilian, 70  
Marquezino, Franklin, 53  
Martins, Lia, 86, 89  
Masquio, Bruno P., 36  
Medeiros, Lívia, 58  
Melo, A. A., 96  
Melo, Marcus Vinicius Martins, 97  
Melquiades, Victor, 74  
Mesquita, Fernanda Neiva, 41  
Miyazawa, Flávio K., 93  
Montanheiro, Gustavo L., 48  
Monte Carmelo, Emerson Luiz, 30  
Mortosa, Otávio S., 35
- Mota, Guilherme O., 51  
Mota, Kelson, 64  
Moyano, Verónica A., 61
- Nascimento, Julliano R., 35, 41  
Neethu, P. K., 88  
Nicodemos, D., 46, 103  
Nigro, Mauro, 31, 76  
Nisse, Nicolas, 20, 21  
Nogueira, Rodrigo, 99  
Nunes, Jonathas, 64
- Oliveira, Carla S., 74  
Oliveira, Deise L., 78  
Oliveira, Fabiano S., 58, 65, 66  
Oliveira, Micael, 64  
Oliveira, Thiago, 73  
Oliveros, Débora, 25  
Omai, M. M., 40, 79
- Pabon, M. V., 45  
Palma, Miguel A. D. R., 87  
Pereira, A. A., 34  
Pinto, Paulo E. D., 36, 66  
Pinto, Renan Vicente, 26  
Pizaña, Miguel, 43, 55, 81, 94  
Portugal, Lucas, 72  
Posner, Daniel, 53, 91  
Proença, Glasielly Demori, 39  
Protti, Fábio, 24
- Ravenna, Gabriela, 56  
Robles, I. A., 55  
Rocha, Aleffer, 54  
Rocha, Lucas S., 51
- Sales, Cláudia Linhares, 20, 21  
Sambinelli, Maycon, 51, 77  
Sampaio Jr., Moysés S., 66  
Sampaio, Rudini, 88  
Santos, Clarice, 64  
Santos, Lanier, 69

- 
- Santos, Vinicius F., 16, 36, 60  
Sasaki, Diana, 31, 38, 50, 76, 103  
Sau, I., 83  
Schmitz, Eber A., 84  
Schneider, Rafael, 68  
Silva, Ana, 17, 28, 96  
Silva, Cândida Nunes, 39  
Soares, Lucas da Penha, 30  
Sousa, A. D. R., 45  
Souza U. S., 96  
Szwarcfiter, Jayme L., 36, 58, 66,  
84, 91  
Sá, Vinícius Gusmão Pereira, 63
- Terra, J. P. M., 65  
Tondatto, Silvia B., 24  
Torres, Antonio, 25  
Tovar, M., 103
- Uchoa, Eduardo, 22
- Ventura, Lara R., 49  
Villarroel-Flores, Rafael, 43
- Wakabayashi, Yoshiko, 93
- Zatesko, Leandro M., 48, 104