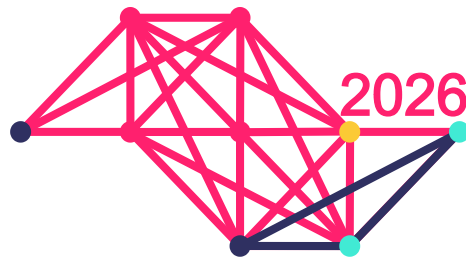


Book of Abstracts of the



LAWCG

12th Latin American Workshop on Cliques in Graphs

Águas de Lindóia, São Paulo, Brazil
October 5–8, 2026

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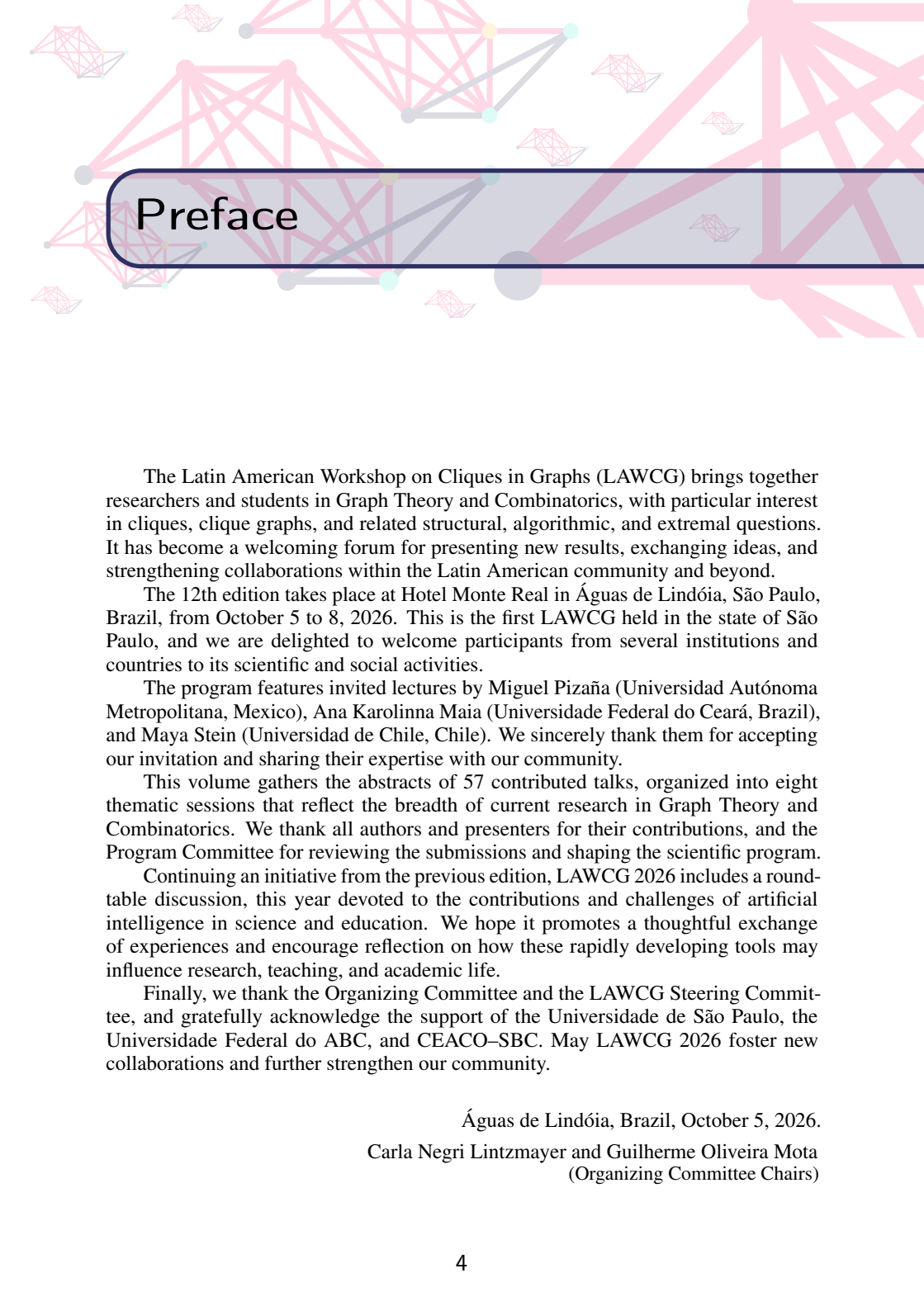
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Preface

The Latin American Workshop on Cliques in Graphs (LAWCG) brings together researchers and students in Graph Theory and Combinatorics, with particular interest in cliques, clique graphs, and related structural, algorithmic, and extremal questions. It has become a welcoming forum for presenting new results, exchanging ideas, and strengthening collaborations within the Latin American community and beyond.

The 12th edition takes place at Hotel Monte Real in Águas de Lindóia, São Paulo, Brazil, from October 5 to 8, 2026. This is the first LAWCG held in the state of São Paulo, and we are delighted to welcome participants from several institutions and countries to its scientific and social activities.

The program features invited lectures by Miguel Pizaña (Universidad Autónoma Metropolitana, Mexico), Ana Karolinnna Maia (Universidade Federal do Ceará, Brazil), and Maya Stein (Universidad de Chile, Chile). We sincerely thank them for accepting our invitation and sharing their expertise with our community.

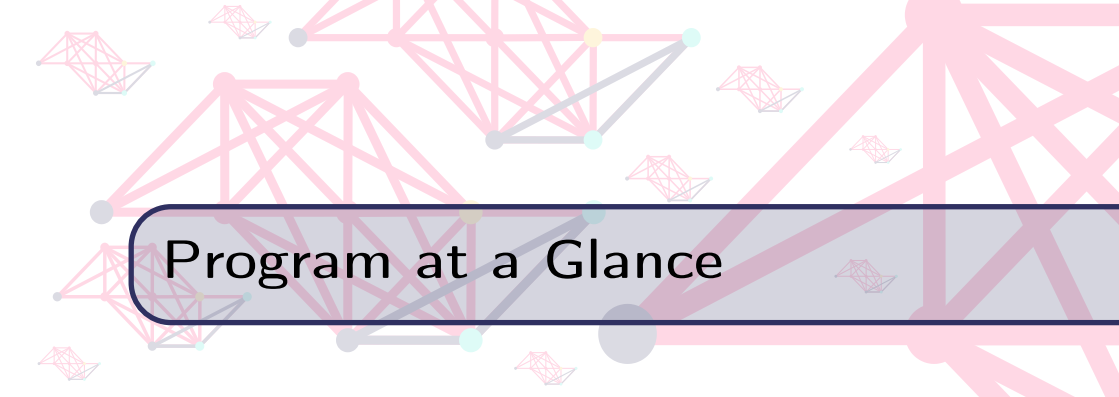
This volume gathers the abstracts of 57 contributed talks, organized into eight thematic sessions that reflect the breadth of current research in Graph Theory and Combinatorics. We thank all authors and presenters for their contributions, and the Program Committee for reviewing the submissions and shaping the scientific program.

Continuing an initiative from the previous edition, LAWCG 2026 includes a roundtable discussion, this year devoted to the contributions and challenges of artificial intelligence in science and education. We hope it promotes a thoughtful exchange of experiences and encourage reflection on how these rapidly developing tools may influence research, teaching, and academic life.

Finally, we thank the Organizing Committee and the LAWCG Steering Committee, and gratefully acknowledge the support of the Universidade de São Paulo, the Universidade Federal do ABC, and CEACO-SBC. May LAWCG 2026 foster new collaborations and further strengthen our community.

Águas de Lindóia, Brazil, October 5, 2026.

Carla Negri Lintzmayer and Guilherme Oliveira Mota
(Organizing Committee Chairs)



Program at a Glance

Monday, October 5

4:00–6:00 p.m. Hotel check-in, workshop registration, and welcome coffee

Tuesday, October 6

8:30–8:50 a.m. Opening ceremony

8:50–9:50 a.m. Invited talk: Miguel Pizaña (chair: Marina Groshaus)

9:50 a.m.–1:00 p.m. Parallel technical sessions 1 and 2; coffee break from 11:10 to 11:40 a.m.

2:30–4:30 p.m. Parallel technical sessions 3 and 4

4:30–5:00 p.m. Coffee break

5:00–6:30 p.m. Round-table discussion

Wednesday, October 7

8:50–9:50 a.m. Invited talk: Ana Karolinnia Maia (chair: Vinicius dos Santos)

9:50 a.m.–1:00 p.m. Parallel technical sessions 5 and 6; coffee break from 11:10 to 11:40 a.m.

1:00 p.m. Official workshop photograph

2:30–4:30 p.m. Collaborative group work or free time at the hotel

4:30–5:00 p.m. Coffee break

5:00–6:30 p.m. Outdoor activities or free time at the hotel

8:00–9:30 p.m. Conference dinner

Thursday, October 8

8:50–9:50 a.m. Invited talk: Maya Stein (chair: Guilherme Mota)

9:50 a.m.–12:40 p.m. Parallel technical sessions 7 and 8; coffee break from 11:10 to 11:40 a.m.

12:40–1:00 p.m. Closing ceremony

By 3:00 p.m. Hotel check-out

Technical Sessions

Tuesday, October 6

9:50 a.m.–1:00 p.m.

Time	Session 1: Identification, location, and domination Chair: Marcia Cappelle	Session 2: Algorithms, complexity, and optimization Chair: Leandro Zatesko
9:50 a.m.	On Locally Identifying Coloring of Cubic Graphs <i>Luiz Carlos de Souza Almeida, Marcia Rodrigues Cappelle Santana and Diane Castonguay</i>	Trie Mergesort: A unified approach to natural mergesorts <i>Théo Araújo Magalhães, Victor Almeida Campos, Pedro Henrique Feitosa Fernandes and Rafael de Paiva Lima</i>
10:10 a.m.	Locally identifying coloring in some split-comparability graphs <i>Murilo Martins Ferreira and Hebert Coelho da Silva</i>	Dijkstra Graphs: Short-circuit evaluation <i>Matheus Telles Batista, Matheus de Souza Feitosa and André L. P. Guedes</i>
10:30 a.m.	On Locally Identifying Coloring of some Circulant Graphs <i>Ryan Fernandes Auder Lopes and Márcia Rodrigues Cappelle Santana</i>	Complexity of the Minimum Labeled Feedback Edge Set Problem <i>Rafael de Paiva Lima, Claudia Linhares Sales and Manoel Campêlo</i>
10:50 a.m.	Independent Locating-Dominating Sets in some Snark Families <i>Dayllon Vinícius Xavier Lemos, Márcia Cappelle, Humberto José Longo, Erika Morais Martins Coelho and Leslie Richard Foulds</i>	Contagem de Triângulos em Fluxo de Grafos Utilizando Complexidade de Amostra <i>Vitor Alencar, Alane da Silva and Murilo da Silva</i>
11:40 a.m.	Locating-dominating partitions and total variants for subcubic graphs <i>Joviro Rocha Neto and Márcia Rodrigues Cappelle Santana</i>	Connectedness in Flip and Twist Reconfiguration Graphs of Colored Triangulations <i>Daniel Debrohim, Diana Sasaki and Patrícia Nunes</i>
12:00 p.m.	Partial Independent k-Dominating Sets in a Superclass of Distance-Hereditary Graphs <i>Vitor Ponciano, Christiane Campos, Simone Dantas and Sylvain Gravier</i>	A Lower Bound on the p-Inversion Diameter of Tournaments <i>Lucas Silva and Caroline Silva</i>
12:20 p.m.	Geodesic mutual visibility in unit interval graphs <i>L. G. S. Gonzaga, L. D. Penso and A. A. Pereira</i>	Optimization Problems on Maya-Tupi Graphs <i>Júlio Araújo, César Hernández-Cruz and Cláudia Linhares Sales</i>
12:40 p.m.	Exact Algorithm for Double Domination with Branch Cutting <i>Iago Arruda Faria and Tanilson Dias dos Santos</i>	Square roots of split graphs via matchings <i>Aleffer Rocha, Renato Carmo and André L. P. Guedes</i>

Tuesday, October 6

2:30–4:30 p.m.

Time	Session 3: Colorings: special classes, complexity, and algorithms Chair: Victor Campos	Session 4: Cliques, bicliques, and perfection Chair: César Hernández-Cruz
2:30 p.m.	Advances on Clustered Colourings of Outerplanar Graphs <i>Luis Gustavo Da Soledade Gonzaga and Christiane Neme Campos</i>	Cliques and Convolutions: An Algebraic Approach to Counting Cliques in Sparse Graphs <i>Miguel Alessandro Mini Huambachano</i>
2:50 p.m.	Jogos de Convexidades de Caminhos em Grafos <i>Eurinaldo Costa, Leon Almeida and Rudini Sampaio</i>	Listing bicliques and cliques: Bipartite Permutation and Proper 2-Thin Graphs <i>Gabriel Godinho, Marina Groshaus and André L. P. Guedes</i>
3:10 p.m.	Minimal Obstructions for Star Coloring in Chordal Graph Families <i>Germán Benítez-Bobadilla, Fernando Esteban Contreras-Mendoza, César Hernández-Cruz, Claudia Linhares Sales and Ana Laura Trujillo-Negrete</i>	Finding Bicliques in Quantum Computing <i>Leandro Zatesko and Marina Groshaus</i>
3:30 p.m.	Algoritmos para o problema da supercoloração em digrafos (acíclicos) com número fixo de cores <i>Pedro Guimarães Caninas, Alvaro Junio Pereira Franco, Rafael de Santiago and Wesly Carmesini Ataide</i>	Reduced clique graphs: characterization, superclasses and subclasses <i>Pablo De Caria</i>
3:50 p.m.	Branch and Bound Algorithm for the Grundy Number Problem <i>Eric Gomes, Kaio Barreto and Wladimir Tavares</i>	A Quadratic Characterization of the Weighted Lovász Theta Number and Related SDP Bounds <i>Luis Higinio and Gabriel Coutinho</i>
4:10 p.m.	On Graceful Colorings of $P(n,k)$ <i>Abias Vidal, Simone Dantas and Atilio Luiz</i>	On the Cost of Perfection in Wheel Graphs <i>Ingrid da Costa Silva Borchert, Diana Sasaki and Diego Nicodemos</i>

Wednesday, October 7

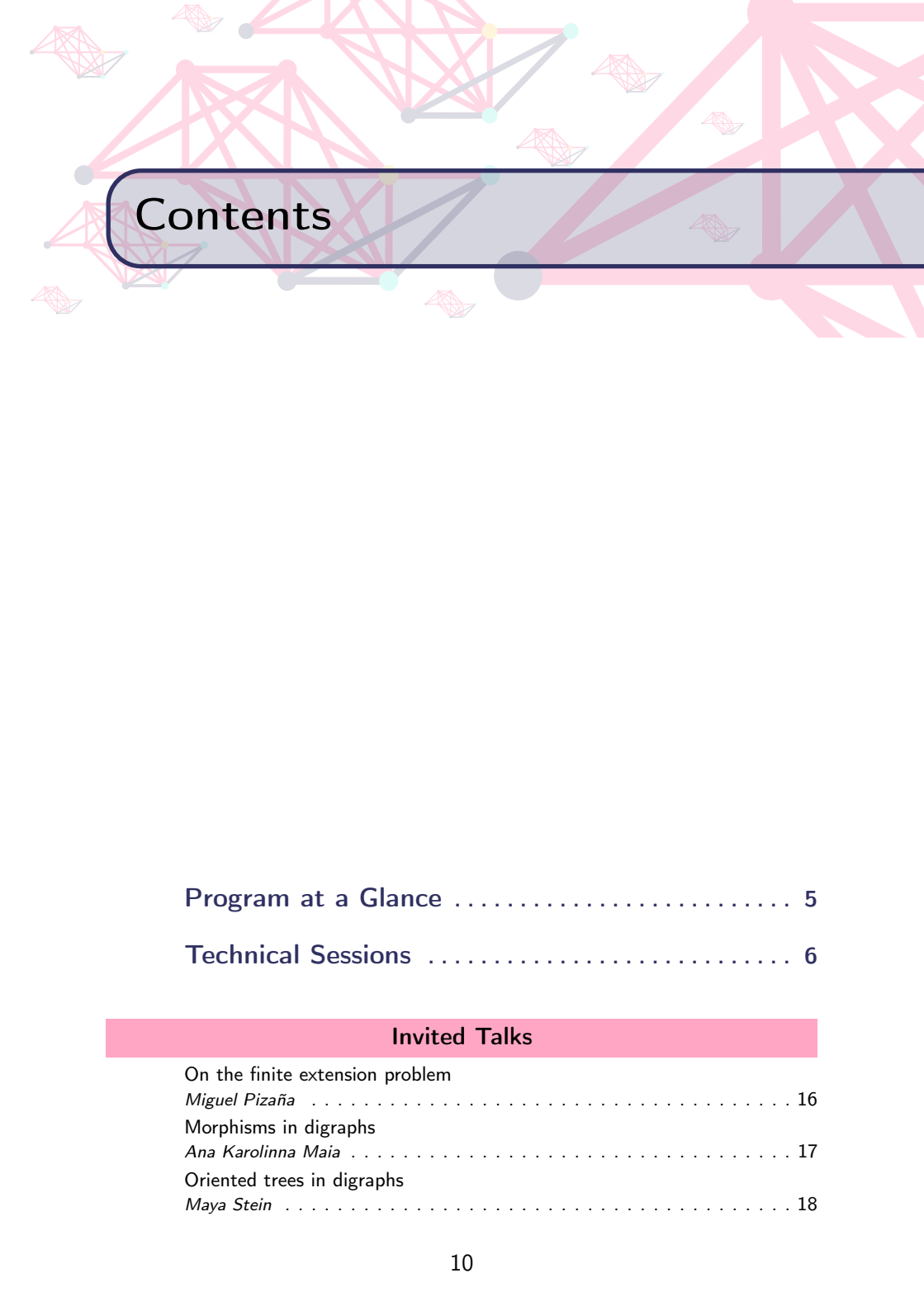
9:50 a.m.–1:00 p.m. (coffee break: 11:10–11:40 a.m.)

Time	Session 5: Total, equitable, and odd colorings Chair: Diana Sasaki	Session 6: Structural classes, digraphs, and convexity Chair: Guilherme Gomes
9:50 a.m.	Adjacent Vertex Distinguishing Total Coloring in Complete 3- and 4-Partite Graphs <i>Juliano Silva do Nascimento, Sheila Almeida and Atilio Gomes Luiz</i>	Balanced distance-hereditary graphs <i>Lucía Busolini, Guillermo Durán and Martín Safe</i>
10:10 a.m.	On the AVD-total chromatic number of cubic graphs <i>Mariana Cruz, Mauro Nigro and Diana Sasaki</i>	Intersection graphs characterizations <i>Marina Groshaus and Gabriel Dantas</i>
10:30 a.m.	Equitable Total Coloring of Complete Multipartite Graphs <i>Nicolas Eusebio, Sheila Almeida and Mauro Nigro</i>	Sobre atribuição de papéis em algumas subclasses de grafos cordais <i>Gustavo Medeiros and Julliano Nascimento</i>
10:50 a.m.	Equitable total chromatic number of graphs with universal vertex <i>Claudiney dos Santos, Arthur Baldoqui, Sheila de Almeida, Matheus Gonçalves, Mayara Omai, Celina de Figueiredo and Diana Sasaki</i>	Atribuição de papéis em produto corona de grafos analisando vértices gêmeos <i>Jarlilson Guajajara and Julliano Rosa Nascimento</i>
11:40 a.m.	Towards the $(2,1)$-total number of complete equipartite graphs <i>M. M. Omai, C. N. Campos and Atilio G. Luiz</i>	Study of Carathéodory Number in Hypercubes <i>Braully Silva, Erika Coelho, Hebert Coelho and Simone Dantas</i>
12:00 p.m.	On total b-chromatic number of power of paths <i>Clayton Santos de Araujo, Mauro Nigro, Luerbio Faria and Ricardo Pinheiro</i>	Strongly chordal digraphs, total domination, and tournaments with possible loops <i>Pavol Hell, César Hernández-Cruz and Jing Huang</i>
12:20 p.m.	Odd coloring of graphs and hypergraphs <i>Carla Negri Lintzmayer, Nicolas de Melo Pedrosa and Maycon Sambinelli</i>	An Approach to Canonical ESD Labelings in Sparse Graphs <i>João Gabriel Moren, Danilo Artigas and Simone Dantas</i>
12:40 p.m.	Parity locally irregular decomposition of graphs <i>Carla Lintzmayer, Maycon Sambinelli and Silas Silva</i>	Triangle-free graphs whose line graph squares are $C_{\geq k}$-free <i>Martín Safe and Martina Vergara</i>

Thursday, October 8

9:50 a.m.–12:40 p.m.

Time	Session 7: Extremal combinatorics, Ramsey theory, and cages Chair: Júlio Araújo	Session 8: Games, domination, alliances, and propagation Chair: Mitre Dourado
9:50 a.m.	Counting edge-colorings of a complete graph that avoid a rainbow K_4 <i>Fabrizio S. Benevides, Josefran de O. Bastos and Rodrigo F. Ribeiro</i>	Normal Domination Partizan Game in Stars <i>João Marcos Brito, Edileudo Moreira Filho, Jeffer Paz and Rudini Sampaio</i>
10:10 a.m.	Anti-Ramsey threshold for $K_{2,r}$ in random graphs <i>Juliane Lima, Walner Mendonça, Guilherme Mota and Henrique Stagni</i>	On the Complexity of Fall Coloring for Subclasses of Chordal Graphs <i>Ana Silva, Julio Araujo, Sheila Almeida and Vinicius dos Santos</i>
10:30 a.m.	Minimum degree threshold for covering bipartite graphs with monochromatic components <i>César Bispo, George Kontogeorgiou, Marcelo Lage, Guilherme Mota and Bruno Skarmeta</i>	O jogo da posição geral em alguns produtos fortes de grafos <i>Douglas Bessa and Hebert Silva</i>
10:50 a.m.	Matching extendability in the Cartesian product of complete graphs <i>A. A. Pereira and C. N. Campos</i>	Global defensive alliance in lexicographic product of stars with paths or cycles <i>Mitre C. Dourado and Alessandra B. Verissimo</i>
11:40 a.m.	König–Egerváry Index under Vertex Coalescence <i>Vadim Levit and Eugen Mandrescu</i>	Contributions toward the Flood-It Problem in co-comparability graphs <i>Luiza Santos, Sheila de Almeida and Mayara Omai</i>
12:00 p.m.	Transversally Transitive Unmatching Edge Graphs <i>Edmilson Pereira da Cruz, Marina Groshaus and André Luiz Pires Guedes</i>	Optimal containment for the firefighter problem on circular and Möbius ladders <i>Wagner da Rocha and Diana Sasaki</i>
12:20 p.m.	Mixed cages <i>Claudia Marlene De la Cruz Torres and Miguel Pizaña</i>	



Contents

Program at a Glance	5
Technical Sessions	6

Invited Talks

On the finite extension problem <i>Miguel Pizaña</i>	16
Morphisms in digraphs <i>Ana Karolína Maia</i>	17
Oriented trees in digraphs <i>Maya Stein</i>	18

Session 1: Identification, location, and domination

On Locally Identifying Coloring of Cubic Graphs <i>Luiz Carlos de Souza Almeida, Marcia Rodrigues Cappelle Santana and Diane Castonguay</i>	20
Locally identifying coloring in some split-comparability graphs <i>Murilo Martins Ferreira and Hebert Coelho da Silva</i>	22
On Locally Identifying Coloring of some Circulant Graphs <i>Ryan Fernandes Auder Lopes and Márcia Rodrigues Cappelle Santana</i>	24
Independent Locating-Dominating Sets in some Snark Families <i>Dayllon Vinícius Xavier Lemos, Márcia Cappelle, Humberto José Longo, Erika Moraes Martins Coelho and Leslie Richard Foulds</i>	26
Locating-dominating partitions and total variants for subcubic graphs <i>Joviro Rocha Neto and Márcia Rodrigues Cappelle Santana</i>	28
Partial Independent k -Dominating Sets in a Superclass of Distance-Hereditary Graphs <i>Vitor Ponciano, Christiane Campos, Simone Dantas and Sylvain Gravier</i> . . .	30
Geodesic mutual visibility in unit interval graphs <i>L. G. S. Gonzaga, L. D. Penso and A. A. Pereira</i>	32
Exact Algorithm for Double Domination with Branch Cutting <i>Iago Arruda Faria and Tanilson Dias dos Santos</i>	34

Session 2: Algorithms, complexity, and optimization

Trie Mergesort: A unified approach to natural mergesorts <i>Théo Araújo Magalhães, Victor Almeida Campos, Pedro Henrique Feitosa Fernandes and Rafael de Paiva Lima</i>	37
Dijkstra Graphs: Short-circuit evaluation <i>Matheus Telles Batista, Matheus de Souza Feitosa and André L. P. Guedes</i> .	39
Complexity of the Minimum Labeled Feedback Edge Set Problem <i>Rafael de Paiva Lima, Claudia Linhares Sales and Manoel Campêlo</i>	41
Contagem de Triângulos em Fluxo de Grafos Utilizando Complexidade de Amostra <i>Vitor Alencar, Alane da Silva and Murilo da Silva</i>	43
Connectedness in Flip and Twist Reconfiguration Graphs of Colored Triangulations <i>Daniel Debrohim, Diana Sasaki and Patrícia Nunes</i>	45
A Lower Bound on the p -Inversion Diameter of Tournaments <i>Lucas Silva and Caroline Silva</i>	47
Optimization Problems on Maya-Tupi Graphs <i>Júlio Araújo, César Hernández-Cruz and Cláudia Linhares Sales</i>	49

Square roots of split graphs via matchings <i>Aleffer Rocha, Renato Carmo and André L. P. Guedes</i>	51
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Session 3: Colorings: special classes, complexity, and algorithms

Advances on Clustered Colourings of Outerplanar Graphs <i>Luis Gustavo Da Soledade Gonzaga and Christiane Neme Campos</i>	54
Jogos de Convexidades de Caminhos em Grafos <i>Eurinaldo Costa, Leon Almeida and Rudini Sampaio</i>	56
Minimal Obstructions for Star Coloring in Chordal Graph Families <i>Germán Benítez-Bobadilla, Fernando Esteban Contreras-Mendoza, César Hernández-Cruz, Claudia Linhares Sales and Ana Laura Trujillo-Negrete</i>	58
Algoritmos para o problema da supercoloração em digrafos (acíclicos) com número fixo de cores <i>Pedro Guimarães Caninas, Álvaro Junio Pereira Franco, Rafael de Santiago and Wesly Carmesini Ataide</i>	60
Branch and Bound Algorithm for the Grundy Number Problem <i>Eric Gomes, Kaio Barreto and Wladimir Tavares</i>	62
On Graceful Colorings of $P(n, k)$ <i>Abias Vidal, Simone Dantas and Atílio Luiz</i>	64

Session 4: Cliques, bicliques, and perfection

Cliques and Convolutions: An Algebraic Approach to Counting Cliques in Sparse Graphs <i>Miguel Alessandro Mini Huambachano</i>	70
Listing bicliques and cliques: Bipartite Permutation and Proper 2-Thin Graphs <i>Gabriel Godinho, Marina Groshaus and André L. P. Guedes</i>	72
Finding Bicliques in Quantum Computing <i>Leandro Zatesko and Marina Groshaus</i>	74
Reduced clique graphs: characterization, superclasses and subclasses <i>Pablo De Caria</i>	76
A Quadratic Characterization of the Weighted Lovász Theta Number and Related SDP Bounds <i>Luis Higino and Gabriel Coutinho</i>	78
On the Cost of Perfection in Wheel Graphs <i>Ingrid da Costa Silva Borchert, Diana Sasaki and Diego Nicodemos</i>	80

Session 5: Total, equitable, and odd colorings

Adjacent Vertex Distinguishing Total Coloring in Complete 3- and 4-Partite Graphs	
<i>Juliano Silva Do Nascimento, Sheila Almeida and Atilio Gomes Luiz</i>	83
On the AVD-total chromatic number of cubic graphs	
<i>Mariana Cruz, Mauro Nigro and Diana Sasaki</i>	85
Equitable Total Coloring of Complete Multipartite Graphs	
<i>Nicolas Eusebio, Sheila Almeida and Mauro Nigro</i>	87
Equitable total chromatic number of graphs with universal vertex	
<i>Claudiney dos Santos, Arthur Baldoqui, Sheila de Almeida, Matheus Gonçalves, Mayara Omai, Celina de Figueiredo and Diana Sasaki</i>	89
Towards the $(2,1)$ -total number of complete equipartite graphs	
<i>M. M. Omai, C. N. Campos and Atilio G. Luiz</i>	91
On total b -chromatic number of power of paths	
<i>Clayton Santos de Araujo, Mauro Nigro, Luerbio Faria and Ricardo Pinheiro</i>	93
Odd coloring of graphs and hypergraphs	
<i>Carla Negri Lintzmayer, Nicolas de Melo Pedroso and Maycon Sambinelli</i>	95
Parity locally irregular decomposition of graphs	
<i>Carla Lintzmayer, Maycon Sambinelli and Silas Silva</i>	97

Session 6: Structural classes, digraphs, and convexity

Balanced distance-hereditary graphs	
<i>Lucía Busolini, Guillermo Durán and Martín Safe</i>	100
Intersection graphs characterizations	
<i>Marina Groshaus and Gabriel Dantas</i>	102
Sobre atribuição de papéis em algumas subclasses de grafos cordais	
<i>Gustavo Medeiros and Julliano Nascimento</i>	104
Atribuição de papéis em produto corona de grafos analisando vértices gêmeos	
<i>Jarlilson Guajajara and Julliano Rosa Nascimento</i>	106
Study of Carathéodory Number in Hypercubes	
<i>Braulio Silva, Erika Coelho, Hebert Coelho and Simone Dantas</i>	108
Strongly chordal digraphs, total domination, and tournaments with possible loops	
<i>Pavol Hell, César Hernández-Cruz and Jing Huang</i>	110
An Approach to Canonical ESD Labelings in Sparse Graphs	
<i>João Gabriel Moren, Danilo Artigas and Simone Dantas</i>	112
Triangle-free graphs whose line graph squares are $C_{\geq k}$ -free	
<i>Martín Safe and Martina Vergara</i>	114

Session 7: Extremal combinatorics, Ramsey theory, and cages

Counting edge-colorings of a complete graph that avoid a rainbow K_4 <i>Fabricao S. Benevides, Josefran de O. Bastos and Rodrigo F. Ribeiro</i>	117
Anti-Ramsey threshold for $K_{2,r}$ in random graphs <i>Juliane Lima, Walner Mendonça, Guilherme Mota and Henrique Stagni</i> . . .	119
Minimum degree threshold for covering bipartite graphs with monochromatic components <i>César Bispo, George Kontogeorgiou, Marcelo Lage, Guilherme Mota and Bruno Skarmeta</i>	121
Matching extendability in the Cartesian product of complete graphs <i>A. A. Pereira and C. N. Campos</i>	123
König-Egerváry Index under Vertex Coalescence <i>Vadim Levit and Eugen Mandrescu</i>	125
Transversally Transitive Unmatching Edge Graphs <i>Edmilson Pereira da Cruz, Marina Groshaus and André Luiz Pires Guedes</i> .	127
Mixed cages <i>Claudia Marlene De la Cruz Torres and Miguel Pizaña</i>	129

Session 8: Games, domination, alliances, and propagation

Normal Domination Partizan Game in Stars <i>João Marcos Brito, Edileudo Moreira Filho, Jefter Paz and Rudini Sampaio</i>	132
On the Complexity of Fall Coloring for Subclasses of Chordal Graphs <i>Ana Silva, Julio Araujo, Sheila Almeida and Vinicius dos Santos</i>	134
O jogo da posição geral em alguns produtos fortes de grafos <i>Douglas Bessa and Hebert Silva</i>	136
Global defensive alliance in lexicographic product of stars with paths or cycles <i>Mitre C. Dourado and Alessandra B. Verissimo</i>	138
Contributions toward the Flood-It Problem in co-comparability graphs <i>Luiza Santos, Sheila de Almeida and Mayara Omai</i>	140
Optimal containment for the firefighter problem on circular and Möbius ladders <i>Wagner da Rocha and Diana Sasaki</i>	142

Invited Talks

On the finite extension problem	
<i>Miguel Pizaña</i>	16
Morphisms in digraphs	
<i>Ana Karolinnna Maia</i>	17
Oriented trees in digraphs	
<i>Maya Stein</i>	18

On the finite extension problem

Plenary talk by Miguel Pizaña

Universidad Autónoma Metropolitana, Mexico

We say that a graph G is an extension of another graph H whenever $N_G(x) \cong H$ for every $x \in G$, i.e. when the subgraph induced by the open neighborhood of any vertex of G is isomorphic to H . For example, the octahedron is an extension of the 4-cycle and the icosahedron is an extension of the 5-cycle. It is easy to verify that the 3-path does not have any extension.

The *finite extension problem* (FEP, also known as the Trahtenbrot-Zykov problem) consists in deciding whether a given finite graph H has a finite extension. It is not known whether the FEP is algorithmically decidable, but several closely related problems are known to be undecidable, namely: the *infinite extension problem* (G must be infinite), the *extension problem* (G can be either finite or infinite), the *mixed finite extension problem* (neighborhoods are required to be isomorphic to some H_i in a given finite set of finite graphs $\mathcal{H} = \{H_1, H_2, \dots, H_r\}$) and the *2-mixed finite extension problem* (as before but with $|\mathcal{H}| = r = 2$). All these results were obtained by V. K. Bulitko.

In this talk we shall present a procedure (we do not know whether it terminates on all inputs) that do find extensions of graphs for small enough graphs. For instance, it is capable of finding an extension of the icosahedron of order 40 (the icosahedron has exactly 3 extensions of orders 40, 60 and 120). The procedure uses quite a range of algorithmic techniques including: backtracking, branch and bound, priority queues, dynamic programming, simplex method, isomorphism reduction, symmetry exploitation, and (exact) heuristic methods.

All the programming was done in GAP (<https://www.gap-system.org/>) plus YAGS (<http://xamanek.izt.uam.mx/yags/>). This is a joint work with R. Villarroel-Flores and N. García-Colín.

Morphisms in digraphs

Plenary talk by Ana Karolinna Maia
Universidade Federal do Ceará, Brazil

There are different ways of determining whether a (di)graph H is contained in a (di)graph G . We may look for an exact copy of H in G , that is, check whether G contains a sub(di)graph isomorphic to H . However, we may also ask whether G contains a graph “similar to” H , or one that can be obtained from H . The meanings of “similar” or “obtained from” may vary. In this work, we address different relations of this kind, involving (strong and weak) immersions and subdivisions. We refer to them as containment relations or (di)graph morphisms.

The study of these containment relations plays a fundamental role in graph theory, since many interesting classes are defined by forbidding certain (induced) subgraphs. From a computational complexity perspective, for undirected graphs, the problem of determining whether H is “contained” in G is usually polynomial-time solvable. However, this is not always the case when H and G are digraphs.

We investigate the complexity of checking the containment relations in different digraphs, as well as a variant of the problem with parity conditions.

Oriented trees in digraphs

Plenary talk by Maya Stein
Universidad de Chile, Chile

In this talk, I will present some old conjectures and new results on the following topic. Given an orientation of a given tree, what condition on a digraph D can ensure that D contains this tree? This could be a condition on the arc-density, or on the minimum out-degree, semi-degree or total degree. Does the answer change if D is known to have no 2-cycles, i.e., is an oriented graph? Some (partial) answers are given in the talk.

Session 1: Identification, location, and domination

On Locally Identifying Coloring of Cubic Graphs <i>Luiz Carlos de Souza Almeida, Marcia Rodrigues Cappelle Santana and Diane Castonguay</i>	20
Locally identifying coloring in some split-comparability graphs <i>Murilo Martins Ferreira and Hebert Coelho da Silva</i>	22
On Locally Identifying Coloring of some Circulant Graphs <i>Ryan Fernandes Auder Lopes and Márcia Rodrigues Cappelle Santana</i>	24
Independent Locating-Dominating Sets in some Snark Families <i>Dayllon Vinícius Xavier Lemos, Márcia Cappelle, Humberto José Longo, Erika Moraes Martins Coelho and Leslie Richard Foulds</i> .	26
Locating-dominating partitions and total variants for subcubic graphs <i>Joviro Rocha Neto and Márcia Rodrigues Cappelle Santana</i> . .	28
Partial Independent k -Dominating Sets in a Superclass of Distance-Hereditary Graphs <i>Vitor Ponciano, Christiane Campos, Simone Dantas and Sylvain Gravier</i>	30
Geodesic mutual visibility in unit interval graphs <i>L. G. S. Gonzaga, L. D. Penso and A. A. Pereira</i>	32
Exact Algorithm for Double Domination with Branch Cutting <i>Iago Arruda Faria and Tanilson Dias dos Santos</i>	34

On Locally Identifying Coloring of Cubic Graphs

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Keywords: locally identifying coloring, cubic graphs, Heawood graph, Fano plane, claw-free graphs

A *locally identifying coloring* (lid-coloring) of a graph $G = (V, E)$ is a proper coloring $c : V \rightarrow \mathbb{N}$ such that, for every edge uv with $N[u] \neq N[v]$, $c(N[u]) \neq c(N[v])$. The *lid-chromatic number* $\chi_{\text{lid}}(G)$ is the minimum number of colors in such a coloring, combining proper coloring with the vertex-distinguishing paradigm of identifying codes. Introduced by Esperet et al. [1], who proved $\chi_{\text{lid}}(G) \leq 4$ for connected bipartite graphs and NP-completeness of deciding $\chi_{\text{lid}}(G) = 3$ for subcubic bipartite graphs, and later bounded by $\chi_{\text{lid}}(G) \leq 12$ for cubic graphs by Foucaud et al. [2]. We investigate χ_{lid} on cubic graphs and present three contributions: a characterization for cubic bipartite graphs via hypergraph 2-colorability; a subgraph obstruction forcing $\chi_{\text{lid}} \geq 6$; and an improved bound $\chi_{\text{lid}}(G) \leq 10$ for claw-free cubic graphs.

Every connected cubic bipartite graph G satisfies $\chi_{\text{lid}}(G) \in \{3, 4\}$ [1]. We characterize which case holds via 2-colorings of the 3-uniform 3-regular hypergraphs $H_U = (U, \{N(v) : v \in V\})$ and $H_V = (V, \{N(u) : u \in U\})$: $\chi_{\text{lid}}(G) \leq 3$ if and only if at least one of H_U, H_V is 2-colorable. By a result of Porschen et al. [3], this fails precisely when every component contains the Fano plane $S(2, 3, 7)$, which leads to the following.

Theorem 1. *The Heawood graph is the unique connected cubic bipartite graph satisfying $\chi_{\text{lid}}(G) = 4$. Consequently, deciding $\chi_{\text{lid}}(G) \in \{3, 4\}$ takes linear time.*

If a cubic graph G is not bipartite, then $\chi_{\text{lid}}(G) \geq 4$ [1]. Moreover, if G contains a triangle whose vertices are pairwise non-twin, the local structure around this triangle forces $\chi_{\text{lid}}(G) \geq 5$. We further exhibit a subcubic graph H on 11 vertices with $\chi_{\text{lid}}(H) = 6$ (Figure 1), and show that any graph containing H satisfies $\chi_{\text{lid}} \geq 6$.

Proposition 2. *Every subcubic graph containing H as a subgraph satisfies $\chi_{\text{lid}}(G) \geq 6$.*

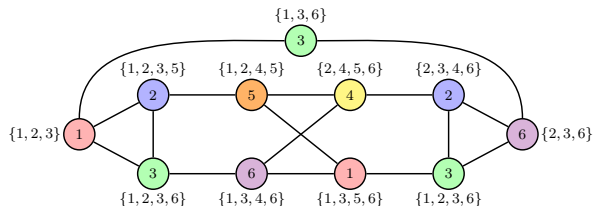


Figure 1: Graph H with $\chi_{lid}(H) = 6$. For each vertex v the set $c(N[v])$ is indicated.

A graph is *claw-free* if it contains no induced $K_{1,3}$. In the cubic case, this means that every vertex has at least one edge among its neighbors. Since Foucaud et al. [2] established $\chi_{lid}(G) \leq 12$ for cubic graphs, their recoloring lemma applied to the claw-free setting yields the following improvement.

Theorem 3. *If G is a claw-free cubic graph, then $\chi_{lid}(G) \leq 10$.*

The results above suggest that 6 is the correct universal upper bound.

Conjecture 4. *Every cubic graph G satisfies $\chi_{lid}(G) \leq 6$.*

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Locally identifying coloring in some split-comparability graphs

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Keywords: Lid-coloring; split graphs; split-comparability; independent set S_t

The locally identifying coloring in a graph is a proper vertex coloring where, for any edge uv , if u and v have distinct closed neighborhoods, then the set of colors used for vertices of the closed neighborhoods of u and v are distinct. The local identification chromatic number of a graph G , represented by $\chi_{lid}(G)$, is the minimum number of colors required in a lid-coloring of G . [Esperet et al. \[2012\]](#) showed that bipartite graphs admit lid-colorings with at most 4 colors, while deciding 3-lid-colorability remains NP-complete, in addition to other important results.

A split graph $G = (K \cup S, E)$ is a graph in which the vertex set can be divided into a clique K and an independent set S . Throughout this work, we consider partitions $K \cup S$ where K has maximum size. A split graph is a chordal graph whose clique number and chromatic number are both equal to $|K|$. [Esperet et al. \[2012\]](#) proposed that every chordal graph G has $\chi_{lid}(G) \leq 2\chi(G)$. They also proved that if G is a split graph, then $\chi_{lid}(G) \leq 2|K| - 1$. In [Oliveira et al. \[2023\]](#) they improve some other results on specific split graphs.

We investigate the behavior of the local identification chromatic number in split-comparability graphs, following the characterization of [Ortiz and Villanueva \[2010\]](#), where the clique K is partitioned into two ordered subsets P and Q , and the independent set S is partitioned into S_t , S_p , and S_q , corresponding respectively to vertices adjacent to both P and Q , only to P , and only to Q . Our goal is to understand to what extent the adjacency structure between S_t and the clique K influences the need for additional colors in a lid-coloring.

We first analyze the case in which the maximum degree of vertices in S_t with respect to the clique satisfies $\Delta(S_t) < |K| - 1$, and the subgraph $G - S_t$ has no twin vertices. In this setting, we show that the presence of S_t does not affect the lid-chromatic number of the graph. We prove that an optimal lid-coloring of the reduced graph $G - S_t$ can be extended to G without

introducing new colors while preserving the locally identifying property. As a consequence, we obtain that the lid-chromatic number remains unchanged, showing that S_t does not contribute essentially to neighborhood distinction.

Next, we consider the case where $\Delta(S_t) = |K| - 1$. Unlike the previous case, we show that the need for an additional color depends on the interaction between S_t and a specific vertex of the clique whose closed neighborhood in $G - S_t$ contains just colors of K . We prove that if there exists a vertex in S_t with $|K| - 1$ neighbors, and it is adjacent to this critical vertex, then the introduction of a new color is necessary, increasing the lid-chromatic number by one. On the other hand, if such adjacency does not occur, the coloring can again be extended without additional cost. This result precisely characterizes when the set S_t affects the coloring considering that the subgraph $G - S_t$ has no twin vertices.

Finally, we study the case in which the reduced graph $G - S_t$ contains exactly one set of twin vertices. In this context, we show that the contribution of S_t is directly related to how its vertices connect to this twin set. In particular, only vertices in S_t that have partial adjacency with the twin set are relevant for local distinction. Based on this, we establish an upper bound for the lid-chromatic number of G , given by the value for $G - S_t$, plus the number of distinct intersection patterns with the twin set, and possibly one additional color. This provides a characterization of the structural impact of S_t on the coloring.

These results show that the effect of S_t on the lid-chromatic number depends on specific adjacency properties, leading to a more precise understanding of locally identifying colorings in split-comparability graphs.

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On Locally Identifying Coloring of some Circulant Graphs

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Keywords: locally identifying coloring, circulant graphs, power of cycles

A *locally identifying coloring* (lid-coloring) of a graph $G = (V, E)$ is a proper coloring $c : V \rightarrow \mathbb{N}$ such that $c(N[u]) \neq c(N[v])$ for every edge uv with $N[u] \neq N[v]$. The *lid-chromatic number* $\chi_{\text{lid}}(G)$ is the minimum number of colors in such a coloring, introduced by Esperet et al. [1] and combining proper coloring with the vertex-distinguishing paradigm of identifying codes. A *circulant graph* $C_n(S)$, for $S \subseteq \{1, \dots, \lfloor n/2 \rfloor\}$, has vertex set $\mathbb{Z}_n$ with i adjacent to j whenever $|i - j| \bmod n \in S$. For convenience, whenever $S = \{s_1, \dots, s_t\}$, we write $C_n(s_1, \dots, s_t)$ instead of $C_n(\{s_1, \dots, s_t\})$. This family includes cycles, complete graphs, and Cayley graphs of cyclic groups, and has been extensively studied under various coloring parameters [2, 3]. Heuberger [2] showed that $C_n(d_1, \dots, d_t)$ is bipartite if and only if n is even and every d_i is odd, yielding $\chi_{\text{lid}} \leq 4$ for such graphs by [1] – though deciding $\chi_{\text{lid}}(G) = 3$ remains NP-complete even for bipartite graphs, while for trees it can be solved in linear time [1]. This complexity gap motivates the study of χ_{lid} in structured graph families such as circulants. We investigate χ_{lid} on circulant graphs and present results for three families.

Power of cycles. The k -th power of the cycle C_n , denoted $C_n^k = C_n(1, \dots, k)$, is $2k$ -regular and twin-free for $n \geq 2k + 2$. We determine $\chi_{\text{lid}}(C_n^k)$ asymptotically for all $k \geq 2$. For the lower bound, C_n^k contains a twin-free clique of order $k + 1$; a careful analysis of the closed neighborhood overlaps in the circulant structure then yields $\chi_{\text{lid}}(C_n^k) \geq 2k + 1$ for all $n \geq 2k + 2$. For the upper bound, we represent colorings as *circular words* and reduce the lid property to local conditions on subwords of bounded length. Blocks of lengths $2k + 2$ and $2k + 3$, each inducing a valid $(2k + 1)$ -lid-coloring and compatible under concatenation, yield a $(2k + 1)$ -lid-coloring for every n in the numerical semi-group $\langle 2k + 2, 2k + 3 \rangle$, which contains all integers $n \geq (2k + 1)(2k + 2)$. We thus obtain that for every $k \geq 2$, there exists n_0 such that $\chi_{\text{lid}}(C_n^k) = 2k + 1$ for all $n \geq n_0$. The case $k = 2$ is fully characterized in a companion paper, where $\chi_{\text{lid}}(C_n^2)$ is determined for all $n \geq 6$.

Cubic circulant graphs. Every cubic circulant graph has the form $C_{2n}(d, n)$ with $1 \leq d < n$ and $n \geq 2$, and we determine $\chi_{\text{lid}}(C_{2n}(d, n))$ completely, with explicit lid-colorings for each case. Every connected cubic circulant of order $2n$ is isomorphic to either $M_n = C_{2n}(1, n)$, connected for all $n > 2$, or $Y_n = C_{2n}(2, n)$, connected only for odd $n > 2$ [4]. For M_n with $n \geq 4$ even, a 4-coloring by parity on each half $\{0, \dots, n-1\}$ and $\{n, \dots, 2n-1\}$, fixed at $i = n-1$ and $i = 2n-2$, gives $\chi_{\text{lid}}(M_n) = 4$. For M_n with $n \geq 5$ odd, splitting the odd-indexed vertices by whether $v \leq n/2$ adds the third color needed, giving $\chi_{\text{lid}}(M_n) = 3$. For Y_n with $n \geq 5$ odd, a 4-coloring that cycles through the colors with period 4 along the two n -cycles induced by the generator 2, fixed at $v = 0$ and $v = n-1$, gives $\chi_{\text{lid}}(Y_n) = 4$.

Circulant graphs $C_n(1, k)$. The family $C_n(1, k)$ for $2 \leq k \leq \lfloor n/2 \rfloor$ consists of 4-regular circulant graphs. We determine $\chi_{\text{lid}}(C_n(1, k))$ for several cases, showing that the values depend on the parities of k and n , and provide explicit lid-colorings.

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Independent Locating-Dominating Sets in some Snark Families

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Keywords: independent-locating dominating, snarks, blanuša, flower snark.

Consider a graph G representing a network in which detection devices are placed at selected vertices. The goal is to determine a minimum set of locations that allows every vertex outside the set to be uniquely identified by its neighbourhood within the selected vertices. This problem is modeled by the notion of *locating-dominating* sets, introduced by Rall and Slater [1984]. An independent locating-dominating (*ILD*) set is a locating-dominating set that is also independent, and its minimum cardinality is denoted by $ILD(G)$.

The independent locating-dominating problem was introduced by Slater and Sewell [2018], who proved that deciding whether a graph admits an ILD set is an *NP*-complete problem. In a recent work, Lemos et al. [2026] proved that the problem remains *NP*-complete even for planar triangle-free subcubic graphs. Motivated by this result, we study the problem in highly structured subclasses of cubic graphs, focusing on two classical families of snarks: flower snarks and generalized Blanuša snarks.

Snarks are bridgeless cubic graphs that are not 3-edge-colourable and represent important extremal objects in cubic graph theory. While exact results are known for some cubic families, such as Möbius ladders and circular ladders Lemos et al. [2025], the exact values of $ILD(G)$ for flower snarks and generalized Blanuša snarks were not known.

For flower snarks J_n , we analyze how an ILD set can be distributed among the consecutive copies of $K_{1,3}$ that form the graph. We prove that if a block contains exactly two selected vertices, then among the next four consecutive blocks at least one must contain at least two selected vertices. Consequently, every five consecutive blocks impose a local density restriction on ILD sets, which yields the lower bound $ILD(J_n) \geq \lfloor \frac{7n+2}{5} \rfloor$. To attain this bound, we construct ILD sets by repeating a pattern over groups of five consecutive blocks. Two alternating patterns are required: after each group of five

blocks, the choice of selected vertices switches to the other pattern in order to preserve both independence and the locating condition between adjacent blocks. The remaining blocks, depending on the value of n modulo five, are then treated separately. Therefore, $ILD(J_n) = \lfloor \frac{7n+2}{5} \rfloor$.

For generalized Blanuša snarks B_n^1 , we partition the graph into an initial block and n consecutive copies of a fixed block. We prove that every non-initial block requires at least two vertices in any ILD set. Moreover, if a block contains exactly two selected vertices, then the next three blocks must contain exactly three selected vertices. This restriction leads to a repeating pattern of period four blocks, where one block contains two selected vertices and the other three contain three selected vertices each. Thus, this construction saves one selected vertex for every four consecutive blocks when compared with the configuration containing three selected vertices in every block, resulting in a total of $3n + 4 - \lfloor \frac{n+2}{4} \rfloor$ selected vertices. When n is not a multiple of four, the remaining blocks require separate adjustments to maintain the locating and independence conditions. Therefore, $ILD(B_n^1) = 3n + 4 - \lfloor \frac{n+2}{4} \rfloor$.

These results provide exact formulas for the independent locating-dominating number of flower snarks and generalized Blanuša snarks. They show that, even within highly constrained cubic graphs, local structural restrictions determine the optimal ILD configurations in these graph families.

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Locating-dominating partitions and total variants for subcubic graphs

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Keywords: Locating-Dominating, Subcubic graphs, Locating-total-dominating

Let $G = (V, E)$ be a simple, undirected graph. For a vertex $v \in V$, denote by $N(v)$ its open neighborhood and by $N[v] = N(v) \cup \{v\}$ its closed neighborhood. A set $S \subseteq V$ is a *locating-dominating set* (LD-set) of G if every vertex $v \in V \setminus S$ satisfies $N(v) \cap S \neq \emptyset$ (domination) and, moreover, $N(u) \cap S \neq N(v) \cap S$ for every pair of distinct vertices $u, v \in V \setminus S$ (location). The *locating-domination number* $\gamma^{LD}(G)$ is the minimum cardinality of an LD-set of G . LD-sets were introduced by Slater [1] in the context of fault detection in networks: placing sensors at the vertices of S allows one to identify the location of any fault occurring at a vertex outside S from the subset of sensors that detect it. A strengthening of this concept is the *locating-total-dominating set* (LTD-set): a set $S \subseteq V$ is an LTD-set if it is an LD-set and, additionally, every vertex in S has at least one neighbor in S , i.e., $G[S]$ contains no isolated vertex. The *locating-total-domination number* $\gamma^{LTD}(G)$ is the minimum cardinality of an LTD-set of G . A central open problem in this area is the following conjecture.

Conjecture 1. [3] *Every twin-free graph G with no isolated vertices satisfies $\gamma^{LD}(G) \leq |V(G)|/2$.*

Conjecture 1 has been confirmed for several graph classes, including subcubic graphs [2]. A powerful strategy for establishing this bound is the notion of an *LD-partition*: a partition (S_1, S_2) of $V(G)$ such that both S_1 and S_2 are LD-sets of G . Whenever such a partition exists, $\min(|S_1|, |S_2|) \leq |V(G)|/2$, immediately yielding the conjectured bound. Beyond its role in the conjecture, an LD-partition provides a structurally richer guarantee: the graph can be simultaneously monitored by two complementary sensor sets, offering built-in redundancy. LD-partitions are known to exist for distance-hereditary graphs, maximal outerplanar graphs, and split graphs [4].

In this work, we investigate LD-partitions and introduce total locating-dominating partitions (LTD-partitions). We study both notions in sev-

eral subclasses of subcubic graphs. Analogously to LD-partitions, an LTD-partition is a partition of the vertex set into two total locating-dominating sets. We verify that circular ladder graphs CL_n , Möbius ladders M_n , subcubic corona graphs, cubic circulant graphs, and certain claw-free subcubic graphs admit LD-partitions, with only a few exceptions. These results provide further evidence toward a general partition theorem for subcubic graphs. For LTD-partitions, the situation differs across classes: some of the graphs above admit LTD-partitions, while others do not. This contrast motivates a finer classification of subcubic graphs according to the partitions they admit. Additionally, we investigate a conjecture concerning the locating-domination number of cubic graphs. All known cubic graphs satisfying $\gamma^{LD}(G) = n/2$ have fewer than 10 vertices, suggesting that this extremal behavior may be restricted to small graphs, which motivates the following conjecture.

Conjecture 2. *Every cubic graph G on $n \geq 10$ vertices satisfies $\gamma^{LD}(G) < n/2$.*

We present results on γ^{LD} for several classes of cubic graphs, contributing to the understanding of Conjecture 2 and the structural properties that influence this parameter.

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Partial Independent k -Dominating Sets in a Superclass of Distance-Hereditary Graphs¹

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Keywords: Domination; independent domination; distance hereditary graphs.

Let $G = (V(G), E(G))$ be a finite, undirected, and simple graph, with $V(G)$ and $E(G)$ denoting the vertex and edge sets of G , respectively. The *distance* between two vertices u and v in G is the number of edges in a shortest path joining u and v . A graph G is called a *distance-hereditary graph* if, for every connected induced subgraph H of G , the distance between any two distinct vertices $u, v \in V(H)$ in H equals their distance in G .

A *twin set* is a subset $S \subseteq V(G)$ such that for all $u, v \in S$, $N(u) \setminus S = N(v) \setminus S$. A set I is *independent* if no two vertices in I are adjacent. A set $I \subseteq V(G)$ is an *independent dominating set* of G if it is independent and every vertex in $V(G) \setminus I$ has at least one neighbor in I . Generalizing this concept, a set I is an *independent k -dominating set* if it is independent and every vertex in $V(G) \setminus I$ has at least k neighbors in I .

In 1997, Chang *et al.* [1] introduced a decomposition theorem for distance-hereditary graphs based on three structural operations as follows. Given two distance-hereditary graphs G_1 and G_2 with twin sets $TS(G_1)$ and $TS(G_2)$, the *false twin* operation consists of taking their disjoint union; the *true twin* operation connects every vertex of $TS(G_1)$ to every vertex of $TS(G_2)$; in both cases the twin set of the resulting graph is $TS(G_1) \cup TS(G_2)$. The *attachment* operation connects every vertex of $TS(G_2)$ to all vertices of $TS(G_1)$,

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preserving $TS(G_1)$ as the twin set of the resulting graph. Each of these binary operations produces a new distance-hereditary graph, yielding a constructive characterization of the class.

We generalize the previous construction by considering a superclass of distance-hereditary graphs constructed from successive applications of these operations, but allowing arbitrary graphs G_1 and G_2 , parameterized by a generic target set of terminal vertices TS , relaxing the original twin set restriction. Over this superclass, we extend the counting framework of Lin [2] – who proposed a unified method for designing linear-time algorithms that count independent dominating sets in distance-hereditary graphs – by introducing a new domination notion, namely a partial independent k -dominating set I of G .

Informally, a partial independent k -dominating set I is a collection of pairwise non-adjacent vertices such that every vertex outside I has sufficiently many neighbours in I , the required number depending on whether the vertex belongs to the distinguished subset $TS \subseteq V(G)$ or not. The term *partial* reflects that this domination is not necessarily uniform over TS since the additional constraints regulate how vertices in TS interact with I .

In this work, we count the number partial independent k -dominating sets in this superclass of distance-hereditary graphs, providing a structural framework that unifies and extends previous algorithmic approaches.

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Geodesic mutual visibility in unit interval graphs

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Keywords: geodesic mutual visibility; unit interval graphs; interval graphs

Let G be a simple connected graph. For a set $S \subseteq V(G)$, we denote by $\bar{S}$ the set $V(G) \setminus S$. Given a set $S \subseteq V(G)$, two vertices $u, v \in S$ are S -visible if there exists a shortest u, v -path in G such that no internal vertex lies in S . If the vertices of S are pairwise S -visible, then S is a *geodesic mutual visibility set* (GMV-set) of G . The *geodesic mutual visibility number* of G , denoted $\mu(G)$, is the maximum cardinality of a GMV-set of G .

The concept of geodesic mutual visibility was introduced by Di Stefano [1] as a graph-theoretic model for coordination and communication constraints in distributed systems. In this setting, shortest paths represent optimal communication routes, while intermediate vertices may act as obstacles, causing interference or introducing security vulnerabilities. Determining $\mu(G)$ is NP-hard for general graphs [1], which motivates the study of this parameter in structured graph classes. In particular, exact results are known for paths, cycles, trees, block graphs, cographs, distance-hereditary graphs, and Cartesian products of graphs.

In this work, we focus on unit interval graphs. A graph is a *unit interval graph* if its vertices can be represented by unit-length intervals on the real line, with two vertices adjacent if and only if the corresponding intervals intersect. The vertices of a unit interval graph admit a canonical ordering $(v_1, \dots, v_n)$ such that, whenever $i < j < k$ and $v_i v_k \in E(G)$, both $v_i v_j$ and $v_j v_k$ belong to $E(G)$. This ordering induces a linear ordering of the maximal cliques of G with strong consecutive-intersection properties.

Our main result characterizes $\mu(G)$ in terms of the intersections of consecutive maximal cliques. Let $\mathcal{C} = (C_1, \dots, C_m)$ denote the sequence of maximal cliques induced by the canonical ordering. A set $R \subseteq V(G)$ *hits the clique intersections* of G if $R \cap (C_i \cap C_{i+1}) \neq \emptyset$ for every $i \in \{1, \dots, m-1\}$ such that $C_i \cap C_{i+1} \neq \emptyset$. Let $\mathcal{R}(G)$ denote the family of all such sets and define

$r(G) = \min\{|R| : R \in \mathcal{R}(G)\}$. We prove that, for every unit interval graph G of order n , it follows that $\mu(G) = n - r(G)$.

The proof combines the canonical ordering of unit interval graphs with the linear structure of their maximal cliques. For the upper bound, we show that if a GMV-set S satisfies $|S| > n - r(G)$, then $\bar{S}$ fails to intersect some non-empty consecutive clique intersection $C_i \cap C_{i+1}$. Hence, $C_i \cap C_{i+1} \subseteq S$. Using the canonical ordering and the maximality of C_i and C_{i+1} , we then find two vertices of S for which every shortest path contains an internal vertex of S , contradicting the assumption that S is a GMV-set.

For the lower bound, we construct a set $R \in \mathcal{R}(G)$ with $|R| = r(G)$ such that $\bar{R}$ is a GMV-set. The construction is inductive on the number of maximal cliques in the clique sequence. In the inductive step, we consider the subgraph obtained by deleting the vertices that belong only to the last maximal clique, and then extend a minimum set intersecting the clique intersections of this subgraph. The choice of R guarantees that any two vertices in $\bar{R}$ are joined by a shortest path whose internal vertices all belong to R , and hence are $\bar{R}$ -visible.

Our result shows that the geodesic mutual visibility number of a unit interval graph is completely determined by the structure of its maximal cliques. A natural direction for further work is to investigate whether similar characterizations hold for broader subclasses of interval graphs, where the maximal cliques still admit a linear ordering but the unit-length restriction is no longer available.

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Exact Algorithm for Double Domination with Branch Cutting

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Keywords: Graphs; Domination Problem; Double Domination Problem; Exact Algorithm

In this paper, we propose an exact algorithm for the problem of the minimum double dominating set. We define a graph as a ordered pair $G = (V, E)$ where V is the set of vertices and E is the set of edges where an edge is an unordered pair (v, w) with $v, w \in V$, also we define the closed neighborhood $N[v]$, of a vertex v , as the set of all the neighbors of v plus v itself, from this we define that a set D is a double dominating set of a graph G if $D \subseteq V$ and $|N[v] \cap D| \geq 2$ for all $v \in V$, as introduced by Harary and Haynes in [3] and equivalent to 2-tuple domination. Thus, the minimum double dominating set problem is the problem of finding the smallest set that is a valid double domination for a graph G .

We found several articles exploring algorithms or heuristics for double domination, either directly or as k -tuple domination or equivalent formulations, some of which are: [2], [6], [4], [5] and [1]. All, with the exception of [1], present heuristics or approximation algorithms and both algorithms in [1] require that $|N[s] \cap D| \geq |N[u] \cap D|$ for $s, u \in V$, thus only applies to a subset of double domination cases, leaving space for exact algorithms that applies for all cases.

With the problem of double domination on mind, we create an algorithm that uses of recursion to iterate through all vertices, creating two branches for every decision to remove or keep a vertex and it then compares the subsets created from both branches, returning the smallest of the subsets. However, not every subset of V is a valid double dominating set or close to the minimum, thus we can make the algorithm take considerably less time by cutting the branches where the leaf either wouldn't be a valid double dominating set or would be a double dominating set above the upper bound for the graph. Finally, we prove that the algorithm outputs optimal answers and the algorithm itself is omitted due to a lack of space.

Moreover, we can calculate the number of steps of the algorithm at medium and worst cases by summing the number of steps at every level of the tree and this can be modeled as the sum of a geometric progression where $a_1 = 1$, the root, and $1 < r < 2$, r cannot be 2 since that would mean every possible subset of V is a valid double dominating set including the empty set, which is never a valid double dominating set. Furthermore, the number of possible subsets of V of size l is the combination $C_l^{|V|}$, then the number of steps at the bottom of the tree must be at most $\sum_{\Gamma_{\times 2}(G)} C_l^{|V|}$, thus r must be:

$$a_{|V|} = r^{|V|-1} \Rightarrow \sum_{\Gamma_{\times 2}(G)} C_l^{|V|} = r^{|V|-1} \Rightarrow r = \sqrt[|V|-1]{\sum_{\Gamma_{\times 2}(G)} C_l^{|V|}},$$

Consequently, for medium and worst cases, the algorithm is $\mathcal{O}(r^n)$, where $1 < r < 2$ and n is the order of the graph, and for the best case the tree is a skewed tree due to no attempt at removing a vertex from double dominating set resulting in a valid double dominating set, thus it is asymptotically $\mathcal{O}(n)$.

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Session 2: Algorithms, complexity, and optimization

Trie Mergesort: A unified approach to natural mergesorts
*Théo Araújo Magalhães, Victor Almeida Campos, Pedro Henrique
Feitosa Fernandes and Rafael de Paiva Lima* 37

Dijkstra Graphs: Short-circuit evaluation
*Matheus Telles Batista, Matheus de Souza Feitosa and André L. P.
Guedes* 39

Complexity of the Minimum Labeled Feedback Edge Set
Problem
Rafael de Paiva Lima, Claudia Linhares Sales and Manoel Campêlo 41

Contagem de Triângulos em Fluxo de Grafos Utilizando
Complexidade de Amostra
Vitor Alencar, Alane da Silva and Murilo da Silva 43

Connectedness in Flip and Twist Reconfiguration Graphs
of Colored Triangulations
Daniel Debrohim, Diana Sasaki and Patrícia Nunes 45

A Lower Bound on the p -Inversion Diameter of Tourna-
ments
Lucas Silva and Caroline Silva 47

Optimization Problems on Maya-Tupi Graphs
Júlio Araújo, César Hernández-Cruz and Cláudia Linhares Sales 49

Square roots of split graphs via matchings
Aleffer Rocha, Renato Carmo and André L. P. Guedes 51

Trie Mergesort: A unified approach to natural mergesorts

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Keywords: Algorithms; Natural Mergesort; Trie; Entropy; Powersort

A **run** in a sequence of values is a sorted or inversely sorted contiguous subsequence. A **mergesort algorithm** sorts an input sequence by iteratively choosing and merging runs until a unique sorted run remains. A **natural mergesort** is a mergesort algorithm which uses runs naturally present on the input. The study of natural mergesorts is currently an active area of research as it allows for instance optimality. **Instance optimality** consists of exploiting structures of the input to have an optimal algorithm for every possible input, and not just for the worst case input.

We consider the input sequence to have size n and it decomposes into r runs of lengths $\ell_1, \dots, \ell_r$. Let $H(p_1, \dots, p_r) = -\sum_{i=1}^r p_i \log_2 p_i$ be the binary Shannon entropy and $\mathcal{H} = H(\ell_1/n, \dots, \ell_r/n)$ be the **run entropy** of the input. We note that $\mathcal{H} \leq \log_2 r \leq \log_2 n$. To prove instance optimality, the run entropy provides a lower bound of $n\mathcal{H} - \Theta(n)$ comparisons [BN13]. To compute precise bounds, we follow the standard cost of $a + b$ used in the literature for merging runs of lengths a and b , which provides a better lower bound of $n\mathcal{H}$ for the total merge cost [Jug20; MW18].

Since its introduction in 2002, *timsort* [Pet02] is arguably the most influential natural mergesort algorithm. It was adopted as the standard core library sorting algorithm for widespread programming languages such as Python and Java due to its practical efficiency. The *ad hoc* conception of timsort presented a challenge for a theoretical explanation to its performance. Proving it has worst case cost of $\mathcal{O}(n \log n)$ was only done in 2015, more than a decade after it was introduced [ANP15]. Instance optimality was later shown by [AJNP18] as they prove timsort has total merge cost of $1.5n\mathcal{H} + \mathcal{O}(n)$. They also show this 1.5 constant cannot be improved.

Nowadays, a variety of timsort-like natural mergesorts have been studied [Jug20]. These algorithms consist of using a stack of runs and a series of rules to decide merges. Merges between runs in the stack happen if a newly

found run violates one of the rules. Out of these *stack based* algorithms, *length-adaptive shivers sort* [Jug20] has a worst-case merge cost of $n\mathcal{H} + 2n$, a linear term away from the $n\mathcal{H}$ lower bound. However, the first algorithm to obtain a worst-case merge cost of $n\mathcal{H} + 2n$ was *powersort* [MW18]. Powersort differs from stack based natural mergesort algorithms as it uses Melhorn’s construction of nearly optimal binary search trees [Meh75]. We note powersort was a breakthrough result and has become the default library sorting implementation of CPython, PyPy and numpy projects.

We provide a novel view of the merge tree of natural mergesort algorithms as a compressed trie of indices assigned to input runs. Furthermore, we provide necessary conditions on the choice of these indices to achieve total merge cost of $n\mathcal{H} + \mathcal{O}(n)$. Using our framework, we model powersort while improving the cost of computing its merge tree from $\Theta(r \log r)$ to $\Theta(r)$ and provide a self contained proof of its $n\mathcal{H} + 2n$ total merge cost. Our construction and conditions for the merge cost are also more flexible than Melhorn’s recursive construction used in powersort.

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Dijkstra Graphs: Short-circuit evaluation [‡]

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Keywords: Structured programming; Short-circuit evaluation; Dijkstra Graphs

One of, if not the major contributor to Structured Programming (SP) was Edsger W. Dijkstra [3, 4]. It can be described as a technique for constructing structured programs, programs where the flow structures are constrained to enforce that a proof of correctness can be easily provided. Fortunately, with Bohm and Jacopini result, it is possible to create any computable function with SP [2]. More recently Bento et al. [1] came with the concept of Dijkstra Graphs (DG), a graph class that represents the Control Flow Graphs (CFG) derived from structured programs. In our work, we found an empirical problem related to the generation of CFGs from structured programs. Compiling with GCC 12.5, the resultant CFG of programs with short-circuit operators was not a DG. Therefore, we introduce the Dijkstra-McCarthy graphs (DMG)*: a more general class that contains the DGs with short-circuit graphs induced on them. Short-circuit evaluation is a change in the semantic of boolean operators that languages admit to avoid unnecessary steps evaluating a boolean expression. For example, consider the expression $x_1 \wedge x_2$. If $x_1 = \text{false}$, we do not need to read x_2 to deduce that the expression is **false**. Equivalently, in case of a short-circuit $x_1 \vee x_2$, and $x_1 = \text{true}$, the expression is always **true** regardless of the x_2 value. In both cases, the short-circuit operator signals to the compiler it should not continue the boolean evaluation, saving execution time. It is possible to achieve a short-circuit behavior from a DG, but it would be necessary to duplicate code.

In order to present DMGs, we introduce two artifacts: the short-circuit graph and the conditional vertex (all such graphs are directed and acyclic). A vertex v is conditional if its outdegree is 2; a short-circuit graph H is one containing two adjacent conditional vertices, s and v sharing exactly one common out-neighbor (Figure 1). A short-circuit graph can be derived from

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*After John McCarthy, who introduced the short-circuit evaluation [6]. We highlight the fact that Dijkstra did not agree with its use [5].

a DG by consecutive expansions on conditional vertices. In each step of the expansion, choose a conditional vertex v and swap the same into a short-circuit graph H , ensuring that the in-neighborhood of s remain the same as v before the expansion (Figure 1). Similarly, a DMG can be reduced into a DG. The process can be done performing a DFS, saving the cycle edges and generating a topological order. After that, traverse the graph in a reverse topological order and contract the short-circuits into conditional vertices. This mapping (from DMG to DG) takes $\mathcal{O}(V + E)$ time, because the DFS is $\mathcal{O}(V + E)$ and the traversal in topological order takes $\mathcal{O}(V)$.



Figure 1. Building blocks of Dijkstra-McCarthy graphs

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Complexity of the Minimum Labeled Feedback Edge Set Problem¹

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Keywords: Edge labeled graphs, Connectivity, Complexity, Hardness, Feedback Edge Set, Parameterized Complexity

An *edge-colored graph* (also known as a labeled graph) is a tuple $G = (V, E, L, \ell)$ where $G = (V, E)$ is a graph, L is a set of colors (labels), and $\ell : E \rightarrow L$ is an edge-coloring function, not necessarily inducing a proper edge-coloring of G . Edge-colored graph problems have diverse applications in computational biology and network design [1], and their complexity and algorithms for solving them have been studied extensively. In general, these problems have as a goal finding a solution with a minimum (or a maximum) number of colors.

The results obtained so far show that many polynomial problems in unlabeled graphs, such as minimum spanning tree [2, 3], minimum $(s-t)$ -path [4] and minimum $(s-t)$ -cut [5], become NP-hard on edge-colored graphs (when minimizing the number of labels instead of the classic objective of the problem). In a previous work, we studied the complexity of finding the cycle with as few labels on arc-colored digraphs and proved that this problem is also NP-hard [6], following the same pattern mentioned above, since it is polynomial on unlabeled graphs.

In the minimum labeled feedback edge set problem, given an edge-colored graph G and a positive integer k , we look for a subset of labels $L' \subseteq L$ with size at most k , such that every cycle of G has at least one edge labeled

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with a color in L' . Its counterpart in unlabeled graphs, the minimum feedback edge set, can be solved in polynomial time. In this work, we prove that the set cover problem is polynomially reducible to the minimum labeled feedback edge set problem. Given a set $A = \{u_1, \dots, u_n\}$ and a family $\mathcal{F} = \{S_1, \dots, S_t\}$ of subsets of A , the set cover problem consists of asking if there are at most k sets of $\mathcal{F}$ whose union covers A . In our reduction, the produced graph G contains a cycle corresponding to each element u_i in A , whose edges are labeled with colors L_j for every j such that S_j covers u_i . It's straightforward checking that A can be covered with at most k subsets if and only if G has a feedback edge set with at most k colors. The graph obtained by the reduction is a union of disjoint cycles, proving the hardness of the problem even for disconnected simple graphs. As a consequence of our reduction, we get that the parameterized decision problem is $W[2]$ -hard when parameterized by k .

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Contagem de Triângulos em Fluxo de Grafos Utilizando Complexidade de Amostra

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Palavras-Chave: Fluxo de Dados; Contagem de Triângulos; Teoria da Complexidade de Amostra

Algoritmos para fluxos de dados (*data streams*) tornaram-se ferramentas fundamentais para o processamento de grandes volumes de dados sob restrições limitadas de memória para que seja viável a análise em tempo real de fluxos contínuos [McGregor et al., 2016]. Os primeiros trabalhos desta área contém estudo de problemas elementares em fluxo de dados, como busca por elementos majoritários, identificação de itens frequentes e estimativa de elementos distintos [Misra e Gries, 1982]. Embora a resolução de muitos problemas em fluxos de dados se reduzam a tais problemas elementares, isso nem sempre ocorre no caso em que o fluxo de dados se refere a uma estrutura mais complexa, como um grafo, mesmo para tarefas simples.

Na literatura, dois modelos principais para representação de um grafo dado como um fluxo de dados têm sido estudados: o modelo de ordem arbitrária, em que um grafo é representado por uma sequência de arestas em ordem arbitrária, e o modelo baseado em listas de adjacência, no qual as arestas são agrupadas por vizinhanças de vértices [McGregor et al., 2016].

Em relação ao problema de contagem de triângulos, Kutzkov e Pagh (2013) propõem um algoritmo de *amostragem monocromática* para os dois modelos acima. O algoritmo mencionado resolve originalmente o problema do coeficiente de agrupamento local, de maneira que a contagem de triângulos pode ser naturalmente resolvida a partir deste problema. Os autores utilizam tamanho de amostra log-linear, obtendo uma (ϵ, δ) -aproximação, onde $0 < \epsilon, \delta < 1$ são os parâmetros de qualidade e confiança da aproximação. Dado um grafo com n vértices e m arestas, o algoritmo executa em tempo $\mathcal{O}(\frac{m}{\alpha \epsilon^2} \log \frac{1}{\epsilon} \log \frac{n}{\delta})$, onde α é a razão entre o coeficiente de transitividade e a quantidade de 2-caminhos deste grafo.

O presente trabalho propõe uma adaptação do algoritmo de Kutzkov

e Pagh (2013), baseando-se em análise de complexidade de amostra para o problema da contagem de triângulos em fluxos de grafos. Para tal adaptação, utiliza-se como referência o trabalho de Lima et al. (2022), que apresenta, com probabilidade $1 - \delta$, uma aproximação com erro no máximo ϵ para o problema do coeficiente de agrupamento local. Dado um grafo G , onde Δ é o seu grau máximo, o algoritmo proposto por estes autores executa em $\mathcal{O}(\Delta \lg \Delta + m)$.

Assim como no trabalho de Lima et al. (2022), propõe-se utilizar conceitos chave de Teoria da Complexidade de Amostra, como dimensão Vapnik-Chervonenkis (VC dimension) e o teorema da ϵ -amostra [Har-Peled e Sharir, 2011]. Para um grafo de entrada G e para o mesmo escopo do trabalho de Kutzkov e Pagh [2013], a linha central de investigação neste trabalho é o uso de tais conceitos para obter um limitante superior para o tamanho de amostra na ordem de $\lg(\lfloor \Delta - 1 \rfloor) + 1$, onde Δ é o grau máximo de G . Este limitante é mais justo do que limitantes obtidos usando técnicas clássicas da literatura, como limitante de Hoeffding e da união, pois o tamanho da amostra não depende diretamente do tamanho de G , mas sim, de uma propriedade estrutural do grafo, no caso Δ .

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Connectedness in Flip and Twist Reconfiguration Graphs of Colored Triangulations

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Keywords: colored triangulations; flip graphs; combinatorial reconfiguration; connectedness; associahedron

Triangulations of a convex polygon are classical objects in combinatorics. A triangulation is a maximal set of pairwise noncrossing diagonals, and the basic local move between two triangulations is the *flip*: in a convex quadrilateral, one diagonal is replaced by the other possible one. The corresponding flip graph has as vertices all triangulations of a fixed polygon, with edges joining triangulations that differ by exactly one flip. In the unconstrained setting, this graph is connected and forms the 1-skeleton of the associahedron; see, for example [2].

In this work, we study cyclic color constraints on polygon triangulations from the viewpoint of combinatorial reconfiguration. We consider convex polygons whose vertices are cyclically colored and triangulations satisfying the condition that no triangle has a monochromatic edge. The associated reconfiguration graph has as vertices all valid triangulations and as edges the admissible local operations preserving validity.

The case of cyclic 3-colorings is particularly interesting. Here, every valid triangle must be trichromatic, and the classical flip is no longer admissible in general. Instead, the natural local operation is a *twist*, a move occurring in a suitable hexagon, as introduced by Acharya, Mütze and Verciani [1] and formalized in our work. They proved that the corresponding twist graph $\mathcal{H}_N$ is connected when $N \equiv 0 \pmod{3}$ and left open the case $N \equiv 2 \pmod{3}$ [1]. We settle this question by proving that $\mathcal{H}_N$ is connected for every $N = 3k + 2$ with $k \geq 4$, that is, for all $N \geq 14$. Thus, in the cyclic 3-color setting, the only disconnected cases are the small exceptional instances $N = 8$ and $N = 11$.

Our proof is based on a structural decomposition of $\mathcal{H}_N$ into classes determined by a distinguished base triangle. We first show that each class induces a connected subgraph, using the known connectedness in the case $N \equiv 0 \pmod{3}$ from [1], and then connect different classes by explicit local twists

in the base case $N = 14$ and an inductive construction for larger values of N .

We also place this phenomenon in a broader context. For cyclic colorings with $j \geq 4$ colors, the larger number of colors restores enough flexibility so that the admissible local operation becomes again the classical flip. In this setting, for polygons with

$$N = jk + r, \quad 0 \leq r \leq j - 1, \quad r \neq 1,$$

we show that the corresponding graph $\mathcal{G}_N^{(j)}$ is connected. Therefore, disconnectedness under cyclic color constraints is highly exceptional: although the coloring restrictions substantially modify the local reconfiguration rules, they do not usually destroy global connectedness.

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A Lower Bound on the p -Inversion Diameter of Tournaments

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Keywords: inversion; subgraph complementation; edge clique cover;

For a graph G and $S \subseteq V(G)$, the graph obtained after the *complementation* of S is the graph G' such that $V(G') = V(G)$ and $E(G')$ is the symmetric difference of $E(G)$ with the set of edges of the complete graph on vertex set S , that is, $E(G') = E(G) \Delta \binom{S}{2}$. The complementation of S in G is a p -complementation if $|S| \leq p$. This operation was introduced by Kaminski [4] in their study of clique-width; they showed that subgraph complementation preserves the boundedness of clique-width. Let G be a graph. If $\mathcal{X}$ is a family of subsets of $V(G)$, then we denote by $\text{Comp}(G, \mathcal{X})$ the graph obtained after the complementation of each $X \in \mathcal{X}$ applied sequentially. Given two labeled graphs G_1, G_2 on vertex set $[n]$ and an integer $2 \leq p \leq n$ it is straightforward to check that there exists a family $\mathcal{X}$ of subsets of $[n]$ such that $|X| \leq p$ for every $X \in \mathcal{X}$ and $\text{Comp}(G_1, \mathcal{X}) = G_2$. This notion naturally leads to the following optimization version of the problem: what is the minimum number of p -complementations needed to transform G_1 into G_2 ? This problem was investigated in the unrestricted case of $p = n$ by Buchanan et al. [1], who found a strong relationship between subgraph complementation and the minimal rank of a class of matrices. Equivalently, the problem amounts to finding a minimum-length sequence of cliques of size at most p in K_n whose edge sets sum modulo 2 to $E(G_1) \Delta E(G_2)$. This optimization problem is also intimately related to the p -inversion diameter problem on tournaments, defined next. Given an oriented graph $\vec{G}$, the *inversion* of a subset X of vertices consists of reversing the orientation of all arcs with both endpoints in X . An inversion of X in $\vec{G}$ is a p -inversion if $|X| \leq p$. If $\vec{G}$ is an oriented graph and $\mathcal{X}$ is a family of subsets of $V(\vec{G})$, then we denote by $\text{Inv}(\vec{G}, \mathcal{X})$ the graph obtained after applying the inversion operation for each $X \in \mathcal{X}$ sequentially. The $(\leq p)$ -inversion graph of a labeled graph G is the graph whose vertices are the labeled orientations of G in which two labeled orientations $\vec{G}_1$ and $\vec{G}_2$ of G are adjacent if and only if there is a set $X \subseteq V(\vec{G}_1)$ with $|X| \leq p$ whose inversion transforms $\vec{G}_1$ into $\vec{G}_2$. The

($\leq p$)-inversion diameter of a graph, denoted by $\text{id}^{\leq p}(G)$, is the diameter of its ($\leq p$)-inversion graph. The ($\leq p$)-inversion diameter of oriented graphs is investigated in [2, 3].

Let G_1 and G_2 be two graphs with vertex-set $[n]$. Let T_1 and T_2 be two orientations of a tournament with vertex set $[n]$ such that T_1 and T_2 disagree exactly on the arcs of endpoints i and j for which $ij \in E(G_1) \triangle E(G_2)$. The equivalence between p -inversions on tournaments and p -complementations on graphs comes from the observation that there exists a family $\mathcal{X}$ of subsets of $[n]$ such that $|X| \leq p$ for every $X \in \mathcal{X}$ and $\text{Comp}(G_1, \mathcal{X}) = G_2$ if and only if $\text{Inv}(T_1, \mathcal{X}) = T_2$. Note that if $LB \leq \text{id}^{\leq p}(T) \leq UB$ for a tournament T then there exist two graphs G_1 and G_2 on the same vertex set such that at least LB p -complementations are necessary to transform G_1 into G_2 , and every pair of graphs on this same vertex set can be transformed into one another using at most UB p -complementations. Here, we are interested in establishing lower bounds on $\text{id}^{\leq p}(T)$ where T is a tournament on n vertices. Buchanan et al. [1] and Havet et al. [2] results imply that $\text{id}^{\leq n}(T) = n - 1$. When $p < n$, it follows that $\text{id}^{\leq p}(T) \geq \binom{n}{2} / \binom{p}{2}$ which is $\Theta(n^2/p^2)$, since any p -inversion reverses the orientation of at most $\binom{p}{2}$ arcs. To the best of our knowledge, no nontrivial lower bounds for $\text{id}^{\leq p}(T)$ were previously known. Our contribution is to show that $\text{id}^{\leq p}(T) \geq \frac{\binom{n}{2}}{4p \log_2 \binom{n}{2}}$ for any $n > 2$, which is $\omega(n^2/p^2)$ as long as $p = \omega(\log n)$.

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Optimization Problems on Maya-Tupi Graphs ¹

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Keywords: Maya-Tupi graphs; vertex coloring problem; fill-in problem

The class of Maya-Tupi graphs (MT-graphs, for short) was introduced by the authors of this abstract in [1] and contains the graphs that admit a partition (A, B) of their vertex sets, such that A induces a complete multipartite graph where each part has a size of at most two, and B induces a graph where every connected component is K_1 or K_2 . As a consequence of the definition of the class, an MT-graph $G = (V, E)$ also admits a partition of $V(G)$ into sets $(S, M, \overline{M}, K)$ where S is a stable set, M induces a matching, $\overline{M}$ induces a complete graph minus a perfect matching, and K is a clique.

The family of MT-graphs is self complementary, generalizes split graphs, falls into the sparse-dense partitioning scheme [3], and is known to have finitely many forbidden induced subgraphs. However, there are over 2000 of them (some having up to 10 vertices), making it visibly inefficient to recognize a graph in this class by checking the existence of one of its forbidden induced subgraphs. In fact, due to [3], an $\mathcal{O}(n^8)$ -time algorithm to recognize Maya-Tupi graphs is known. In [1], more efficient recognition algorithms are presented for some particular families of graphs.

Let $G = (V, E)$ be a graph. Recall that the independence number of G is the size of the largest independent set of G , and its complementary parameter is the clique number of G , which is the size of the maximum clique of G . The chromatic number of G is the smallest integer k such that G admits a proper

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coloring of $V(G)$ with k colors, whereas the clique cover number of G is the minimum number of cliques that cover $V(G)$. Another two interesting and related parameters are the fill-in number of G , which is the minimum number of edges to be added to G in order to turn it chordal, and the treewidth of G , which is the size (minus one) of the largest bag of $\mathcal{T}$, $\mathcal{T}$ being an optimal tree decomposition of G . The decision versions of all these problems are NP -complete in general graphs [4], while some of them are polynomial in perfect graphs [2]. We observe that MT-graphs are not necessarily perfect, as they can have cycles of 5 vertices as induced subgraphs.

In this work, we efficiently solve the optimization problems cited above by presenting polynomial algorithms to compute all the aforementioned parameters for the MT-graphs. The most expensive cost of all algorithms is the preprocessing step of finding a partition $(S, M, \bar{M}, K)$ of $V(G)$. Since all these problems are also polynomial time solvable in split graphs, we are left with the interesting question of finding problems that are polynomial time solvable in split graphs and hard to solve on MT-graphs.

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Square roots of split graphs via matchings[‡]

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Keywords: square root; split graphs; matching

The *square* of a graph G is the graph G^2 obtained from G by adding an edge between any two vertices at distance at most 2 in G . We say that H is a *square root* of G if $H^2 = G$. Deciding whether a graph has a square root is an $\mathcal{NP}$ -complete problem [3]. A *split partition* of a graph G is a pair (Q, S) , with $Q \cup S = V(G)$, $Q \cap S = \emptyset$, where Q is a clique and S is a maximal independent set (called *satellites*) in G . A *split graph* is a graph that admits a split partition. The complexity of deciding whether a split graph has a square root is open. The problem is $\mathcal{NP}$ -complete when restricted to chordal graphs [1], a superclass of split graphs, and is solvable in linear time for threshold graphs [2], split-anchored, split-paired graphs and split-indifference graphs [4], which are subclasses of split graphs.

In this work, we investigate split graphs which have square roots where the satellites have degree exactly 1 (called 1-roots). This restriction is motivated by the fact that, in a square root H of a split graph, there always exists a matching between Q and S that covers S . When H is a 1-root, this matching is precisely $\partial_H(S)$, the boundary of S in H . Note that not every split graph that admits a square root has one 1-root (see Figure 1(b) and (c)).

Theorem 1. *Let G be a 2-connected split graph where the neighborhoods of the satellites are pairwise intersecting and let M be a matching in (the bipartite graph) $G[\partial_G(S)]$ that covers S . Let $\pi_M: V(M) \rightarrow V(M)$ such that $\pi_M(u) = v$ iff $uv \in M$. Let also $Q_M = V(M) \cap Q$ and let H be a spanning subgraph of G with $E(H) = M \cup \{qx \in E(G): q \in Q_M, x \in N_G(\pi_M(q)) \setminus \{q\}\} \cup \binom{Q \setminus Q_M}{2}$. Then H is a square root of G if and only if the following conditions hold:*

1. *for every pair of vertices $q \in Q_M$ and $\bar{q} \in Q \setminus Q_M$, at least one of the following holds: (a) $\bar{q} \in N_G(\pi_M(q))$; (b) $\exists q' \in Q_M$ such that $q, \bar{q} \in N_G(\pi_M(q'))$; (c) $\exists \bar{q}' \in Q \setminus Q_M$ such that $\bar{q}' \in N_G(\pi_M(q))$.*

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2. for every pair of edges $\{uv, wx\} \subseteq M$, the graph $G[\{u, v, w, x\}]$ is not a 3-pan graph (Figure 1(a)).

The requirements that G be 2-connected and the neighborhoods of the satellites be pairwise intersecting are necessary conditions for the existence of a square root [4]. These, along with condition 1 and the definition of H , imply that Q is a clique in H^2 . Condition 2 ensures that the neighborhoods of the satellites in H^2 coincide with its neighborhoods in G .

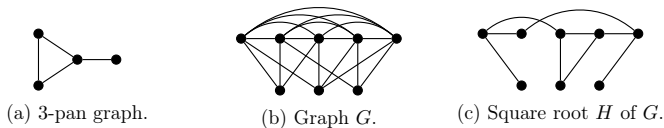


Figure 1: The 3-pan graph, a graph G that does not admit a 1-root, and a square root H of G .

The exhaustive search algorithm that results straightforwardly from Theorem 1 takes $\mathcal{O}(\Delta^{|S|} \cdot |S|^3)$ time, where Δ is the maximum degree of the satellites. The total cost is $\mathcal{O}(n^{|S|+3})$, where $n = |Q| + |S|$. Thus, the algorithm is polynomial for fixed $|S|$. Note that when $\Delta = \mathcal{O}(1)$ and $|S| = \mathcal{O}(\log n)$, the total cost of this algorithm is polynomial as well.

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Session 3: Colorings: special classes, complexity, and algorithms

Advances on Clustered Colourings of Outerplanar Graphs
Luis Gustavo Da Soledade Gonzaga and Christiane Neme Campos
54

Jogos de Convexidades de Caminhos em Grafos
Eurinaldo Costa, Leon Almeida and Rudini Sampaio 56

Minimal Obstructions for Star Coloring in Chordal Graph Families
Germán Benítez-Bobadilla, Fernando Esteban Contreras-Mendoza, César Hernández-Cruz, Claudia Linhares Sales and Ana Laura Trujillo-Negrete 58

Algoritmos para o problema da supercoloração em digrafos (acíclicos) com número fixo de cores
Pedro Guimarães Caninas, Álvaro Junio Pereira Franco, Rafael de Santiago and Wesly Carmesini Ataide 60

Branch and Bound Algorithm for the Grundy Number Problem
Eric Gomes, Kaio Barreto and Wladimir Tavares 62

On Graceful Colorings of $P(n, k)$
Abias Vidal, Simone Dantas and Atílio Luiz 64

Advances on Clustered Colourings of Outerplanar Graphs

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Keywords: improper colourings; clustered colourings; outerplanar graphs

A *vertex colouring* of a simple graph G is an assignment of colours from a set $\mathcal{C}$ to $V(G)$. Given a vertex colouring π , a *monochromatic subgraph* is a maximal subgraph of G induced by the vertices of a single colour. A connected component of a monochromatic subgraph is referred to as a *monochromatic component*; it is these components whose order and maximum degree we seek to control. A (k, d) -*colouring* is a vertex colouring using at most k colours such that the maximum degree of any monochromatic component is at most d ; a $(k, 0)$ -colouring corresponds to a *proper colouring*, which is a colouring in which every pair of adjacent vertices are assigned different colours. A (k, c) -*clustering* is a vertex colouring using at most k colours in which each monochromatic component has at most c vertices. A $(k, 1)$ -clustering corresponds to a proper colouring, while a (k, d) -clustering is equivalent to a $(k, d - 1)$ -colouring since restricting each monochromatic component to at most d vertices implies that its maximum degree is at most $d - 1$.

Clustered colourings were originally introduced by Kleinberg et al. under the term (k, c) -*fragmentation*. The terminology (k, c) -*clustering* has since been widely adopted. In 2018, van den Heuvel [4] proved that there exist outerplanar graphs that do not admit $(2, c)$ -clusterings for any finite c . Additionally, the author showed that for every c , there exist planar graphs that do not admit $(3, c)$ -clusterings. On the other hand, as an immediate consequence of proper colourings results, every planar graph admits a $(4, 1)$ -clustering and every outerplanar graph has a $(3, 1)$ -clustering.

Cowen et al. [1] studied defective colourings for graphs embedded in the plane and characterised the pairs (k, d) for which every planar or outerplanar graph admits a (k, d) -colouring. On the positive side, every outerplanar graph admits a $(2, 2)$ -colouring and every planar graph has a $(3, 2)$ -colouring. On the negative side, there exist outerplanar graphs that do not admit $(2, 1)$ -colourings, and there is no value of d for which every planar graph admits a $(2, d)$ -colouring.

While (k, d) -colourings have been extensively studied, (k, c) -clusterings have received comparatively less attention [5]. The literature highlights the central role of triangles in (k, d) -colourings, with numerous papers studying their properties in relation to triangle adjacency to small cycles. Jonck Jr. and Campos [3] showed that every outerplanar graph with at most three triangular faces admits $(2, 1)$ -colourings. Since $(2, 1)$ -colourings are MSO_1 -expressible, the existence of these colourings with fixed parameters for outerplanar graphs is decidable in linear time. However, the authors questioned whether restrictions on the appearance of triangles in an outerplanar graph could yield the minimum value of c for which every outerplanar graph admits a $(2, c)$ -clustering. Recently, Gonzaga and Campos [2] showed that deciding whether a planar graph G has a $(2, 2)$ -clustering is NP-complete even when every pair of triangles in G is vertex-disjoint.

In this work, we focus on $(2, d)$ -clusterings of outerplanar graphs in which triangular faces are vertex-disjoint, which we call apart-outerplanar graphs. Our main result establishes that every apart-outerplanar graph admits a $(2, 3)$ -clustering, settling the question of Jonck Jr. and Campos for this class. In establishing this result, we first exhibit an infinite family of apart-outerplanar graphs that do not admit $(2, 2)$ -clusterings, showing that $(2, 3)$ is tight for this class. We also show that every apart-outerpath graph – a subclass of apart-outerplanar graphs – admits a $(2, 2)$ -clustering.

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Jogos de Convexidades de Caminhos em Grafos

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Keywords: Convexidade em Grafos, Teoria Combinatória dos Jogos, Jogos de Convexidade em Grafos; Teoria de Sprague-Grundy; Convexidade P_3

Toda convexidade $\mathcal{C}$ sobre um grafo G pode ser definida a partir de uma função de intervalo $I_{\mathcal{C}}(\cdot)$ de modo que $S \subseteq V(G)$ é $\mathcal{C}$ -convexo se $I_{\mathcal{C}}(S) = S$. As convexidades mais investigadas são as convexidades de caminhos em que, dada uma família $\mathcal{P}$ de caminhos, $I_{\mathcal{C}}(S)$ consiste de S mais todo vértice em algum caminho de $\mathcal{P}$ entre dois vértices de S . Por exemplo, nas convexidades geodésica, ℓ_k [4] e P_3 , consideram-se caminhos mínimos, caminhos induzidos com no máximo k arestas e caminhos P_3 . O fecho $\mathcal{C}$ -convexo $\text{hull}_{\mathcal{C}}(S)$ de S é o menor conjunto $\mathcal{C}$ -convexo que contém S . Sabe-se que $\text{hull}_{\mathcal{C}}(S)$ pode ser obtido da aplicação sucessiva da função de intervalo $I_{\mathcal{C}}(\cdot)$ até se obter um conjunto $\mathcal{C}$ -convexo. Dizemos que $S \subseteq V(G)$ é um conjunto de intervalo (resp. de fecho) em $\mathcal{C}$ se $I_{\mathcal{C}}(S) = V(G)$ (resp. $\text{hull}_{\mathcal{C}}(S) = V(G)$).

Frank Harary [5] escreveu em 1981 o 1º artigo sobre convexidade em grafos e, em 1984, propôs os primeiros jogos de convexidade [6]. O Jogo do Fecho ($\text{CHG}_{\mathcal{C}}$: *Hull Game*) e o Jogo do Intervalo ($\text{CIG}_{\mathcal{C}}$: *Interval Game*) estão entre os primeiros investigados [2, 7], considerando a convexidade geodésica. Apesar disso, só recentemente provou-se que o jogo CHG_g é PSPACE-completo [1], enquanto a PSPACE-completude do jogo CIG_g ainda é um problema em aberto, onde g indica a convexidade geodésica. Neste artigo, iniciamos o estudo desses jogos nas convexidades ℓ_k .

Nos jogos $\text{CHG}_{\mathcal{C}}$ e $\text{CIG}_{\mathcal{C}}$ em G , dois jogadores, Alice e Bob, começando com Alice, alternam-se selecionando um vértice *jogável* de G em cada jogada. No jogo $\text{CHG}_{\mathcal{C}}$ (resp. $\text{CIG}_{\mathcal{C}}$), um vértice é *jogável* se não pertence a $\text{hull}_{\mathcal{C}}(S)$ (resp. $I_{\mathcal{C}}(S)$), onde S é o conjunto de vértices selecionados antes da jogada atual. Assim, o jogo $\text{CHG}_{\mathcal{C}}$ (resp. $\text{CIG}_{\mathcal{C}}$) termina quando o conjunto S de vértices selecionados é um conjunto de fecho (resp. de intervalo), ou seja, quando $\text{hull}_{\mathcal{C}}(S) = V(G)$ (resp. $I_{\mathcal{C}}(S) = V(G)$). Na variante normal, perde o jogador incapaz de mover em sua jogada, ou seja, com exceção de jogo vazio, vence quem jogar por último. Com isso, tais jogos não possuem empate.

Provamos que o jogo CHG_{ℓ_k} é PSPACE-completo para todo $k \geq 2$. Investigamos os jogos CHG_{ℓ_k} em ciclos C_n e caminhos P_n , usando a Teoria de Sprague-Grundy, que permite atribuir a cada jogo um número natural, chamado *nimber* do jogo. Em caminhos e ciclos, os jogos CHG_{ℓ_k} e CIG_{ℓ_k} são equivalentes. Obtemos assim um algoritmo de programação dinâmica de tempo polinomial para decidir o vencedor do jogo CHG_{ℓ_k} em qualquer caminho P_n ou ciclo C_n para qualquer $k \geq 2$. Provamos que, quando $k \geq 3$ é ímpar, Alice vence em P_n se e só se n é ímpar e vence em C_n se e só se $n = 3$ ou $n = 2k+1+i \cdot (k+1)$ para $i \geq 0$. Mostramos que, para $k = 2$, a sequência de *nimbers* de caminhos P_n no jogo CHG_{ℓ_2} converge para a sequência do clássico jogo *Couples-are-Forever* de J. H. Conway a partir de $n \geq 68$. Ainda é um problema em aberto se tal sequência é periódica ou não. Considerando os primeiros 50 milhões de *nimbers*, não apresentou periodicidade [3] e mostrou que Alice perde (resp. vence) o jogo CHG_{ℓ_2} em P_n (resp. C_n) apenas para 13 valores de n . Mostramos ainda que, para $k = 4, 12, 28$ e 60 , as sequências de *nimbers* do jogo CHG_{ℓ_k} em P_n e em C_n são periódicas e que Alice perde (resp. vence) em P_n (resp. C_n) para no máximo 31 valores de n .

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Minimal Obstructions for Star Coloring in Chordal Graph Families

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Keywords: star chromatic number; minimal obstructions; chordal graphs;
split graphs; 2-paths; certifying algorithms

A *star coloring* of a graph G is a proper coloring in which no path on four vertices is bicolored; equivalently, for every pair of colors, the subgraph induced by the corresponding color classes is a star forest. An *acyclic coloring* of G is defined analogously, requiring only that each pair of color classes induces a forest. The *star chromatic number* $\chi_*(G)$ and the *acyclic chromatic number* $\chi_a(G)$ are the smallest k for which G admits an acyclic or a star coloring with k colors, respectively, and satisfy $\chi(G) \leq \chi_a(G) \leq \chi_*(G)$.

Both parameters were introduced by Grünbaum [4], who proved that every planar graph admits an acyclic coloring with 9 colors and conjectured that 5 suffice. Borodin [2] proved this conjecture, obtaining as a corollary that every planar graph admits a star coloring with 80 colors. Fertin, Raspaud, and Reed [3] introduced the name *star coloring*; they determined χ_* for several classical graph families and established bounds for others. The upper bound for planar graphs was later improved to 20 [1], where NP-completeness of star 3-coloring for planar bipartite graphs was also established.

In this work we study χ_* from the perspective of minimal obstructions, a viewpoint not previously considered for this parameter. For a hereditary class $\mathcal{C}$, a graph $F \notin \mathcal{C}$ is a *minimal obstruction* for $\mathcal{C}$ if every proper induced subgraph of F belongs to $\mathcal{C}$; since star k -colorable graphs form a hereditary class, they admit such a characterization. We obtain minimal obstruction characterizations for several subfamilies of chordal graphs, and use them to

design certifying algorithms returning either a star coloring (yes-certificate) or a minimal obstruction (no-certificate).

Chordal graphs. Within chordal graphs, χ_a coincides with χ , making χ_* the more interesting parameter. We exhibit the complete family $\mathcal{F}_3^{\text{ch}} = \{K_4, \text{kite, gem}, X_{127}\} \cup \{D_i : i \in \mathbb{Z}_{\geq 4}\}$ of minimal obstructions for star 3-colorability in chordal graphs. A key step is a structural characterization of $\mathcal{F}_3^{\text{ch}}$ -free chordal graphs as *crowned forests* (graphs whose blocks are all crowns, obtained from a forest via triangle substitutions and edge augmentations) in which every path joining two nontrivial blocks has a vertex of degree 2. This characterization yields a certifying algorithm that determines, in time $O(|V| + |E|)$, whether an input chordal graph is star 3-colorable.

Split graphs. For every split graph G with clique number $\omega(G)$, we have $\chi_*(G) \in \{\omega(G), \omega(G) + 1\}$. We identify the complete families $\mathcal{F}_4^{\text{sp}}$ and $\mathcal{F}_5^{\text{sp}}$ of minimal obstructions for star 4- and 5-colorability in split graphs, consisting of 6 and 13 graphs, respectively, and derive certifying $O(|V| + |E|)$ algorithms in both cases. With computer assistance, we obtain the complete list of 68 minimal obstructions for star 6-colorability; completeness follows from a structural bound: any such obstruction has clique number 6 and independent set of size at most 20 by Sperner's theorem, giving at most 26 vertices.

2-paths. A *2-path* is a 2-tree whose clique graph is a path. We show there are exactly two minimal obstructions for star 4-colorability in this class. We further provide a star 5-coloring for any 2-path, establishing 5 as a tight upper bound for χ_* within this class. This contrasts with 2-trees in general, for which the maximum star chromatic number is 6; we exhibit a 2-tree on 21 vertices with star chromatic number 6 such that every proper induced subgraph has star chromatic number 5, improving upon the previously known smallest such constructions on 48 [3] and 41 [1] vertices.

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Algoritmos para o problema da supercoloração em digrafos (acíclicos) com número fixo de cores

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Keywords: Supercoloração; Algoritmos em Grafos; Redes de Parentesco

No problema da supercoloração, é dado um digrafo $D = (V, A)$, com V sendo o conjunto de vértices, A o conjunto de arcos e cada vértice colorido com uma cor de $\{1, 2, \dots, k\}$. Não há nenhuma restrição sobre a forma como as cores aparecem nos vértices de D . O problema da *supercoloração* deseja encontrar, para cada vértice v de D , o valor $super(v)$, isto é, o número de cores de um caminho com o número máximo de cores que se inicia em v .

A origem dos algoritmos para caminhos supercoloridos na antropologia foi motivada para melhorar a eficiência da busca por anéis cromáticos (Franco; Vendramin, 2021). O problema dos anéis cromáticos constitui uma variação do clássico problema antropológico de enumeração de circuitos matrimoniais, que consiste em identificar ciclos estruturais resultantes da combinação de relações de consanguinidade (filiação) e alianças matrimoniais (casamentos) (White, 2004), (Hamberger; Houseman; Douglas, 2011). No entanto, o problema da supercoloração possui outras aplicações no estudo das relações matrimoniais na antropologia permitindo, por exemplo, a análise da transmissão de características e propriedades de ancestrais para descendentes, a observação da movimentação no espaço ao longo de uma genealogia e a visualização das relações entre clãs ou diferentes comunidades.

Os algoritmos de supercoloração são aplicados sobre redes de parentesco, que são caracterizadas por três princípios sociais e biológicos: a filiação (consanguinidade), o casamento (aliança matrimonial) e o gênero. As redes de parentesco são representadas por grafos mistos, onde os vértices representam os indivíduos, as relações de filiação são mapeadas como arcos direcionados (partindo dos pais para os filhos) e as alianças matrimoniais (casamentos) são representadas por arestas. Importante notar que, pelo fato de que nenhum indivíduo pode ser ancestral biológico de si mesmo, a rede induzida pelos arcos forma um digrafo acíclico (DAG).

Observamos que o problema é difícil quando D é geral e que há um algoritmo parametrizado com tempo $O(2^k k^2 |A|)$ quando D é um digrafo acíclico (Franco; Vendramin, 2021). Portanto, se o número de cores k é fixo, então tal solução é linear em $|A|$, mas haverá uma constante exponencial em k multiplicando $|A|$. Os autores desse resumo desenvolveram novos algoritmos que resolvem o problema da supercoloração para k fixo igual a 2, 3 e 4 cores com complexidade $O(|V| + |A|)$, $O(|V|^2)$ e $O(|V|^3)$, respectivamente. Em seguida, destacamos o algoritmo de supercoloração para $k = 4$ cores.

O algoritmo recebe um digrafo acíclico D com 4 cores, considera o fecho transitivo $\tilde{D}$ de D e visita duas vezes cada 3-clique multicolorida (clique sem considerar a direção dos arcos e com três cores distintas) em $\tilde{D}$. Vale a pena dizer que cada 3-clique $t = (u, v, w)$ em $\tilde{D}$, quando observada isoladamente, possui um vértice fonte (u), um vértice intermediário (v) e um vértice sorvedouro (w). Para cada 3-clique multicolorida $t = (u, v, w)$, o algoritmo guarda a clique t em duas tabelas de espalhamento com as seguintes chaves: $T_{inter}[u, v, \bar{c}] = T_{inter}[v, w, \bar{c}] = t$ e $T_{serv}[v, w, \bar{c}] = t$, sendo $\bar{c}$ a cor que não está na 3-clique t . Novamente, para cada 3-clique multicolorida $t = (u, v, w)$, seja c_v a cor de v e c_w a cor de w , se $T_{inter}[u, w, c_v] = t'$ ou $T_{serv}[u, v, c_w] = t'$, então a existência de t e t' garante a existência de uma 4-clique em $\tilde{D}$. Com isso, temos que o valor de supercoloração da fonte da 3-clique multicolorida t' será igual a 4. O tempo de execução do algoritmo é $O(|V|^3)$, melhor que o de um algoritmo ingênuo que resolve o problema em $O(|V|^4)$ visitando todas as 4-cliques em $\tilde{D}$.

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Grundy Number

Branch and Bound Algorithm for the Grundy Number Problem

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Keywords: Grundy number, greedy coloring, graph theory, vertex coloring, chromatic number, combinatorial optimization.

The Grundy number of a graph, denoted by $\Gamma(G)$, is a parameter in graph theory associated with vertex coloring using the greedy algorithm. In this process, the vertices are colored in a given order, and each vertex receives the smallest color available that is not used by its previously colored neighbors. Because the result depends on the chosen ordering, the algorithm may use more colors than necessary.

The concept originated in the work of Grundy [1] in the context of combinatorial game theory and was later formalized as a graph-theoretic parameter by Christen and Selkow [2]. The Grundy number $\Gamma(G)$ represents the worst-case behavior of the greedy coloring algorithm, being defined as the maximum number of colors that can be used on all possible vertex orderings of the graph. This concept is closely related to the chromatic number $\chi(G)$, which corresponds to the minimum number of colors required for a proper coloring, satisfying the relation $\chi(G) \leq \Gamma(G)$.

Therefore, the Grundy number measures how sensitive a graph is to vertex ordering and evaluates the worst-case performance of greedy graph coloring. Determining the Grundy number of graphs belongs to the NP-complete class [3] and is subject to some limits established in the literature.

In this work, we propose an exact solver for the Grundy Coloring problem based on a branch-and-bound framework combined with bit parallelism. The recursive characterization of the Grundy number $\Gamma(G)$ is based on the enumeration of all maximal independent sets of the graph, proposed by Bonnet et al. [5]. To efficiently perform this task, we employ a bit-parallel adaptation of the Bron-Kerbosch algorithm [4], which generates maximal independent

sets on demand through generators (lazy evaluation), avoiding the need to store all sets in memory simultaneously.

This approach significantly accelerates set operations while reducing memory consumption. To further enhance performance, the proposed algorithm integrates multiple cascading upper bounds for pruning, including degree-based bounds [8], the stair factor [7]. Computational experiments indicate that these techniques substantially reduce the search space and provide competitive performance in benchmark instances.

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SUBMISSION 4 — On Graceful Colorings of $P(n, k)$
by Vidal, Dantas and Luiz.

Dear organizers of LAWCG 2026,

Thank you for your email dated June 15th, 2026 with one referee report accepting our abstract for presentation at LAWCG 2026. We are grateful to the referee for the diligent reading of our abstract. The reviewer suggested modifications in our abstract, so we have revised the text addressing all suggestions as listed below.

With our best regards,

Vidal, Dantas and Luiz.

Reviewer 1

1. Since there is substantial unused space on the second page, the authors should include at least a brief explanation of the main ideas of the proofs. In particular, it would be useful to indicate how the colorings are constructed and why the congruence conditions on (n) and (k) allow the stronger bound. A complete proof is not necessary, but some information about the techniques would make the contribution more informative and easier to assess.

Answer: We have added the required explanation in the last two paragraphs of the abstract.

2. The definition of $(P(n, k))$ must include the admissible ranges of (n) and (k) , and it must state that all subscripts are taken modulo (n) . Without these restrictions, the claim that all the defined graphs are 3-regular is not correct.

Answer: We have clarified the ranges $n \geq 3$ and $k < \frac{n}{2}$ in the definition of the generalized Petersen graph $P(n, k)$. We also specified that all subscripts are taken modulo (n) as suggested.

3. The authors should cite and discuss the previous paper specifically devoted to graceful colorings of generalized Petersen graphs, namely the work of Kristiana et al. on the cases $(k = 1, 2, 3, 4)$. See <https://doi.org/10.1142/S1793830923500970>. The abstract currently states that previous results are extended, but it neither cites these results nor explains precisely what is generalized. The relationship between the new theorems and the known cases should be made explicit.

Answer: We have added paragraphs explaining previous results and the relation with our generalization as required.

4. The abstract should include at least a brief description of the proof strategy. In particular, the authors should indicate how the graceful 10-coloring is constructed, which structural properties of generalized Petersen graphs are used, and why the congruence conditions ($n \equiv 0, 1 \pmod k$) allow the number of colors to be reduced to 8. A complete proof is not required, but the present statement-only presentation does not provide enough information to assess the results.

Answer: We did as suggested.

5. The phrase “the first upper bound” is imprecise, since nonconstant upper bounds depending on the order of the graph are already immediate or may follow from general results. The authors seem to mean the first uniform constant upper bound for all generalized Petersen graphs. Please reformulate the claim accordingly.

Answer: We did as suggested and replaced “the first upper bound” with “the first uniform constant upper bound”.

6. Replace “Rio de janeiro” with “Rio de Janeiro.”

Answer: Done.

7. The introductory discussion of proper coloring could be shortened and made more precise. Since the definition of graceful coloring directly involves a proper vertex coloring and an induced proper edge coloring, the authors could define these two notions directly, instead of first introducing a general and somewhat vague description of proper coloring in terms of adjacent “elements.”

Answer: We did as suggested and incorporated the required definitions in the second paragraph.

8. Consider stating the second result as: “For every admissible pair (n, k) , if $n \equiv 0$ or $1 \pmod k$, then $(\chi_g(P(n, k)) \leq 8)$.” The expression “for any fixed integer (k) ” is unnecessary unless fixing (k) is important to the proof.

Answer: We have modified the text.

9. The title could be made more informative, for example, “Bounds on the Graceful Chromatic Number of Generalized Petersen Graphs.”

Answer: We thank for the recommendation but, for this abstract, we prefer to maintain the title in the current form.

10. In Reference [3], please distinguish between the year of the conference and the year of publication. The symposium was held in Rome in July 1966, whereas the proceedings were published in 1967. A more complete reference would be: A. Rosa, “On certain valuations of the vertices of a graph,”

in Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, New York, and Dunod, Paris, 1967, pp. 349–355.

Answer: We did as suggested and replaced the reference.

On Graceful Colorings of $P(n, k)$ ¹

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Keywords: graceful coloring; generalized Petersen graphs; graceful labeling

Graph labeling is an area of graph theory that studies the assignment of labels to the elements of a graph. In 1967, Rosa [4] introduced the concept of *graceful labeling* of a graph G , that is, a one-to-one function $f : V(G) \rightarrow \{0, 1, \dots, |E(G)|\}$ that induces on each edge $uv \in E(G)$ the label $f'(uv) = |f(u) - f(v)|$, such that the labels of the edges of G are pairwise distinct.

In 2017, Bi et al. [1] introduced the *graceful coloring*: a type of coloring that combines *proper vertex coloring* (adjacent vertices are colored with distinct colors), *proper edge coloring* (adjacent edges must have distinct colors), and graceful labelings. A *graceful k -coloring* of a graph G is a proper vertex coloring $c : V(G) \rightarrow \{1, \dots, k\}$, $k \geq 1$, that induces a proper edge coloring $c' : E(G) \rightarrow \{1, \dots, k-1\}$ defined by $c'(uv) = |c(u) - c(v)|$ for each edge $uv \in E(G)$. The minimum positive integer k for which it is possible to construct a graceful k -coloring of G is the *graceful chromatic number* of G , denoted by $\chi_g(G)$.

In 2025, Pantoja et al. [3] conjectured that, with the exception of the Petersen graph, every other 3-regular graph G has $\chi_g(G) \leq 8$. Motivated by this conjecture, in this work, we investigate the graceful chromatic number of a family of 3-regular graphs called *generalized Petersen graphs*, denoted by $P(n, k)$, for positive integers n and k , with $n \geq 3$ and $k < \frac{n}{2}$, these graphs are 3-regular with vertex set $V(P(n, k)) = \{x_i : 0 \leq i \leq n-1\} \cup \{y_i : 0 \leq i \leq n-1\}$, and edge set $E(P(n, k)) = \{x_i x_{i+1} : 0 \leq i \leq n-1\} \cup \{x_i y_i : 0 \leq i \leq n-1\} \cup \{y_i y_{i+k} : 0 \leq i \leq n-1\}$ (indices taken modulo n).

In 2023, Suparta et al. [5] studied $P(n, 1)$ and proved that $\chi_g(P(n, 1)) = 5$ if $n \equiv 0 \pmod{4}$, and $\chi_g(P(n, 1)) = 6$ otherwise. Kristiana et al. [2]

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analyzed this class for $k \in \{2, 3, 4\}$ and proved the following bounds: (i) $5 \leq \chi_g(P(n, 2)) \leq 6$ for $n \equiv 0 \pmod{3}$; (ii) $5 \leq \chi_g(P(n, 2)) \leq 8$ for $n \not\equiv 0 \pmod{3}$; (iii) $5 \leq \chi_g(P(n, 3)) \leq 8$; and (iv) $5 \leq \chi_g(P(n, 4)) \leq 8$.

In our work, we first extend these results from the previous fixed values of k to arbitrary values of $k \geq 3$ and $n \equiv 0, 1 \pmod{k}$. Furthermore, we establish the first uniform constant upper bounds for $\chi_g(P(n, k))$. More precisely, if $P(n, k)$ has no internal cycle isomorphic to C_5 then $\chi_g(P(n, k)) \leq 10$; otherwise, except for the Petersen graph, we have $\chi_g(P(n, k)) \leq 8$. In the former case, the proof considers two disjoint sets of colors, one for the external cycle (vertices x_i), and one for the internal cycles (vertices y_i). In the later case, we first consider the graceful colorings of the following three graphs: $P(5, 1)$ [5], $P(5, 2)$ (Petersen graph) [3], and $P(10, 2)$. For the remaining graphs, we partition the set of internal vertices into blocks of consecutive vertices. We assign a single color to all the vertices of each block so that each internal vertex is gracefully colored. Finally, for the external cycle, we assign a color sequence that is compatible with the internal coloring. If $n \equiv 0 \pmod{k}$ or $n \equiv 1 \pmod{k}$, then the subgraphs induced by the internal vertices y_i have a structure that allows the application of similar proof techniques, yielding $\chi_g(P(n, k)) \leq 8$. Therefore, Pantoja's conjecture [3] holds for these cases.

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Session 4: Cliques, bicliques, and perfection

Cliques and Convolutions: An Algebraic Approach to
Counting Cliques in Sparse Graphs

Miguel Alessandro Miní Huambachano 70

Listing bicliques and cliques: Bipartite Permutation and
Proper 2-Thin Graphs

Gabriel Godinho, Marina Groshaus and André L. P. Guedes . . . 72

Finding Bicliques in Quantum Computing

Leandro Zatesko and Marina Groshaus 74

Reduced clique graphs: characterization, superclasses and
subclasses

Pablo De Caria 76

A Quadratic Characterization of the Weighted Lovász
Theta Number and Related SDP Bounds

Luis Higino and Gabriel Coutinho 78

On the Cost of Perfection in Wheel Graphs

Ingrid da Costa Silva Borchert, Diana Sasaki and Diego Nicodemos 80

Cliques and Convolutions: An Algebraic Approach to Counting Cliques in Sparse Graphs

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Keywords: clique counting; vertex ordering; subset zeta transform; exact enumeration; sparse graphs

Abstract

Exact clique counting on sparse graphs is often organized around a vertex ordering: every clique has a unique earliest vertex in that order, and once that vertex is fixed, the rest of the clique must lie among its later neighbors and must form a clique inside the subgraph induced by those later neighbors. Classical exact methods then process one vertex at a time and solve that induced neighborhood by recursive local search [1, 2]. We keep that outer counting framework, but replace the local search by a hybrid exact routine. For each chosen vertex, we split its forward neighborhood into two parts: one part is still explored explicitly, and the other part is summarized once with the subset zeta transform [3]. In practice, that transform builds a table over all subsets of the summarized part, so the repeated compatibility checks on that side become lookups instead of being recomputed many times. The explicit side is the part we are willing to branch on, while the preprocessed side is the part where many compatibility checks repeat and can be reused. If the preprocessed side has size s , building the table costs $O(s^2 2^s)$ time and $O(s 2^s)$ memory, and each clique count is then recovered by table lookup after the explicit side has been enumerated. This keeps the computation exact, because the final count is still assembled vertex by vertex in the same ordering, but it replaces repeated local branching with batched reuse wherever that reuse is worthwhile. We compare this hybrid method against *Pivoter*, a strong exact sequential baseline that builds a Succinct Clique Tree via pivoting and reads exact clique statistics from that compressed structure [4]. We choose $k \in \{6, 8, 10, 12\}$ to cover a range where exact counting is still feasible but the local subproblems vary enough to

stress the method, and we use six sparse SNAP graphs [5], which span a wide range of sizes and densities. For these single-size queries and for full-range counting with $k = 4, \dots, 12$, the method matched all reported counts, was faster on every query that fixed one clique size k by about $1.5\times$ to more than $12\times$, and used about $5\times$ to $10\times$ less memory. For the full range, it remained competitive and won on four of the six graphs while still using far less memory than *Pivoter*. The contribution is an exact hybrid strategy that trades local recursion for batched algebraic reuse when that is profitable, without changing the outer ordering-based framework. The adaptive split lets the method use the algebraic table only when reuse is worthwhile. The single-size queries isolate the effect of the reuse step, while the full-range queries check that the same strategy remains stable across clique sizes.

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Listing bicliques and cliques: Bipartite Permutation and Proper 2-Thin Graphs[‡]

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Keywords: Proper 2-thin graphs; Cliques; Bicliques; Listing algorithms

Listing the cliques and bicliques in graphs is a fundamental problem in Graph Theory, with applications in various areas of Computer Science and Mathematics. Despite their practical relevance, finding cliques and bicliques in arbitrary graphs is a computationally hard problem. In general, a graph with n vertices can have up to $3^{n/3}$ cliques [5]. However, when restricted to certain graph classes, the problem becomes easier not only due to the polynomial number of cliques but also because of specific structural properties that lead to more efficient clique detection. Examples include interval graphs, chordal graphs, and permutation graphs.

A graph $G = (V, E)$ is called k -thin if, for some total ordering of its vertices, there exists an associated partition of V into k parts $(V_1, V_2, \dots, V_k)$ satisfying the following condition: for any triple (r, s, t) with $r < s < t$, if vertices r and s belong to the same part and $\{t, r\} \in E$, then $\{t, s\} \in E$. The smallest value of k for which this property is satisfied is called the *thinness* of the graph G [1, 2, 3, 4]. Graphs with bounded *thinness* were introduced by Mannino et al. [4] as a generalization of interval graphs, which correspond exactly to 1-thin graphs. A graph G is called *proper k -thin* if it is k -thin under both the total ordering of the vertices $<$ and its reverse ordering $>$. Since k -thin graphs are defined as a generalization of interval graphs, proper k -thin graphs naturally arise as a generalization of proper interval graphs. Bonomo and Brito [3] defines the subclass of independent proper k -thin graphs in which all parts of the partition of V are independent sets. It is important to note that the class of independent proper 2-thin graphs coincides with the class of bipartite permutation graphs [2]. Cliques in proper 2-thin graphs are closely related to bicliques in the independent proper 2-thin subgraphs. This work presents an algorithm to list all bicliques in bipartite permutation

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graphs in $\mathcal{O}(n^2)$ time. Our main contribution is an algorithm for listing all cliques in proper 2-thin graphs with time complexity of $\mathcal{O}(c + n^2)$, where c is the total number of cliques. This complexity is optimal in many cases, particularly when the number of cliques is large ($c > n^2$). The algorithm’s design, which combines the listing of bicliques in the independent subgraph with a search for “local” cliques within each part, alongside an indexing mechanism to prevent redundant computations, was crucial to avoiding a brute-force approach and minimizing unnecessary work.

Our research opens several paths for future work. One natural direction is to address the enumeration of cliques in general 2-thin graphs. Our current algorithm relies on the symmetric structure provided by the “proper” condition; removing this constraint introduces significant challenges. Furthermore, a more ambitious goal is the generalization of our method to proper k -thin graphs for any fixed $k > 2$. This would represent a significant extension of our work and would be of great interest, given the wide range of applications for these graph classes. An algorithm for this more general case would likely need to handle a more complex interplay between the k vertex parts.

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Finding Bicliques in Quantum Computing

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Keywords: graph algorithms; bicliques; quantum algorithms; quantum walk

We consider the k -BICLIQUE problem, which consists of, given a simple graph G and an integer k , finding a (not necessarily maximal) subgraph of G with k vertices that induces a complete bipartite graph, or reporting that no such subgraph exists. This problem is the bipartite version of the k -CLIQUE problem, which consists of finding a subgraph with k vertices that induces a complete graph. Both problems are NP-hard [1]. The best known classical algorithm for k -CLIQUE has $O(n^{0.792k})$ time [4], and, to the best of our knowledge, no $o(n^k)$ classical algorithm for k -BICLIQUE is known. Actually, both problems are W[1]-hard parameterized by k [2].

Recently, a novel quantum algorithm for k -CLIQUE was proposed [5]. This method is the first to use quantum walks on the Johnson graph of the k -subsets of vertices of G , outperforming previous standard quantum amplification approaches and yielding a quantum circuit of $O(\sqrt{\binom{n}{k}} n \log n)$ depth and $O(n \log n)$ qubits. We show how to adapt this method to generate a quantum circuit for k -BICLIQUE with asymptotically the same depth and number of qubits.

The method proceeds briefly as follows. The first step consists of preparing a uniform superposition of the vertices of the $k(n-k)$ -regular Johnson graph $J(n, k)$, i.e. the graph whose vertices are all $\binom{n}{k}$ subsets with k elements of a set with n elements, such that two vertices are adjacent if and only if the intersection of their corresponding subsets has cardinality $k-1$. The second step consists of preparing, in another register, a uniform superposition of the edges of $J(n, k)$, i.e. the state

$$\sqrt{\frac{2}{nk(n-k)}} \sum_{uv \in E(J(n,k))} |u\rangle |v\rangle.$$

Finally, an *MNRS* quantum walk search [3] is performed on $J(n, k)$. Regardless of the function evaluated by the oracle on the candidate k -subsets during

the search, the structure of the candidate graph remains the same, and so does its spectral gap, which determines the number of search iterations required for the underlying Markov chain to converge to the desired k -subsets with high probability. The spectral gap of $J(n, k)$ is known to be $\Theta(1/k)$ [6].

Let A_G be the adjacency matrix of G and, for $S_1, S_2 \subseteq V(G)$, let $A_G[S_1, S_2]$ be the submatrix of A_G induced by the rows corresponding to the vertices in S_1 and the columns corresponding to those in S_2 . Our algorithm for k -BICLIQUE differs from Perriello's for k -CLIQUE only in the oracle implementation. Since k -subsets are treated as binary strings of Hamming weight k , the set S of the k selected vertices is used as a control register in Perriello's oracle, which is simply an efficient parallel adder of the columns of $A_G[S, S]$, evaluating S as a clique if and only if the sum in every column is $k - 1$.

In our oracle, we first show how it is possible to construct a circuit, using the adjacency matrix of G , that identifies a vertex $v \in S$, identifies the set L of all its neighbours, and also the set R of all neighbours of a vertex $u \in L$, allocating (via duplication using CNOT gates) separate registers for L and for R . We then apply: 1. an adder controlled by L to check whether the sum in every column of $A_G[L, L]$ is 0; 2. an adder controlled by R to check whether the sum in every column of $A_G[R, R]$ is 0; 3. an adder controlled by both L and R to check whether the sum in every column of $A_G[L, R]$ is $|L|$. We show that the resulting circuit has depth and number of qubits at most three times the corresponding values for the circuit for k -CLIQUE.

Considering the k -BICLIQUE problem restricted to bipartite graphs, we further show how to implement the oracle using the biadjacency matrix B_G of G , achieving similar depth and number of qubits.

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Reduced clique graphs: characterization, superclasses and subclasses

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Keywords: Chordal graph; clique tree; reduced clique graph; split

Given a chordal graph G , the *reduced clique graph* of G is the graph $C_r(G)$ whose vertices are the maximal cliques of G , such that two cliques C_1 and C_2 are adjacent in $C_r(G)$ if $C_1 \cap C_2 \neq \emptyset$ and there exists a clique tree of G that has the edge C_1C_2 .

Reduced clique graphs were introduced in [1] with the name of clique graphs of chordal graphs. Later, the name of the class was changed to its modern form in [2] to avoid confusion with dually chordal graphs. In that paper, an algorithmic use of reduced clique graphs was found to solve problems on chordal graphs efficiently. This trend continued in [3], where reduced clique graphs proved advantageous for the study of parameters like the asteroidal number and the leafage of chordal graphs.

The papers cited in the previous paragraph construct the reduced clique graph from a given chordal graph to implement the applications. Moreover, reduced clique graphs are always considered there as labeled graphs, where every edge CC' has label $C \cap C'$, without addressing the problem of determining if an unlabeled graph can be labeled to become the reduced clique graph of a chordal graph. An effort in this direction appears in [4]. There, reduced clique graphs are compared to chordal graphs and clique graphs, and it is proved that the wheels W_n , where n is even, are isomorphic to the reduced clique graph of a chordal graph, that C_n is not the reduced clique graph of a chordal graph ($n \geq 4$) and that no reduced clique graph has an induced C_5 , also conjecturing that the same is true for every odd hole.

In this work, we show that reduced clique graphs can be recursively decomposed to form a binary tree that is called the *R-split tree*, which records splits of the original graph and the resulting subgraphs whose corresponding cut-sets consist of edges all with the same label. Moreover, we prove that every graph with an *R-split tree* can be represented as the reduced clique graph of a chordal graph, thus obtaining a characterization of reduced clique graphs.

As a first application of this characterization, we prove that reduced clique graphs cannot contain odd holes, and that they cannot contain any induced $\overline{C}_n$, for $n \geq 7$. As a consequence, we conclude from the strong perfect graph theorem that perfect graphs form a superclass of reduced clique graphs. We also see that R -split trees are useful to find several other graphs that are neither reduced clique graphs nor induced subgraphs of one.

As for subclasses of reduced clique graphs, we prove that cographs and clique graphs of ptolemaic graphs are reduced clique graphs.

Next, we consider graphs that have special R -split trees to define the subclass of *split trivial reduced clique graphs*, characterize it and find a characterization of it restricted to split graphs. Furthermore, we use this subclass to derive new characterizations of block graphs and comparability graphs with a universal vertex.

We also find other ways to define new subclasses of reduced clique graphs. Much like clique trees are used to define the reduced clique graph of a chordal graph, for dually chordal graphs we can use the compatible trees to define *reduced dually chordal graphs*. We prove that every reduced dually chordal graph is a reduced clique graph, and that the converse is not necessarily true.

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A Quadratic Characterization of the Weighted Lovász Theta Number and Related SDP Bounds

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Keywords: Lovász theta function; semidefinite programming; Graph invariants; Spectral graph theory

Given a graph G with n vertices we denote the set of real symmetric weighted adjacency matrices for G by $\mathcal{M}(G)$. Then, given a non-negative vector $w \in \mathbb{R}^n$ and a matrix $A \in \mathcal{M}(G)$ we define the following parameter:

$$\alpha_w(G, A) = \sup\{\|v\|^2 \mid v \in \mathbb{R}^n, \langle v, \sqrt{w} \rangle = \|v\|^2, \langle v, Av \rangle = 0\}$$

where $\sqrt{w}$ denotes taking the square root entrywise. The infimum of $\alpha_w(G, A)$ for all $A \in \mathcal{M}(G)$ is denoted by $\alpha_w(G)$.

It was recently shown by Wolff 2025 that $\alpha_w(G)$ equals the Lovász Theta number when $w = \mathbf{1}$, the all ones vector of appropriate size. We significantly simplify and generalize this proof, showing that, for any graph G and non-negative vector w , the weighted Lovász Theta number satisfies:

$$\vartheta(G; w) = \alpha_w(G)$$

We also prove that, for any non-negative vector w and $A \in \mathcal{M}(G)$, the value of $\alpha_w(G, A)$ equals the optimum value of the following optimization:

$$\min_{t \in \mathbb{R}} \lambda_{\max}(\sqrt{w}\sqrt{w}^\top + t \cdot A)$$

This provides a formulation of $\alpha_w(G, A)$ more akin to other famous formulations of the weighted Lovász Theta number.

Furthermore, we compare $\alpha_w(G)$ with another parameter established by Luz and Schrijver 2005, defined for any $A \in \mathcal{M}(G)$:

$$v(G, A) = \max\{2 \cdot \langle \mathbf{1}, v \rangle - \langle v, (H_A + I)v \rangle \mid v \in \mathbb{R}^n, v \geq 0\}$$

where $H_A = \frac{1}{-\lambda_{\min}(A)}A$. Luz and Schrijver 2005 showed that the infimum of $v(G, A)$ for all $A \in \mathcal{M}(G)$ equals $\vartheta(G)$. We prove that whenever the optimum

vector of either $v(G, A)$ or $\alpha_1(G, A)$ is feasible in the other optimization, both optimum values collapse to $\alpha(G)$, providing a novel characterization of graphs where $\vartheta(G)$ exactly equals their independence number.

Building upon these analytical insights, we also explore a related SDP parameter defined in Oliveira 2024. We consider:

$$\mu(G) = \min \{ \langle I, Y \rangle \mid Y \in \mathbb{R}^{n \times n}, Y = Y^\top, \langle J, Y \rangle = n, I \succeq Y \succeq 0, Y \circ A = 0 \},$$

where A is the adjacency matrix of G , $J = \mathbf{1}\mathbf{1}^\top$ and $\circ$ is the Hadamard product. Its optimum value corresponds to the smallest k such that $\vartheta_k(G) = n$, a parameter defined in Narasimhan 1989. Oliveira 2024 proved that, for any graph G , $\mu(G) \geq \frac{n}{\vartheta(G)}$.

We prove that there are no rank one feasible matrices for $\mu(G)$, a fact we use to prove that $\mu(G) > \frac{n}{\vartheta(G)}$ for all complete bipartite graphs with partitions of different sizes.

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On the Cost of Perfection in Wheel Graphs

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Keywords: Wheel graph; matching; perfect matching; edge weighting; cost of perfection

We investigate the relationship between maximum weight perfect matchings and maximum weight matchings in graphs. This problem is motivated by applications in computational geometry, particularly in the transformation of triangular meshes into quadrilateral meshes, where it is important to quantify the loss incurred when restricting to maximum weight perfect matchings.

To address this, we consider the parameter $\eta(G)$ [1], defined as

$$\eta(G) = \min_{w: E \rightarrow [0, \infty)} \frac{w(P^*(G))}{w(M^*(G))}.$$

where $G = (V, E)$ is a graph that admits a perfect matching and $w : E \rightarrow [0, \infty)$ is a nonnegative edge-weight function such that $w(e) > 0$ for at least one edge $e \in E(G)$. For any subset $E' \subseteq E$, we define $w(E') = \sum_{e \in E'} w(e)$. We denote by $M^*(G)$ a maximum weight matching and by $P^*(G)$ a maximum weight perfect matching. Thus, $w(M^*(G))$ and $w(P^*(G))$ represent the total weights of these matchings, respectively.

The study of $\eta(G)$ has predominantly focused on cubic graphs, mainly due to their connection with the dual graphs of triangular meshes. However, wheel graphs provide a natural and meaningful extension. Although they are not cubic, they share structural similarities with the duals of n -prism graphs, capturing both cyclic and radial features that are relevant in mesh configurations. Moreover, wheel graphs constitute a well-studied and fundamental class, making them a suitable setting for extending known results.

In particular, when n is odd, the wheel graph W_n , formed by a cycle of length n and a central vertex adjacent to all cycle vertices, admits a perfect matching, ensuring that $\eta(W_n)$ is well defined. We prove that for every odd wheel graph W_n with $n \geq 5$, $\eta(W_n) = \frac{1}{2}$.

Our proof is constructive. We begin by analyzing the perfect matchings of the graphs W_5 , W_7 , and W_9 , identifying a pattern that extends to arbitrary

odd wheels. We show that every radius of W_n belongs to exactly one perfect matching, implying that W_n has precisely n perfect matchings. Using this structure, we construct two perfect matchings whose cycle edges are disjoint. This decomposition enables us to determine the minimum value of the ratio defining $\eta(W_n)$. From this characterization, we establish that

$$\eta(W_n) = \frac{1}{2}.$$

Our proof is inspired by the general framework of Mazzuoccolo and Mella [4], but instead of using maximal matchings, it relies on structural properties of the wheel graph.

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Session 5: Total, equitable, and odd colorings

Adjacent Vertex Distinguishing Total Coloring in Complete 3- and 4-Partite Graphs <i>Juliano Silva Do Nascimento, Sheila Almeida and Atilio Gomes Luiz</i>	83
On the AVD-total chromatic number of cubic graphs <i>Mariana Cruz, Mauro Nigro and Diana Sasaki</i>	85
Equitable Total Coloring of Complete Multipartite Graphs <i>Nicolas Eusebio, Sheila Almeida and Mauro Nigro</i>	87
Equitable total chromatic number of graphs with universal vertex <i>Claudiney dos Santos, Arthur Baldoqui, Sheila de Almeida, Matheus Gonçalves, Mayara Omai, Celina de Figueiredo and Diana Sasaki</i>	89
Towards the $(2,1)$ -total number of complete equipartite graphs <i>M. M. Omai, C. N. Campos and Atilio G. Luiz</i>	91
On total b -chromatic number of power of paths <i>Clayton Santos de Araujo, Mauro Nigro, Luerbio Faria and Ricardo Pinheiro</i>	93
Odd coloring of graphs and hypergraphs <i>Carla Negri Lintzmayer, Nicolas de Melo Pedroso and Maycon Sambinelli</i>	95
Parity locally irregular decomposition of graphs <i>Carla Lintzmayer, Maycon Sambinelli and Silas Silva</i>	97

Adjacent Vertex Distinguishing Total Coloring in Complete 3- and 4-Partite Graphs

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Keywords: Adjacent Vertex Distinguishing Total Coloring; AVD Total Coloring; Complete Multipartite Graphs

Let $G = (V(G), E(G))$ be a simple graph. A *proper total coloring* of G is an assignment of colors to the vertices and edges of G such that adjacent or incident elements (vertices or edges) receive distinct colors. Given a proper total coloring φ of G , the *label* of a vertex $v \in V(G)$ is defined as $L(v) = \{\varphi(v)\} \cup \{\varphi(e) : e \text{ is incident to } v\}$.

A proper total coloring is called an *adjacent vertex distinguishing total coloring* if every pair of adjacent vertices receives distinct labels. The minimum number of colors required for such a coloring is called the *adjacent vertex distinguishing total chromatic number* of G , denoted by $\chi''_a(G)$.

Adjacent vertex distinguishing total colorings were introduced by Zhang et al. [4] and have been investigated for several graph classes, including complete multipartite graphs. For every simple graph G , we have $\chi''_a(G) \geq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of G . Moreover, if G contains adjacent vertices of maximum degree, then $\chi''_a(G) \geq \Delta(G) + 2$. Zhang et al. [4] conjectured that every simple graph satisfies $\chi''_a(G) \leq \Delta(G) + 3$.

Several results are known for complete multipartite graphs. Luiz, Campos and Mello [2] determined the adjacent vertex distinguishing total chromatic number of complete equipartite graphs of even order and established upper bounds for graphs of odd order. Additional results for complete tripartite graphs were obtained by Luiz [1] and Tiburcio [3]. In this work, we complete the characterization of complete tripartite graphs by proving that if G is a complete tripartite graph with partition $[A_1, A_2, A_3]$ satisfying $|A_1| < |A_2| = |A_3|$, then $\chi''_a(G) = \Delta(G) + 1$.

We also investigate adjacent vertex distinguishing total colorings of complete 4-partite graphs. Let G be a complete 4-partite graph with partition

$[A_1, A_2, A_3, A_4]$ such that $|A_1| \leq |A_2| \leq |A_3| \leq |A_4|$. Previous works already determined $\chi''_a(G)$ whenever G contains adjacent vertices of maximum degree. Thus, the remaining open cases occur when $|A_1| < |A_2|$. Among these graphs, the case in which all part sizes are distinct is already known. Therefore, the remaining open cases can be divided into these three subclasses: Case 1) $|A_2| < |A_3| = |A_4|$; Case 2) $|A_2| = |A_3| = |A_4|$; and Case 3) $|A_2| = |A_3| < |A_4|$.

In this work, we completely solve Case 1 by proving that if $|A_1| < |A_2| < |A_3| = |A_4|$, then $\chi''_a(G) = \Delta(G) + 1$. We also obtain substantial progress on Case 2 by proving that if $|A_1| + 1 < |A_2| = |A_3| = |A_4|$, then $\chi''_a(G) = \Delta(G) + 1$; and also that if $|A_1| + 1 = |A_2| = |A_3| = |A_4|$ and $|A_4|$ is even, then $\chi''_a(G) = \Delta(G) + 1$. Consequently, the only remaining unresolved situation in Case 2 occurs when $|A_1| + 1 = |A_2| = |A_3| = |A_4|$ and $|A_2|$ is odd. Finally, for Case 3, we prove that if $|A_1| + 1 < |A_2| = |A_3| < |A_4|$, then $\chi''_a(G) = \Delta(G) + 1$. Thus, the only remaining unresolved situation in Case 3 occurs when $|A_1| + 1 = |A_2| = |A_3| < |A_4|$.

Our proofs are constructive and are based on decomposition techniques involving suitable matching removals, and edge colorings of auxiliary graphs. In several cases, the constructions produce auxiliary graphs whose cores are acyclic, allowing the application of classical results on optimal edge colorings.

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On the AVD-total chromatic number of cubic graphs

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Keywords: cubic graphs; total-coloring; adjacent-vertex-distinguishing

An *AVD- k -total coloring* of a simple graph G is a mapping $\pi : V(G) \cup E(G) \rightarrow \{1, \dots, k\}$, with $k \geq 1$ such that: for each pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$, $\pi(x) \neq \pi(y)$; and for each pair of adjacent vertices $x, y \in V(G)$, sets $C_\pi(x) : \{\pi(x)\} \cup \{\pi(xv) \mid xv \in E(G) \text{ and } v \in V(G)\}$ and $C_\pi(y) : \{\pi(y)\} \cup \{\pi(yv) \mid yv \in E(G) \text{ and } v \in V(G)\}$ are distinct. For a vertex $u \in V(G)$, the *complementary* of the set $C_\pi(u)$ is defined as $\overline{C_\pi(u)} = \{1, \dots, k\} \setminus C_\pi(u)$. The *AVD-total chromatic number*, denoted by $\chi''_a(G)$, is the smallest k for which G admits an AVD- k -total-coloring. In 2005, Zhang et al. [5] conjectured that any graph G has $\chi''_a(G) \leq \Delta + 3$, where Δ is the maximum degree of G and this conjecture is known as AVD-Total Coloring Conjecture (AVD-TCC). Hulgan [3] presented a concise proof for AVD-TCC and stated the Conjecture 1.

Conjecture 1 ([3]). *Every graph G with $\Delta = 3$ has $\chi''_a(G) \leq 5$.*

Luiz et al. [4] proved the Hulgan's conjecture for graphs with $\Delta = 3$, where vertices with maximum degree are not adjacent, and for infinite families of bridgeless cubic graphs known as snarks. The generation of cubic graphs can be considered a benchmark problem in structure enumeration. In 2011, Brinkmann et al. [1] described a new algorithm for the efficient generation of all non-isomorphic connected cubic graphs, which is more than 4 times faster than previous generators.

A key ingredient of this generation algorithm is the definition of expansion operations. Historically, the graphs obtained by the three primary expansions, according to subfigures 1a, 1b, and 1c, denoted here by $E_a(G, uv)$, $E_b(G, uv)$, and $E_c(G, uv, wz)$, respectively, have been used to recursively construct all non-isomorphic cubic graphs from smaller graphs, called *prime graphs* (see Figure 1). Together with the *insertion edge* operation, these expansions generate every cubic graph. In this work, we study the conditions

under which these expansions preserve the AVD-total chromatic number of cubic graphs (see Theorem 1).

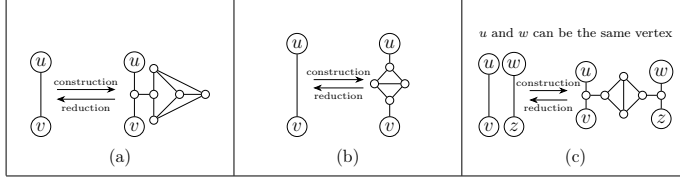


Figure 1: Expansion operations defined by de Vries [2].

Theorem 1. Let G be a cubic graph with an AVD-5-total coloring π and $uv, uz \in E(G)$.

1. If $\pi(uv) = i$ and $\overline{C_\pi(w)} \neq \{i\}$ for all $w \in N(u) \cup N(v)$, then $E_a(G, uv)$ has an AVD-5-total coloring.
2. Then $E_b(G, uv)$ has an AVD-5-total coloring.
3. If $\pi(uv) = i$ and $\overline{C_\pi(z)} = \{i\}$, then $E_c(G, uv, uz)$ has an AVD-5-total coloring.

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Equitable Total Coloring of Complete Multipartite Graphs

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Keywords: equitable total coloring; total coloring; complete multipartite graphs; tripartite graphs

A *proper total coloring* of a graph G is an assignment of colors to the vertices and edges of G , such that any two adjacent or incident elements (vertices and/or edges) receive distinct colors. The minimum number of colors required for a total coloring of G is the *total chromatic number*, denoted by $\chi''(G)$. By definition, $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G . The *Total Coloring Conjecture*, posed by Behzad [1] and Vizing [5], states that for any simple graph G , $\chi''(G) \leq \Delta(G) + 2$.

An *equitable total coloring* of G is a proper total coloring where the cardinality of any two color classes differs by at most one. The minimum number of colors required for an equitable total coloring of G is the *equitable total chromatic number*, denoted by $\chi_e''(G)$. In 1994, Fu [3] proposed the *Equitable Total Coloring Conjecture* (ETCC), which asserts that every simple graph G admits an equitable total coloring with k colors for every $k \geq \max\{\chi''(G), \Delta(G) + 2\}$.

A *complete multipartite graph* is a graph whose vertex set can be partitioned into r independent sets (parts), such that every vertex in a part is adjacent to all vertices in the other parts. If $r = 2$, the graph is a *complete bipartite graph* K_{p_1, p_2} , where p_i represents the size of each part. Similarly, when $r = 3$, the graph is a *complete tripartite graph*, K_{p_1, p_2, p_3} . A complete multipartite graph with r parts, each of size p , is called a *balanced complete multipartite graph*, denoted by $K_{r \times p}$.

The *equitable total chromatic number* for complete multipartite graphs has been determined progressively in the literature. Sasaki, Silva, and Dantas [2] established that $\chi_e''(K_{p_1, p_2, p_3}) = \Delta(K_{p_1, p_2, p_3}) + 1$ for the following

classes: (i) $p_1 < p_2 = p_3$; (ii) $p_1 = p_2$ and $p_3 \geq (p_2)^2$ if $p_2 \neq 1$, or $p_3 \geq 2$ if $p_2 = 1$; and (iii) $p_1 < p_2$ and $p_3 \geq (p_2)^2$. In 2021, Sasaki, Silva, and Dantas [4] completely determined the equitable total chromatic number of balanced complete multipartite graphs $K_{r \times p}$.

In this work, we extend the structural properties of balanced graphs to a non-balanced configuration with an odd number of parts. We prove that if $K_{p_1, p_2, \dots, p_r}$ is a complete multipartite graph with r odd and $p_1 < p_2 = \dots = p_r$, then $\chi_e''(K_{p_1, p_2, \dots, p_r}) = \Delta(K_{p_1, p_2, \dots, p_r}) + 1$. We also extend the results of Sasaki, Silva, and Dantas [2] by proving that $\chi_e''(K_{p_1, p_2, p_3}) = \Delta(K_{p_1, p_2, p_3}) + 1$ for complete tripartite graphs satisfying $p_1 < p_2 \leq p_3$. Unlike previous matrix-based approaches, our proof for tripartite graphs combines equitable edge colorings with the removal of a suitably chosen matching and the construction of an auxiliary graph through the addition of a temporary vertex. More precisely, we transform the equitable total coloring problem into a restricted equitable edge-coloring problem on an auxiliary graph whose maximum-degree vertices induce an edgeless subgraph. This allows the application of classical edge-coloring results to obtain an optimal equitable total coloring for the original complete tripartite graph. Moreover, the proposed strategy appears to be promising for the remaining open cases of complete tripartite graphs.

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Equitable total chromatic number of graphs with universal vertex

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Keywords: equitable total coloring; universal vertex; equitable total chromatic number

Let $G = (V(G), E(G))$ be a simple graph of order $n = |V(G)|$ and let $\Delta(G)$ denote the maximum degree of G . A vertex $v \in V(G)$ is called a *universal vertex* if it is adjacent to every vertex of G , i.e., $d(v) = n - 1$, in which $d(v)$ denotes the degree of v . A *proper total k -coloring* of G is an assignment of k colors to its vertices and edges, such that every pair of adjacent or incident elements receive distinct colors. The *total chromatic number* of G , denoted $\chi''(G)$, is the minimum k for which G admits a proper total k -coloring. The *Total Coloring Problem* consists of determining $\chi''(G)$, for a given graph G . Behzad [1] and Vizing [5] independently stated the *Total Coloring Conjecture* (TCC), which claims that $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$.

It is well-known that complete graphs K_n have $\chi''(K_n) = \Delta(K_n) + 1$, when n is odd; and $\chi''(K_n) = \Delta(K_n) + 2$, when n is even. So, when n is odd, any subgraph G of K_n with $\Delta(G) = \Delta(K_n)$ has $\chi''(G) = \Delta(G) + 1$. However, when n is even, there are subgraphs of K_n with maximum degree equal to $\Delta(K_n)$ that has $\chi''(G) = \Delta(G) + 1$. Hilton [3] proves that a graph G of even order and with universal vertex has $\chi''(G) = \Delta(G) + 1$ if and only if $\alpha'(\overline{G}) + |E(\overline{G})| \geq \frac{n}{2}$, where $\alpha'(\overline{G})$ is the maximum matching in $\overline{G}$. Thus, the total chromatic number of graphs with universal vertex is known.

Given a total k -coloring of a graph G , a *color class* C_i is the set of all elements assigned color c_i , for $1 \leq i \leq k$. A *proper equitable total k -coloring* of G is a proper total k -coloring in which the cardinalities of any two color

classes C_i and C_j , $1 \leq i < j \leq k$, differ by at most 1. The *equitable total chromatic number* of G , denoted $\chi_e''(G)$, is the minimum k for which G admits a proper equitable total k -coloring. The *Equitable Total Coloring Problem* consists of determining $\chi_e''(G)$, for a given graph G . The *Equitable Total Coloring Conjecture* (ETCC), posed by Fu [2], states that for every graph G , G has an equitable total k -coloring for each $k \geq \max\{\chi''(G), \Delta(G) + 2\}$.

Fu [2] showed that if G is a graph with universal vertex, then G admits an equitable total k -coloring for every $k \geq \chi''(G)$. The proposed proof relies on the assumption that if $\chi''(G) = \Delta(G) + 1$, then it is possible to construct a graph G^* from G by adding a new vertex v^* adjacent to all vertices of G , such that G^* has an edge-coloring with $\Delta(G^*) = \Delta(G) + 1$ colors. However, this proof fails when adding v^* causes the graph to be overfull. We present an example of a graph G with $\chi''(G) = \Delta(G) + 1$, for which the minimum number of colors to an edge-coloring of G^* is $\Delta(G^*) + 1 = \Delta(G) + 2$. In such cases, the equitable total-coloring proposed by Fu is not optimal.

Omai et al. [4] presented an optimal proper total coloring for graphs G with a universal vertex, when G has odd order and when G has even order and $|E(\overline{G})| \geq \frac{n}{2}$. The open cases are graphs G with even order and $|E(\overline{G})| < \frac{n}{2}$. We provide a polynomial solution for the Equitable Total Coloring Problem for graphs with universal vertex.

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Towards the $(2,1)$ -total number of complete equipartite graphs ¹

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Keywords: graph labelling; $(2,1)$ -total labelling; complete equipartite graphs

In the Frequency Channel Assignment Problem, integers (channels) are assigned to transmitters so that nearby transmitters receive channels that differ by at least two units, while transmitters at distance two must receive distinct channels. The objective is to minimize the range of integers used. Although this problem was introduced in 1980, it remains relevant today, particularly in the context of modern communication networks.

This problem led to the introduction of $L(2,1)$ -labellings, in which adjacent vertices must receive integers whose absolute difference is at least two and vertices at distance two receive distinct integers. Havet and Yu [2] later extended this concept to $(2,1)$ -total labellings, in which labels are assigned to both vertices and edges. A k - $(2,1)$ -total labelling of a simple graph G is a function $\pi: V(G) \cup E(G) \rightarrow \{0, 1, \dots, k\}$ for which the following properties hold: $|\pi(uv) - \pi(u)| \geq 2$ and $|\pi(uv) - \pi(v)| \geq 2$ for every edge uv ; $\pi(uv) \neq \pi(uw)$ for adjacent edges; and $\pi(u) \neq \pi(v)$ for adjacent vertices. The least integer k for which G admits such a labelling is denoted by $\lambda_2^t(G)$ and is called the $(2,1)$ -total number of G .

The problem of determining the $(2,1)$ -total number of a graph is, in general, computationally hard. For bipartite graphs G , Havet and Thomassé [1] showed that $\lambda_2^t(G) \in \{\Delta(G) + 1, \Delta(G) + 2\}$, so the problem reduces to deciding whether $\lambda_2^t(G) = \Delta(G) + 1$. They further established a complete complexity dichotomy for this decision problem on bipartite graphs: it is polynomial-time solvable when $\Delta(G) = 3$, and NP-complete for $\Delta(G) \geq 4$. For general graphs, the problem remains NP-complete even when $\Delta(G) = 3$. In spite of this hardness, the $(2,1)$ -total number has been determined for

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a number of classes of graphs such as complete graphs, near-ladder graphs, generalized Petersen graphs, among others.

In their original work, Havet and Yu [2] proved that $\lambda_2^t(G) \leq 6$ for graphs with $\Delta(G) \leq 3$, and that $\lambda_2^t(K_4) = 6$. The authors also proved that $\lambda_2^t(G) \geq \Delta(G) + 2$ for regular graphs, and that $\lambda_2^t(G) \leq 2\Delta(G)$ for graphs with $\Delta(G) \geq 2$. Additionally, Havet and Yu conjectured that $\lambda_2^t(G) \leq \Delta(G) + 3$ for every graph G with $\Delta(G) > 3$, tightening their upper bound of $2\Delta(G)$.

Motivated by this conjecture, we investigate $(2, 1)$ -total labellings on *complete equipartite graphs* $K_{r \times n}$, a class that generalizes complete bipartite graphs by allowing r equal parts of n vertices each. For $r \geq 3$, these graphs are regular and non-bipartite, a case for which the complexity of determining the $(2, 1)$ -total number remains open. For a graph $K_{r \times n}$, if $r = 1$, then $\lambda_2^t(K_{1 \times n}) = 0$ since $E(K_{1 \times n}) = \emptyset$. Moreover, if $r = 2$, graph $K_{2 \times n}$ is a complete bipartite graph, for which it is known that $\lambda_2^t(K_{2 \times n}) = \Delta(K_{2 \times n}) + 2$. On the other hand, if $n = 1$, $K_{r \times 1}$ is isomorphic to the complete graph K_r , for which the $(2, 1)$ -total number is also known. Therefore, we focus on complete equipartite graphs $K_{r \times n}$ with $r \geq 3$ and $n \geq 2$.

We show that Havet and Yu's Conjecture holds for $K_{r \times n}$ with $r \geq 3$ and $n \geq 2$ in three of the four cases arising from the parity of r and n . Specifically, we prove that $\lambda_2^t(K_{r \times n}) \leq \Delta(K_{r \times n}) + 3$ whenever r is odd or n is even, by constructing explicit $(\Delta(K_{r \times n}) + 3)$ - $(2, 1)$ -total labellings. For the remaining case, in which r is even and n is odd, we establish the upper bound $\lambda_2^t(K_{r \times n}) \leq \Delta(K_{r \times n}) + 6$. Despite these upper bounds, our computational experiments on 27 instances of $K_{r \times n}$, obtained via an integer linear programming formulation, found a $(\Delta(K_{r \times n}) + 2)$ -labelling in every case. Thus, all tested instances satisfy $\lambda_2^t(K_{r \times n}) = \Delta(K_{r \times n}) + 2$, suggesting that our upper bounds, although rigorously established, are unlikely to be tight. These observations motivate the following conjecture.

Conjecture 1 *For every complete equipartite graph $K_{r \times n}$ with $r \geq 3$ and $n \geq 2$, $\lambda_2^t(K_{r \times n}) = \Delta(K_{r \times n}) + 2$.*

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On total b -chromatic number of power of paths

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Keywords: total b -chromatic coloring; total coloring; path power.

A k -total coloring of a graph G is an assignment of k colors to the vertices and edges of G , such that no two adjacent elements receive the same color. The total chromatic number of G , denoted by $\chi''(G)$, is the smallest integer k for which G admits a k -total coloring. The total neighbourhood of x , denoted by $N_t(x)$, is defined as follows: $N_t(x) = \{y \in V \cup E : y \text{ is adjacent to } x\}$. The total degree of an element $x \in V \cup E$, denoted by $d_t(x)$, is $|N_t(x)|$. Therefore, the total degree of a vertex $v \in V(G)$ is given by $d_t(v) = 2d(v)$, and the total degree of an edge $e \in E$, where $e = (u, v)$, is given by $d_t(e) = d(u) + d(v)$. Given a graph G and a k -total colouring of G , $\mathbf{c} : V \cup E \rightarrow \{1, \dots, k\}$ we say that an element $x \in V \cup E$ of colour $i \in \{1, \dots, k\}$ is a total b -chromatic element if $\mathbf{c}(N_t(x)) = \{1, \dots, k\} \setminus \{i\}$. A k -total b -chromatic coloring of a graph G is a k -total coloring $\mathbf{c}$ such that there is a total b -chromatic element x of color c for every $c \in \{1, \dots, k\}$. The total b -chromatic number of G , denoted by $\chi''_b(G)$, is defined as the maximum integer k for which there exists a k -total b -chromatic coloring of G . In 2025, Granada and Manlove [1] introduced and studied the total b -chromatic number of graphs. The authors established the NP-completeness of the problem in general graphs, and they produced a polynomial-time algorithm for path, star, and caterpillar graphs. A natural upper bound for total b -chromatic number is $\chi''_b(G) \leq 2\Delta + 1$. Let G be a graph and $L = \{x \in V \cup E : d_t(x) < 2\Delta\}$. In this paper, we show that if $|L| < 2\Delta + 1$, then $\chi''_b(G) \leq 2\Delta$. Furthermore, we present our results on lower and upper bounds for the power p of paths P_n^p . In order to illustrate the technique, we sketch here a proof that $\chi''_b(P_4^2) = 5$. In Figure 1 (a) we show a 5-total b -coloring for P_4^2 . In Figures 1 (b), (c) and (d) we consider the cases of a possible 6-total b -coloring what we prove to be not possible. For this purpose, we classify 6 elements to be the elements, one in each color adjacent to all elements with different colors. Since there are 5

edges and 4 vertices, and vertices a and c are adjacent to only 4 elements, the vertices belonging to the set with the 6 elements only can be b and d . Hence, we have only possible either 4 edges and 2 vertices, or 5 edges and 1 vertex. Therefore, the case with 4 edges and 2 vertices is reduced to either contain or not contain the central edge bd .

The case in Figure 1 (b) shows the case with 4 edges and 2 vertices containing the central edge. In this case, element dc requires that vertex c , or edge cb be assigned with color 2, however they can not because element b is assigned with color 2. The case in Figure 1 (c) shows the case with 4 edges and 2 vertices not containing central edge. In this case, element dc requires that edge db be assigned with color 2, however it can not because element b is assigned with color 2. The case in Figure 1 (d) shows the case with 5 edges and 1 vertex among b, d . Say d . In this case, element cd requires color 2 assigned to vertex c . Hence, element ab requires colors 4 and 6 assigned to vertices a or b , and element d requires color 3 assigned to vertices a or b . A contradiction.

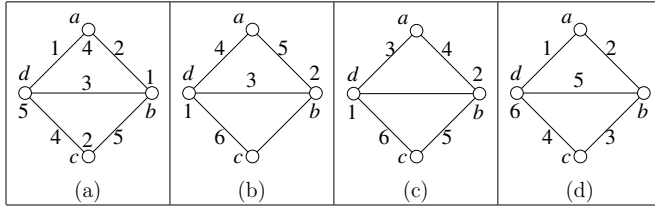


Figure 1: 5- b -coloring for P_4^2 in (a). Three cases for the six elements adjacent all remaining colors. Case in (b) where the 6 elements on colors 1, 2, 3, 4, 5, 6 are 4 edges, containing the central edge, and 2 vertices. Case in (c) where the 6 elements on colors 1, 2, 3, 4, 5, 6 are 4 edges, not containing the central edge, and 2 vertices. Case in (d) where the 6 elements on colors 1, 2, 3, 4, 5, 6 are 5 edges and 1 vertex.

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Odd coloring of graphs and hypergraphs

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Keywords: Conflict-free; Odd coloring; Bipartite graph

A coloring of the vertices of a hypergraph $\mathcal{H}$ is called **conflict-free** if, for every hyperedge $f \in E(\mathcal{H})$, there exists a color that occurs in only one vertex of f . The coloring is called **odd** if, for every hyperedge $f \in E(\mathcal{H})$, there exists a color that occurs an odd number of times among the vertices of f . The **conflict-free chromatic number** (resp. **odd chromatic number**) of a hypergraph $\mathcal{H}$, denoted by $\chi_{\text{cf}}(\mathcal{H})$ (resp. $\chi_o(\mathcal{H})$), is the smallest cardinality of a conflict-free (resp. odd) coloring for $\mathcal{H}$.

The study of conflict-free coloring was introduced by Even, Lotker, Ron, and Smorodinsky [4] considering geometric hypergraphs and motivated by a frequency assignment problem for cellular networks. Pach and Tardos [5] proved that, for every hypergraph $\mathcal{H}$ such that every hyperedge $f \in E(\mathcal{H})$ has size at least $2t - 1$ and intersects at most Γ hyperedges, it follows that $\chi_{\text{cf}}(\mathcal{H}) \in O(t\Gamma^{1/t} \log \Gamma)$. The study of odd coloring was introduced by Cheilaris *et al.* [2] as a weakening of conflict-free coloring. Note that, for every hypergraph $\mathcal{H}$, it follows that $\chi_o(\mathcal{H}) \leq \chi_{\text{cf}}(\mathcal{H})$, since a conflict-free coloring of $\mathcal{H}$ is also an odd coloring of $\mathcal{H}$. Therefore, we have $\chi_o(\mathcal{H}) \in O(t\Gamma^{1/t} \log \Gamma)$. We prove the following result.

Theorem 1 *Let $\mathcal{H}$ be a hypergraph, $t \in \mathbb{N}$ and $E \subseteq E(\mathcal{H})$ be the set of hyperedges of $\mathcal{H}$ with even cardinality. If every hyperedge $f \in E$ has at least $2t$ vertices and intersects at most Γ hyperedges of E , then $\chi_o(\mathcal{H}) \in O(t\Gamma^{1/t})$.*

To prove Theorem 1, we use the Lovász Local Lemma to show that if $V(\mathcal{H})$ is colored randomly with at most $t(e(\Gamma + 1))^{1/t}$ colors, then with positive probability, every hyperedge $f \in E(\mathcal{H})$ contains at least one color appearing an odd number of times.

The concepts of conflict-free and odd coloring were adapted to graphs. In particular, we say that a proper coloring of the vertices of a graph G is **odd open-neighborhood** if, for every non-isolated vertex $v \in V(G)$, there exists

a color that occurs an odd number of times in $N(v)$. We denote by $\chi_o(G)$ the smallest cardinality of an odd open-neighborhood coloring for G . Caro, Petruševski, and Škrekovski [1] conjectured that if G is a connected graph with $\Delta(G) \geq 3$, then $\chi_o(G) \leq \Delta(G) + 1$. Moreover, this bound is tight. To see this, let K'_k be the bipartite graph formed by subdividing every edge of the complete graph K_k with a new vertex. Since each pair of original vertices in $V(K_k)$ form a neighbourhood of a new vertex in K'_k , they must receive distinct colors in any odd open-neighborhood coloring. Thus, $\chi_o(K'_k) \geq k = \Delta(K'_k) + 1$. In the same paper, Caro, Petruševski, and Škrekovski [1] showed that $\chi_o(G) \leq 2\Delta(G)$, if G is a connected graph such that $G \neq C_5$. Dai, Ouyang, and Pirot [3] showed that $\chi_o(G) \leq \Delta(G) + O(\ln \Delta(G))$, thus proving that the conjecture is asymptotically true. Our next theorem shows that we can prove a better asymptotic bound under some conditions.

Theorem 2 *Let G be a graph and $t \in \mathbb{N}$. If $d(v) \geq 2t$ for every $v \in V(G)$ with even degree, then $\chi_o(G) \in O(\chi(G)\Delta(G)^{2/t})$.*

We prove Theorem 2 by showing that $\chi_o(G) \leq \chi(G)\chi_o(\mathcal{H}_G)$, where $\mathcal{H}_G$ is the neighborhood hypergraph of G , defined by $V(\mathcal{H}_G) = V(G)$ and $E(\mathcal{H}_G) = \{N_G(v) : v \in V(G)\}$. This also yields the following corollary.

Corollary 3 *Let G be a graph with $\chi(G) \in O(1)$. If $\delta(G) \geq 7$, then $\chi_o(G) \in O(\sqrt{\Delta(G)})$. Moreover, if $\delta(G) \geq 5$, then $\chi_o(G) \in O(\Delta(G)^{2/3})$.*

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Parity locally irregular decomposition of graphs

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Keywords: graph; decomposition; edge-coloring

Given a vertex v of a graph G , we denote the degree of v in G by $d_G(v)$ and the set of edges in G by $E(G)$. The *size* of a graph G is the cardinality of $E(G)$. A collection of subgraphs $\{H_1, \dots, H_\ell\}$ of G is a *decomposition* of G if $\{E(H_1), \dots, E(H_\ell)\}$ is a partition of $E(G)$.

A graph G is *locally irregular* if, for any edge $uv \in E(G)$, it holds that $d_G(u) \neq d_G(v)$. We say that $\{H_1, \dots, H_k\}$ is a *locally irregular decomposition* of G if H_i is a locally irregular graph for all $i \in [k]$. The *locally irregular chromatic index* of a graph G , denoted by $\chi'_{irr}(G)$, is the smallest size of a locally irregular decomposition of G . Not all graphs have a locally irregular decomposition, but for those that do, it has been conjectured that $\chi'_{irr}(G) \leq 3$ at first. Later, a single counterexample where $\chi'_{irr}(G) = 4$ has been found, updating the conjecture to $\chi'_{irr}(G) \leq 4$ [5]. It has been proved that $\chi'_{irr}(G) \leq 7$ if G is bipartite [4], $\chi'_{irr}(G) \leq 6$ if G is bipartite and has even size [4], $\chi'_{irr}(G) \leq \min\{3d, 6(\lceil \log_2(d+1) \rceil + 1)\}$ if G is d -degenerate [1] and $\chi'_{irr}(G) \leq 3$ if G is a split-graph [3]. It also has been proved that $\chi'_{irr}(G) \leq 220$ for any decomposable graph [2].

In this work, we define a subset of locally irregular graphs: a graph G is *parity locally irregular (PLI)* if, for any edge $uv \in E(G)$, it holds that $d_G(u) \neq d_G(v) \pmod{2}$. A decomposition $\{H_1, \dots, H_k\}$ is a *PLI-decomposition* of G if H_i is a PLI graph for all $i \in [k]$. The *PLI chromatic index* of a graph G , denoted by $\chi'_{PLI}(G)$, is the smallest size of a PLI-decomposition of G . Not all graphs have a PLI-decomposition, but for those that do, since PLI graphs are a subset of locally irregular graphs, we have that $\chi'_{irr}(G) \leq \chi'_{PLI}(G)$. This claim gives us a good reason to characterize and study PLI graphs. We aim to find $\chi'_{PLI}(G)$ for some classes of graphs and try to understand how far $\chi'_{PLI}(G)$ is from $\chi'_{irr}(G)$ in some cases.

Here we show our main results obtained so far.

Theorem 1 *A graph G is PLI-decomposable if, and only if, the size of all components of G is even.*

Theorem 2 *Let G be a PLI-decomposable graph. The following statements are true:*

1. *if G is bipartite, then $\chi'_{PLI}(G) \leq 8$.*
2. *if G is a tree, then $\chi'_{PLI}(G) \leq 3$.*
3. *if G is a cactus graph, then $\chi'_{PLI}(G) \leq 4$.*
4. *if G is d -degenerate, then $\chi'_{PLI}(G) \leq 8\lceil \log_2(d+1) \rceil + 1$.*
5. *if G is claw-free subcubic, then $\chi'_{PLI}(G) \leq 4$.*
6. *if G is isomorphic to K_n , where $n = 4k$, then $\chi'_{PLI}(G) \leq k + 3$.*
7. *if G is isomorphic to K_n , where $n = 4k + 1$, then $\chi'_{PLI}(G) \leq k + 4$.*

Theorem 3 *Let G be a PLI-decomposable graph. Then $\chi'_{PLI}(G) \geq \lceil \log_2 \chi(G) \rceil$.*

Using the computer, we calculated that $\chi'_{PLI}(K_4) = 3$, $\chi'_{PLI}(K_5) = \chi'_{PLI}(K_8) = 4$ and $\chi'_{PLI}(K_9) = \chi'_{PLI}(K_{12}) = \chi'_{PLI}(K_{13}) = \chi'_{PLI}(K_{16}) = 5$ and also calculated that $\chi'_{PLI}(G) \leq 5$ for all graphs with number of vertices between 1 and 9, and $\chi'_{PLI}(G) = 5$ only if G is isomorphic to K_9 .

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Session 6: Structural classes, digraphs, and convexity

Balanced distance-hereditary graphs <i>Lucía Busolini, Guillermo Durán and Martín Safe</i>	100
Intersection graphs characterizations <i>Marina Groshaus and Gabriel Dantas</i>	102
Sobre atribuição de papéis em algumas subclasses de grafos cordais <i>Gustavo Medeiros and Julliano Nascimento</i>	104
Atribuição de papéis em produto corona de grafos analisando vértices gêmeos <i>Jarlilson Guajajara and Julliano Rosa Nascimento</i>	106
Study of Carathéodory Number in Hypercubes <i>Braulio Silva, Erika Coelho, Hebert Coelho and Simone Dantas</i>	108
Strongly chordal digraphs, total domination, and tournaments with possible loops <i>Pavol Hell, César Hernández-Cruz and Jing Huang</i>	110
An Approach to Canonical ESD Labelings in Sparse Graphs <i>João Gabriel Moren, Danilo Artigas and Simone Dantas</i>	112
Triangle-free graphs whose line graph squares are $C_{\geq k}$ -free <i>Martín Safe and Martina Vergara</i>	114

Balanced distance-hereditary graphs

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Keywords: balanced graph; distance-hereditary graph; clique-matrix; forbidden induced subgraph; linear-time algorithm

A $\{0, 1\}$ -matrix is *balanced* [1] if it contains no square submatrix of odd order with exactly two ones in each row and in each column. Given an ordering $Q_1, \dots, Q_k$ of the maximal cliques of a graph G and an ordering $v_1, \dots, v_n$ of the vertices of G , a clique-matrix of G is the matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if $v_j \in Q_i$, and $a_{ij} = 0$ otherwise. A graph G is *balanced* [2] if any clique-matrix of G is balanced. Balanced graphs are perfect and clique-perfect [2]. They are also hereditary clique-Helly [1].

Bonomo et al. [3] proved that a graph is balanced if and only if it contains no generalized odd sun as an induced subgraph. However, this is not a minimal forbidden induced subgraph characterization, since some generalized odd suns contain other generalized odd suns as proper induced subgraphs. Although characterizations by minimal forbidden induced subgraphs are known for balanced graphs within several graph classes (see, e.g., [4]), finding a minimal forbidden induced subgraph characterization for the whole class of balanced graphs remains open.

A graph G is *distance-hereditary* [6] if, for every connected induced subgraph H of G and every pair of vertices $x, y \in V(H)$, the distance between x and y in H is the same as in G . Distance-hereditary graphs are perfect [6] and clique-perfect [7]. However, they are not all balanced. In fact, they are not even all hereditary clique-Helly, as witnessed by the graph $\overline{3K_2}$.

In this work, we prove that, restricted to the class of distance-hereditary graphs, balancedness is characterized by a single forbidden induced subgraph. More precisely, we prove that a distance-hereditary graph is balanced if and

only if it contains no induced $\overline{3K_2}$. The proof proceeds by induction along a one-vertex-extension tree, as defined by Chang et al. for dynamic programming on distance-hereditary graphs [5]. To track balancedness along the construction operations encoded by the tree, we rely on a proper 2-coloring criterion for subsets of vertices with respect to maximal cliques, derived from Berge's bicoloring characterization of balanced hypergraphs [1].

This characterization yields a linear-time algorithm for deciding balancedness within the class of distance-hereditary graphs. Given a graph G , the algorithm first recognizes whether G is distance-hereditary, returning a forbidden induced subgraph for the class if it is not. Otherwise, it processes a one-vertex-extension tree of G and either returns the six vertices of an induced $\overline{3K_2}$ as a negative certificate or concludes that G is balanced.

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Intersection graphs characterizations [‡]

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Keywords: biclique graph; clique graph; graph operators

Intersection graphs are graphs in which each vertex represents a set belonging to a family $\mathcal{F}$ of a certain type of sets. These sets may represent intervals, subgraphs of a graph, or other structures. Two vertices in an intersection graph are adjacent if and only if their corresponding sets have a non-empty intersection. More formally, a graph $G = (V, E)$ is said to be an intersection graph of a family $\mathcal{F}$ when there exists a bijection f between the vertices of G and the elements of $\mathcal{F}$, such that $f(u) \cap f(v) \neq \emptyset$ if and only if $uv \in E(G)$. A natural question is whether a graph is an intersection graph of a certain type of family.

The *clique graph* of a graph G is the intersection graph of the family of all cliques of G [2]. Roberts and Spencer [3] provided a characterization of clique graphs based on the existence of a family of complete subgraphs satisfying specific properties. This type of characterization is referred to as “Krausz-type”. Specifically, the characterization states that a graph G is the clique graph of some graph H if and only if there exists a family of complete subgraphs that covers all edges of G and satisfies the Helly property. The properties required for the family of complete subgraphs are typically checkable in polynomial time. Consequently, while such a characterization does not generally lead to a polynomial-time algorithm, it can still be useful for showing that the recognition problem belongs to NP .

Similarly, a Krausz-type characterization for biclique graphs (the intersection graphs of bicliques of a graph) was provided in [1]. In this case, the characterization is based on the existence of a family of complete subgraphs that covers all edges of G and satisfies the bipartite-Helly property together with some additional simple properties. Both characterizations are closely related to the one of the (clique/biclique) incidence matrix [1]. In particular, the Helly and bipartite-Helly properties, respectively, play an important role in guaranteeing that every clique/biclique of the graph is

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represented in the incidence matrix. The goal of this work is to find Krausz-type characterizations for the intersection graph of different families $\mathcal{F}$. We adapt the ideas used in the mentioned characterizations and present a general guideline for obtaining Krausz-type characterizations for intersection graphs of families $\mathcal{F}$ of subgraphs of a graph.

Guideline:

- Consider A as the incidence matrix of $\mathcal{F}$ and characterize A . Properties involving intersections of columns are desirable, in particular, variants of the Helly property.
- Consider the transpose A^T and let H be its induced graph and consider the family of complete subgraphs induced by each line. Consider the properties of A^T obtained previously and translate them into properties on the vertices of H and related to the complete subgraphs.
- A Krausz-type characterization for intersection graphs of $\mathcal{F}$ is then obtained: A graph H is an intersection graph of a family $\mathcal{F}$ of subgraphs of some graph G if and only if there exists a family of complete subgraphs of H that covers the edges and such vertices of H together with these complete subgraphs satisfies the properties found in previous step.

In this work, we apply the aforementioned steps to characterize intersection graphs when $\mathcal{F}$ is a family of complete subgraphs of fixed size, complete bipartite subgraphs of type $K_{k,\ell}$, and cycles of fixed size.

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Sobre atribuição de papéis em algumas subclasses de grafos cordais

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Keywords: Atribuição de papéis; Grafo cordal; Grafo fortemente cordal; Grafo split; Grafo k -sun

Em muitas áreas, grafos podem ser usados como ferramentas de modelagem para representar sistemas complexos, como redes de comunicação, interações sociais e estruturas biológicas. À medida que esses sistemas crescem em tamanho e complexidade, a análise de suas representações pode se tornar difícil do ponto de vista computacional. Desta forma, Everett e Borgatti [2] formalizaram o conceito de *atribuição de papéis*, onde uma r -atribuição de papéis de um grafo G é uma função de mapeamento $p : V(G) \rightarrow \{1, \dots, r\}$ de forma que dois vértices com o mesmo papel têm o mesmo conjunto de papéis nas suas adjacências.

Consideramos um grafo *fortemente cordal* se ele é cordal e todo ciclo par de tamanho pelo menos 6 tem uma corda ímpar, isto é, uma aresta conectando vértices a uma distância ímpar no ciclo. Um grafo $G(k, \ell)$ é um grafo k -sun, com uma clique K de tamanho k e um conjunto independente I com ℓ vértices, onde $K = \{v_1, v_2, \dots, v_k\}$ e $I = \{u_1, u_2, \dots, u_\ell\}$, de forma que para todo $u_i \in I$, $N_G(u_i) = \{v_i, v_{i+1}\}$, e, para $i = k$, $N_G(u_i) = \{v_i, v_1\}$. O grafo $G(k, \ell)$, é classificado como k -sun *total* se $\ell = k$, e k -sun *parcial* se $\ell = k - 1$. Observamos que o grafo $G(k, \ell)$ é cordal e *split*. Além disso, o grafo $G(k, k - 1)$ é fortemente cordal.

Para grafos fortemente cordais, o problema de atribuição de papéis é solúvel em tempo linear quando $r = 2$, e um grafo fortemente cordal admite uma r -atribuição de papéis, com todos os papéis adjacentes entre si, se, e somente se $\delta(G) + 1 = \omega(G) = r$ [3], sendo $\delta(G)$ e $\omega(G)$ o grau mínimo e a maior clique do grafo G , respectivamente. Para grafos *split*, o problema é solúvel em tempo constante para $r = 2$, em tempo linear para $r = 3$, mas se torna NP-completo para $r \geq 4$ [1].

Pelos resultados para grafos cordais, fortemente cordais e *split*, temos que $G(k, \ell)$ admite 2- e 3-atribuições de papéis. Mostramos, neste trabalho, que $G(k, k)$ admite uma 4-atribuição de papéis se, e somente se, $k = 3t$, com

$t \in \mathbb{N}^*$. Além disso, se $G(k, \ell)$, com $\ell \in \{k, k - 1\}$ admite r -atribuição de papéis p , sem laços, e com $p(K) \cap p(I) = \emptyset$, então $G(k', \ell')$ também admite, onde $k' = k + \ell t$ e $\ell' \in \{k', k' - 1\}$, para todo $t \in \mathbb{N}$. Por fim, para $k \geq 5$, o grafo k -sun parcial $G(k, k - 1)$ não admite $(k + 1)$ -atribuição de papéis, enquanto o grafo k -sun total $G(k, k)$ admite. Em contrate, o grafo k -sun total $G(k, k)$ não admite k -atribuição de papéis quando k é ímpar, diferente do k -sun parcial $G(k, k - 1)$, que sempre admite.

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Atribuição de papéis em produto corona de grafos analisando vértices gêmeos

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Palavras-chave: atribuição de papéis, produto corona, redes sociais

A atribuição de papéis em grafos, introduzida por Everett e Borgatti [1], formaliza a representação de uma rede por uma estrutura menor, preservando o comportamento local dos vértices. Consideramos grafos finitos, simples e sem laços como grafos de entrada, enquanto grafos de papéis podem possuir laços, mas não arestas múltiplas. Dado um grafo simples G e um grafo R , uma R -atribuição de papéis de G é uma função sobrejetora $p : V(G) \rightarrow V(R)$ tal que $p(N_G(v)) = N_R(p(v))$, para todo $v \in V(G)$. Investigamos atribuições de papéis no produto corona de grafos [2]. Dados grafos G e H , o *produto corona* $G \circ H$ é obtido a partir de uma cópia de G e, para cada vértice $v_i \in V(G)$, uma cópia H_i de H , tornando v_i adjacente a todos os vértices de H_i . Se $V(G) = v_1, \dots, v_n$ e $V(H) = u_1, \dots, u_m$, onde $[n] = 1, \dots, n$, então $V(G \circ H) = V(G) \cup u_{i,j} : i \in [n], j \in [m]$. Assim, G atua como o *núcleo* do produto, enquanto as cópias de H compõem sua *periferia*. O produto corona pode ser visto como caso particular do produto enraizado: se H^+ é obtido de H acrescentando um novo vértice universal r , escolhido como raiz, então $G \circ_r H^+$ é isomorfo a $G \circ H$. Nesse contexto, Komathi e Ragukumar [3] estudam colorações de papéis em produtos enraizados, especialmente para 2-atribuições em certas classes. Nosso objetivo é compreender como a estrutura *núcleo-periferia* influencia a formação de grafos de papéis. Dizemos que dois vértices x, y de um grafo F são *gêmeos abertos* se $N_F(x) = N_F(y)$, e são *gêmeos fechados* se $N_F[x] = N_F[y]$. Mostramos que essas equivalências, quando presentes em H , são preservadas em cada cópia H_i : se $u_j, u_{j'} \in V(H)$ possuem a mesma vizinhança aberta em H , então $u_{i,j}$ e $u_{i,j'}$ possuem a mesma vizinhança aberta em $G \circ H$, para todo $i \in [n]$. O mesmo vale para vizinhanças fechadas. Definimos $Q(F)$ como o quociente de F pela relação de gêmeos abertos: seus vértices são as classes $[x]$ de vértices com a mesma vizinhança aberta, e duas classes são adjacentes quando há uma aresta entre representantes dessas classes em F . Chamamos $Q(F)$ de *grafo de papéis canônico* associado à contração por gêmeos abertos. Se $V(H) \neq$

$\emptyset$, então cada vértice central v_i forma uma classe unitária em $Q(G \circ H)$, classes periféricas de cópias distintas não se confundem, e as classes dentro de cada H_i correspondem às classes de $Q(H)$. Consequentemente, $Q(G \circ H) \cong G \circ Q(H)$. Em particular, se $s(H) = |V(Q(H))|$, então $|V(Q(G \circ H))| = |V(G)|(s(H) + 1)$. Também estudamos uma direção inversa. Se um grafo de papéis R , possivelmente com laços, admite uma decomposição $R \simeq X \circ Y$, então existem grafos simples G e H tais que $G \circ H$ admite uma R -atribuição de papéis. Uma R -atribuição $p : V(G \circ H) \rightarrow V(R)$, com $R = X \circ Y$, é dita compatível com essa decomposição se $p(V(G)) \subseteq V(X)$ e, para cada cópia periférica H_i , vale $p(V(H_i)) \subseteq p(v_i) \times V(Y)$. Para realizações compatíveis, obtemos uma construção vértice-mínima em cada fator. Se $\lambda(X)$ e $\lambda(Y)$ denotam o número de vértices com laço em X e em Y , respectivamente, construímos grafos simples G e H tais que $|V(G)| = |V(X)| + \lambda(X)$ e $|V(H)| = |V(Y)| + \lambda(Y)$. A construção substitui cada papel sem laço por um único vértice e cada papel com laço por dois vértices adjacentes. A minimalidade decorre do fato de que, em grafos simples, um laço no grafo de papéis exige pelo menos dois vértices adjacentes recebendo o mesmo papel. Por fim, mostramos que a contração de classes no fator periférico do grafo de papéis preserva a existência de atribuições no produto corona. Mais precisamente, se S' é o quociente de S obtido pela identificação de vértices com a mesma vizinhança aberta, então toda $(R \circ S)$ -atribuição de papéis de $G \circ H$ induz uma $(R \circ S')$ -atribuição de papéis de $G \circ H$. O argumento usa a projeção canônica $\pi : V(S) \rightarrow V(S')$, estende essa projeção a $R \circ S$ e a compõe com a atribuição original.

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Study of Carathéodory Number in Hypercubes¹

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Keywords: graph convexity; Carathéodory sets; hypercubes; Cartesian product; Hamming graphs

In the P_3 -convexity, the family of paths considered is the set of all paths with exactly three vertices. Thus, a set $S \subseteq V(G)$ is P_3 -convex if, for every pair of vertices $u, v \in S$, every vertex lying on a path of length 2 between u and v also belongs to S . Given $S \subseteq V(G)$, the P_3 -convex hull of S , denoted by $H(S)$, is the smallest P_3 -convex set containing S . The P_3 -Carathéodory number $c(G)$ is the smallest integer c such that, for every set S and every vertex $v \in H(S)$, there exists a subset $S' \subseteq S$ with $|S'| \leq c$ and $v \in H(S')$.

Barbosa et al. [1] proved that deciding whether the P_3 -Carathéodory number $c(G)$ of a graph G is at least a given integer k is NP-complete, even when restricted to bipartite graphs. This result motivated the study of this parameter in special graph classes and graph products, including Cartesian products.

Cappelle, Coelho and Coelho [2] investigated the P_3 -Carathéodory number of Cartesian products of graphs, and more recently, Dratman, González and Grippo [3] obtained formulas and upper bounds for the Carathéodory number of Hamming graphs whose factors of the Cartesian product are complete graphs of order at least three.

In this work, we study this remaining case, with emphasis on hypercubes. We establish new bounds for the P_3 -Carathéodory number of Hamming graphs containing factors isomorphic to K_2 , and we construct large

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Carathéodory sets in Q_n . In particular, our results show that the P_3 -Carathéodory number of the n -dimensional hypercube grows at least linearly with n . These bounds extend the preliminary results presented by Silva et al. [4] at SBPO 2025 and complement the known formulas for denser Hamming graphs.

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Strongly chordal digraphs, total domination, and tournaments with possible loops

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Keywords: strongly chordal graphs; strongly chordal digraphs; total domination; simplicial orderings; tournaments; certifying recognition algorithm; forbidden subgraph characterization

A graph is *chordal* if it has a vertex ordering $v_1, v_2, \dots, v_n$ such that for all $i < j < k$, if $v_i v_j, v_i v_k \in E(G)$ then $v_j v_k \in E(G)$. Such an ordering is called a *simplicial ordering* of G . The existence of a simplicial ordering allows many intractable optimization problems such as the maximum clique and the minimum colouring problems to be solved in linear time. Nonetheless, the domination problem is NP-hard for chordal graphs, a fact which motivated Farber to introduce a smaller class of strongly chordal graphs [1] where the problem can be solved in linear time [2].

A graph G is *strongly chordal* if it has a vertex ordering $v_1, v_2, \dots, v_n$ such that for all $i < j$ and $k < \ell$, if $v_i \in N[v_k], v_i \in N[v_\ell], v_j \in N[v_k]$ then $v_j \in N[v_\ell]$. Such an ordering is called a *strong ordering* of G . Every strong ordering is a simplicial ordering.

The *augmented adjacency matrix* of a graph is obtained from its adjacency matrix by replacing each diagonal entry by 1. Given a square matrix M , a *symmetric ordering* of M is obtained from M by simultaneously permuting rows and columns of M . Both simplicial and strong orderings can be defined by absences of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, the Γ -matrix. We say that M is (*principal*) Γ -free if it does not contain the Γ -matrix as a (principal¹) submatrix. Thus, each (simplicial) strong ordering of G corresponds to a symmetric ordering of its augmented adjacency matrix that is (principal-) Γ -free.

These definitions have a natural extension to digraphs, even when loops are allowed (but not required). *Chordal* digraphs have been defined (in an equivalent way) and investigated in [3, 4]. We define a digraph D to be

¹Where the top-left entry 1 is on the main diagonal.

strongly chordal if its adjacency matrix has a symmetric ordering that is Γ -free. Such an ordering of M corresponds to a vertex ordering $v_1, v_2, \dots, v_n$ of D such that for all $i < j$ and $k < \ell$, if $(v_i, v_k), (v_i, v_\ell), (v_j, v_k) \in A(D)$ then $(v_j, v_\ell) \in A(D)$. We again call such a vertex ordering a *strong ordering*.

Somewhat surprisingly, Farber's idea of using strong orderings to find minimum dominating sets can be applied in this general setting. We say that a set S of vertices of a digraph D is *total dominating* if for every vertex v we have $(u, v) \in A(D)$ for some vertex u (possibly $u = v$) in S . The *total domination problem* on digraphs asks for a total dominating set of minimum size. When D is obtained from a graph G by replacing each edge by two arcs and adding all loops, the domination problem on G becomes the total domination problem on D ; thus the total domination problem on digraphs generalizes the classical domination problem for graphs. We extend Farber's algorithm to solve the total domination problem for strongly chordal digraphs in linear time. Simultaneously the algorithm also finds a largest set of vertices whose in-neighbourhoods are disjoint. We show that the size of the largest set with disjoint in-neighbourhoods equals the size of the smallest total dominating set, in any strongly chordal digraph. However, for the algorithm, we need to assume a strong ordering is given. It is an interesting problem to decide if a strong ordering of a digraph can be found in polynomial time, if one exists.

In this research, we restrict our attention to *tournaments* with possible loops, where we provide a forbidden subdigraph characterization, a structural description, and a linear time certifying recognition algorithm for strong chordality; this algorithm does compute a strong ordering if one exists.

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An Approach to Canonical ESD Labelings in Sparse Graphs

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Keywords: Graph Labeling; ESD Labeling; Unicyclic Graphs; Graph Theory.

Graph labeling problems are among the most extensively studied topics in graph theory. Early references on the subject include the works of Kotzig and Rosa [4] and Graham and Sloane [3]. A comprehensive survey of graph labelings can be found in Gallian [2]. Let $G = (V, E)$ be a simple undirected and connected graph with $|V| = n$ and $|E| = m$.

In [3], Graham and Sloane introduced the notion of a harmonious labeling of G . It is an injective function $f : V \rightarrow \mathbb{Z}_m$ that assigns to each edge $uv \in E$ the label

$$\phi(uv) = f(u) + f(v) \pmod{m},$$

with all induced edge labels being distinct. The authors conjectured that every tree admits a harmonious labeling, and this problem remains open.

In 2017, Tuza [6] introduced another labeling based on sums of vertex labels, called an edge-sum distinguishing (ESD) labeling. An ESD labeling of G is an injective function

$$f : V \rightarrow \{1, 2, \dots, s\}$$

that induces the edge labeling $\phi(uv) = f(u) + f(v)$, with all edge labels being distinct. When $s = n$, the labeling is called canonical.

Tuza also proposed a game version of the ESD labeling problem. In this version, given a graph and a set of labels $L = \{1, \dots, s\}$, two players alternately assign labels to the vertices. One player aims to construct an ESD labeling, whereas the other attempts to prevent this. This game version has been studied in recent works such as [1, 5].

Bok and Jedlickova [1] investigated canonical ESD labelings and proved that every tree admits such a labeling. They also established partial results for subclasses of unicyclic graphs, leaving open the problem of determining whether every unicyclic graph admits a canonical ESD labeling.

In this work, we address this open problem. We prove that every unicyclic graph admits a canonical ESD labeling and present a linear-time algorithm for constructing such a labeling. Furthermore, we establish new structural results related to the problem. Finally, motivated by the known bound that graphs with more than $2n - 3$ edges cannot admit canonical ESD labelings, we perform a computational search for fixed values of n and identify minimal non-canonical examples.

The construction is based on a linear-time adaptation of breadth-first search (BFS), whose correctness follows from structural properties of BFS trees.

Our exhaustive computational search identifies the smallest graphs, in terms of their number of vertices, that do not admit canonical ESD labelings. We find exactly eight non-canonical graphs on six vertices with $m = 2n - 3$, exactly two on six vertices with $m = 2n - 4$, and exactly two on eight vertices with $m = 2n - 5$.

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Triangle-free graphs whose line graph squares are $C_{\geq k}$ -free

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Keywords: line graph square; chordless cycle; triangle-free graph; minimal forbidden induced subgraph; deterministic finite automaton

Let $L(G)$ denote the line graph of G and let $L(G)^2$ denote its square, whose vertices are the edges of G , with two of them adjacent whenever their distance in $L(G)$ is at most 2. For $k \geq 4$, call a graph $C_{\geq k}$ -free if it has no chordless cycle of length at least k . We write $\mathcal{G}_{\geq k}$ for the class of all graphs G such that $L(G)^2$ is $C_{\geq k}$ -free.

Since induced matchings of G are exactly the independent sets of $L(G)^2$, the polynomial-time algorithm for MAXIMUM WEIGHT INDEPENDENT SET on $C_{\geq 5}$ -free graphs [1] and its quasipolynomial-time counterpart on $C_{\geq k}$ -free graphs [2] imply that the MAXIMUM WEIGHT INDUCED MATCHING problem is polynomial-time solvable in $\mathcal{G}_{\geq 5}$ and quasipolynomial-time solvable in $\mathcal{G}_{\geq k}$ for each fixed $k \geq 6$.

Scheidweiler and Wiederrecht [5] characterized, for each $\ell \geq 4$, the graphs G for which $L(G)^2$ avoids chordless cycles of length ℓ by forbidden induced subgraphs. Applying this result for all $\ell \geq k$ characterizes $\mathcal{G}_{\geq k}$. However, it does not directly lead to a *minimal* forbidden induced subgraph characterization because some obstructions contain others as proper induced subgraphs.

In this work we give, for every $k \geq 4$, an explicit minimal forbidden induced subgraph characterization of $\mathcal{G}_{\geq k}$ restricted to triangle-free graphs. The obstructions fall into three kinds. First, Hatzel and Wiederrecht [3] proved that every graph in $\mathcal{G}_{\geq k}$ is $C_{\geq \lceil 3k/2 \rceil}$ -free. Second, we prove that the circular ladder CL_k , obtained from two disjoint chordless cycles of length k by joining corresponding vertices, is a minimal forbidden induced subgraph of $\mathcal{G}_{\geq k}$ precisely when k is not a multiple of 4.

To describe the remaining obstructions, we extend to chordless cycles the finite-automaton approach developed in [4] for chordless paths in line graph squares. We view each chordless cycle in $L(G)^2$ as an L^2 -cycle of G ,

namely as a cyclic sequence of edges of G . We encode L^2 -cycles by strings over a finite alphabet, according to the adjacencies between endpoints of consecutive edges along the cycle. With each string s we associate a graph and a weight $w(s)$, the latter being the length of the chordless cycle in the line graph square represented by s .

We construct a deterministic finite automaton $\mathcal{A}_c^{\Delta\text{-free}}$ that recognizes strings that encode L^2 -cycles in triangle-free graphs and satisfies the following for each $k \geq 4$: if $\mathcal{L}(\mathcal{A}_c^{\Delta\text{-free}})$ denotes the language accepted by $\mathcal{A}_c^{\Delta\text{-free}}$, L^+ denotes the set of nonempty strings consisting only of letters L , and

$$\mathcal{F}_{\geq k}^{\Delta\text{-free}} = \{H : H \text{ is associated with } s \in \mathcal{L}(\mathcal{A}_c^{\Delta\text{-free}}) \setminus L^+ \text{ and } w(s) = k\},$$

then a triangle-free graph G belongs to $\mathcal{G}_{\geq k}$ if and only if G is $C_{\geq \lceil 3k/2 \rceil}$ -free, G is CL_k -free whenever k is not a multiple of 4, and G contains no induced subgraph in $\mathcal{F}_{\geq k}^{\Delta\text{-free}}$. Together, for each $k \geq 4$, these are precisely the minimal forbidden induced subgraphs for $\mathcal{G}_{\geq k}$ among triangle-free graphs.

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Session 7: Extremal combinatorics, Ramsey theory, and cages

Counting edge-colorings of a complete graph that avoid a rainbow K_4

Fabício S. Benevides, Josefran de O. Bastos and Rodrigo F. Ribeiro
117

Anti-Ramsey threshold for $K_{2,r}$ in random graphs

Juliane Lima, Walner Mendonça, Guilherme Mota and Henrique Stagni 119

Minimum degree threshold for covering bipartite graphs with monochromatic components

César Bispo, George Kontogeorgiou, Marcelo Lage, Guilherme Mota and Bruno Skarmeta 121

Matching extendability in the Cartesian product of complete graphs

A. A. Pereira and C. N. Campos 123

König-Egerváry Index under Vertex Coalescence

Vadim Levit and Eugen Mandrescu 125

Transversally Transitive Unmatching Edge Graphs

Edmilson Pereira da Cruz, Marina Groshaus and André Luiz Pires Guedes 127

Mixed cages

Claudia Marlene De la Cruz Torres and Miguel Pizaña 129

Counting edge-colorings of a complete graph that avoid a rainbow K_4

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Keywords: counting; edge-coloring; Container Method; density boosting.

An r -coloring of a graph G is any function $\varphi : E(G) \rightarrow \{1, \dots, r\}$. It is called *rainbow- K_k -free* if every copy of K_k in G has at least two edges of the same color. Let $\rho_{r,k}(G)$ be the number of rainbow- K_k -free r -colorings of G .

Theorem 1. *Fix an integer $r > 5^{10}$. For every real $\varepsilon > 0$, there exists $n_0 = n_0(r, \varepsilon)$ such that for all $n \geq n_0$, we have*

$$r^{\lfloor n^2/4 \rfloor + 1} \leq \rho_{r,4}(K_n) \leq r^{(1+\varepsilon)n^2/4}. \quad (1)$$

Let us discuss the history and motivation behind the problem of estimating $\rho_{r,k}(K_n)$. When $k = 3$, the value of $\rho_{r,3}(K_n)$ represents the number of *Gallai colorings* and was computed asymptotically in [1] and [2]. Trivially, every 2-coloring contains no rainbow K_3 . These articles have shown, independently, that almost all Gallai colorings of K_n , for n large, are 2-colorings.

These articles are at the crossroads of two trends: the study of (i) Gallai coloring and its variations, and (ii) counting colorings that avoid subgraphs colored in prescribed ways as in [3]. While the former is inspired by a seminal result of Gallai from 1967, the latter is inspired by a classical problem posed by Erdős and Rothschild in 1974, who asked for the maximum (over graphs with n vertices) number of 2-edge-colorings with no *monochromatic* copies of the K_k . Confirming their conjecture, Yuster (for $k = 3$), and Alon, Balogh, Keevash, and Sudakov (for $k \geq 4$) showed that this maximum is attained only by the Turán graph with $k - 1$ parts, which we denote by $T_{k-1}(n)$.

In the same spirit, Hoppen, Lefmann, and Odermann [6] asked which graphs maximize $\rho_{r,k}(G)$, for any given number of vertices. They show that for fixed $k \geq 3$, $r \geq \binom{k}{2}^{8k-4}$ and n large enough, we have $\rho_{r,k}(G) \leq r^{\text{ex}(n, K_k)}$, with equality only when $G = T_{k-1}(n)$. Here $\text{ex}(n, H)$ is the maximum number of edges in a graph with n vertices that does not contain H as subgraph.

While the lower bound, $\binom{k}{2}^{8k-4}$, on r is not optimal, the result does not hold when r is small compared to k . When $k = 4$, this bound was improved in [5] to $r \geq 12$, which is best possible (see Problem 8.3 in [4]).

While studying $\max\{\rho_{r,k}(G) : |V(G)| = n\}$ is closely related to the Erdős-Rothschild problem, the direct question of estimating $\rho_{r,k}(K_n)$ is just as natural, interesting and closer to the problem of counting Gallai colorings.

Based on the result for $k = 3$, one might have hoped that $\rho_{r,4}(K_n)$ would be near to the number of 5-colorings of K_n (since K_4 has 6 edges), which is approximately $\binom{r}{5} 5^{\binom{n}{2}}$; or to be related to $r^{\text{ex}(n, K_4)}$. Our result shows that this is not the case, for r larger than an absolute constant. In fact, the bounds in Theorem 1 are related to $r^{\text{ex}(n, K_4 - e)}$, where $K_4 - e$ is the graph obtained by removing an edge from K_4 . While the constant (5^{10}) is not optimized, equation (1) does not hold for $5 \leq r < 25$ as, in this range, $5^{\binom{n}{2}}$ is larger than the upper bound in (1).

Our proof uses the Container Method, as in [1], to group colorings into few templates. For templates that generate many colorings we use a boosting argument for the density of edges with only one color in their palette.

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Anti-Ramsey threshold for $K_{2,r}$ in random graphs

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Keywords: anti-Ramsey property; random graphs; complete bipartite graphs; rainbow edge-colouring

A graph G satisfies the *anti-Ramsey* property for a graph H , denoted by $G \xrightarrow{\text{ar}} H$, if for every proper edge-colouring of G there is a *rainbow* copy of H , that is, no two edges in H have the same colour. We are interested in the case where G is the binomial random graph $G(n, p)$. Even though Rödl and Ruciński determined the threshold for the Ramsey property for any graph H in 1995, the same is still not known for the anti-Ramsey property, even for some very simple graphs.

Kohayakawa, Konstadinidis, and the third author proved that $p = n^{-1/m_2(H)}$ is an upper bound for the anti-Ramsey threshold of any graph H [2]. In 2017, Nenadov, Person, Škorić, and Steger presented a framework that reduces the proof of the lower bound to a deterministic verification for strictly 2-balanced graphs. Using this framework as a basis, the threshold was established for cliques, cycles, and some other interesting families of graphs [1]. On the other hand, it is known that $n^{-1/m_2(H)}$ is not the threshold for all graphs. Very recently, Kuperwasser [3] showed that for any graph H with maximum 2-density $m_2(H) \geq 19$, the anti-Ramsey threshold is indeed $n^{-1/m_2(H)}$.

We proved that for complete bipartite graphs $K_{2,r}$ with $r \geq 4$, the anti-Ramsey threshold is given by $n^{-1/m(K_{2,3r-2})}$, which equals to $n^{-3r/(6r-4)}$, where $m(K_{2,3r-2})$ stands for the *maximum density* of $K_{2,3r-2}$.

Theorem 1. *Let $r \geq 4$ be an integer. Then the following holds:*

$$\lim_{n \rightarrow \infty} \mathbb{P}[G(n, p) \xrightarrow{\text{ar}} K_{2,r}] = \begin{cases} 0, & \text{if } p \ll n^{-3r/(6r-4)}; \\ 1, & \text{if } p \gg n^{-3r/(6r-4)}. \end{cases}$$

By a result of Bollobás and Thomason, $p = n^{-1/m(K_{2,3r-2})}$ is the threshold for $G(n, p)$ to contain a copy of $K_{2,3r-2}$. Therefore, we know that if $p \gg n^{-1/m(K_{2,3r-2})}$, then w.h.p. $G(n, p)$ contains a copy of $K_{2,3r-2}$. To prove the 1-statement of Theorem 1, it suffices to show that $K_{2,3r-2} \xrightarrow{\text{ar}} K_{2,r}$.

Lemma 2. *Let $r \geq 1$ be an integer. Then, $K_{2,3r-2} \xrightarrow{\text{ar}} K_{2,r}$.*

Proof. The result follows by induction on r . The statement clearly holds for $r = 1$, since any proper edge-colouring of $K_{2,1}$ is rainbow. Fix $r \geq 2$ and assume that $K_{2,3r-5} \xrightarrow{\text{ar}} K_{2,r-1}$. Let $G := K_{2,3r-2}$ be the bipartite graph with parts X and Y with $|X| = 2$ and $|Y| = 3r - 2$, and consider any proper edge-colouring of its edges. Since G contains a copy of $K_{2,3r-5}$ and $K_{2,3r-5} \xrightarrow{\text{ar}} K_{2,r-1}$, there is a rainbow copy of $H_A := K_{2,r-1}$ with parts X and $A \subseteq Y$ in G . Now, consider the subgraph $H_B := K_{2,r-1}$ of G , with parts X and $B := Y \setminus A$. Since H_A is coloured with $2r - 2$ colours and B has $2r - 1$ vertices, there is at least one vertex $b \in B$ such that the two edges incident to b have colours not used in H_A (note that no colour can be assigned to more than two edges of G). Therefore, b together with H_A gives the desired rainbow $K_{2,r}$. $\square$

It remains to prove the 0-statement, which is given below.

Lemma 3. *Let $r \geq 4$ be an integer. If $p \ll n^{-1/m(K_{2,3r-2})} = n^{-3r/(6r-4)}$, then $G(n, p) \not\xrightarrow{\text{ar}} K_{2,r}$.*

Basic idea of the proof of Lemma 3. By using arguments similar to those of [4], we are able to show that it is enough to prove that $G \not\xrightarrow{\text{ar}} K_{2,r}$ for every graph G with maximum density $m(G) < (6r - 4)/3r$. Given any such graph G , we can avoid rainbow copies of $K_{2,r}$ by colouring G as follows. First, we iteratively colour some edges that create no problem, like edges that belong to no copies of $K_{2,r}$. Then, we carefully colour the remaining edges, exploring the intersection between copies of $K_{2,r}$ to make sure we create no rainbow copies of $K_{2,r}$, which completes the proof. $\square$

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Minimum degree threshold for covering bipartite graphs with monochromatic components

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Keywords: Bipartite Graph; Ramsey Theory; Monochromatic Covering

The study of monochromatic coverings and partitions in edge-coloured graphs is a central topic in Ramsey theory [4, 6, 10]. Given an r -colouring of the edges of a graph G , let $\text{tc}_r(G)$ (resp. $\text{tp}_r(G)$) denote the minimum number of monochromatic connected components needed to cover (resp. partition) $V(G)$. A celebrated conjecture of Erdős, Gyárfás and Pyber [5] states that $\text{tp}_r(K_n) \leq r - 1$, while Gyárfás and Lehel conjectured that $\text{tc}_r(K_{n,m}) \leq 2r - 2$ [2, 3, 9].

Considerable attention has recently turned to graphs satisfying a minimum degree condition, a direction initiated by Bal and DeBiasio [1]. For two colours, Girão, Letzter and Sahasrabudhe [8] proved that $\delta(G) \geq (2n - 5)/3$ implies $\text{tp}_2(G) \leq 2$. The bipartite analogue presents distinct structural challenges: Fernández, Pavez-Signé and Stein [7] showed that any bipartite graph G with parts of size n and $\delta(G) \geq (13/16 + o(1))n$ satisfies $\text{tp}_2(G) \leq 3$. The number of components needed cannot be improved to $\text{tc}_2(G) \leq 2$, as one can see with a simple coloring of some suitable bipartite graph with parts of size n and minimum degree $n - 2$ requires 3 monochromatic components to be covered.

In this work, we establish an essentially sharp minimum degree condition under which every 2-coloured balanced bipartite graph admits a cover by three monochromatic components.

Theorem 1.1 *For every integer n , if G is a 2-coloured spanning subgraph of $K_{n,n}$ with $\delta(G) > 2n/3$, then $\text{tc}_2(G) \leq 3$.*

The minimum degree condition in Theorem 1.1 is essentially sharp, as we can colour a balanced bipartite graph G with minimum degree $\delta(G) = \lfloor (2n - 4)/3 \rfloor$, which requires 4 monochromatic components to be covered.

Theorem 1.2 *Let $n \geq 2$ be an integer. There exists a 2-coloured subgraph G of $K_{n,n}$ with $\delta(G) = \lfloor (2n - 4)/3 \rfloor$ such that $\text{tc}_2(G) \geq 4$.*

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Matching extendability in the Cartesian product of complete graphs

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Keywords: matching extendability; k -fully-extendable graphs; unlocked matching; Cartesian product; complete graphs

A matching M of a graph G *extends* to a perfect matching if some perfect matching of G contains M . A classical notion in matching theory is that of a k -*extendable* graph, introduced by Plummer [3] in 1980: a graph G is k -*extendable* if every matching of size k extends to a perfect matching of G . Since then, k -*extendability* has been investigated from several perspectives, including structural characterisations, degree conditions, criticality, connections with other graph parameters, behaviour under graph operations and results for specific graph classes [4].

Denote by $S(M)$ and $U(M)$ the sets of M -saturated and M -unsaturated vertices of G , respectively. A natural local obstruction to extendability arises from the neighbourhood of an unsaturated vertex: if some $u \in U(M)$ satisfies $N(u) \subseteq S(M)$, then u cannot be matched in any extension of M and, hence, M does not extend to a perfect matching. In this work, we study the extendability of matchings without this type of neighbourhood obstruction. More precisely, a vertex $v \in U(M)$ is *unlocked* if $N(v) \cap U(M) \neq \emptyset$, and the matching M is *unlocked* if every vertex in $U(M)$ is unlocked. We say that G is k -*fully-extendable* if every unlocked matching of size k in G extends to a perfect matching of G . Note that every k -*extendable* graph is also k -*fully-extendable*; however, the converse is not necessarily true.

This concept, which has appeared under different names in the literature, was recently unified and extended by the present authors [1] under the term k -*fully-extendability*. Results in this direction include characterisations for fixed values of k , as well as findings on specific graph families such as hypercubes, d -fold Cartesian product of cycles and Cartesian product of hypercubes with paths [1, 2, 5]. We broaden this line of research by investigating k -*fully-extendability* in the Cartesian product of graphs, focusing in particular on product of complete graphs.

We first establish a general obstruction for the Cartesian product, based

on two parameters: $\lambda(G)$ and $\sigma(G)$. For a graph G and a vertex $u \in V(G)$, let $\lambda_G(u) = \min\{2|M| : M \text{ is a matching of } G, N_G(u) \subseteq S(M) \text{ and } u \in U(M)\}$, with $\lambda_G(u) = \infty$ if no such matching exists. Define $\lambda(G) = \min\{\lambda_G(u) : u \in V(G)\}$. Also, let $\sigma(G) = \min\{|N_G(x) \cup N_G(y)| : \{x, y\} \subseteq V(G), xy \notin E(G) \text{ and } N_G(x) \cap N_G(y) \neq \emptyset\}$, with $\sigma(G) = \infty$ if no such pair exists.

We prove that if G and H are non-trivial connected graphs satisfying $\lambda(G) < \infty$ and $\sigma(H) < \infty$, then $G \square H$ is not k -fully-extendable for $k = \lambda(G) + \sigma(H) - 1$. We further characterise the connected graphs for which the condition $\lambda(G) < \infty$ does not hold: for $|V(G)| \geq 5$, we show that $\lambda(G) = \infty$ if and only if $G \cong K_n$ with n even or $G \cong K_{n,n}$. For these exceptional families, a separate construction shows that, when $|V(G)| \geq 6$ and $\sigma(H) < \infty$, the product $G \square H$ is not k -fully-extendable for $k = |V(G)| + \sigma(H) - 1$.

Cartesian product of complete graphs require a different approach since $\sigma(K_n) = \infty$ for every n . For this family, we obtain matching upper and lower bounds for k -fully-extendability. Let $5 \leq m \leq n$, with mn even, and define $w = 1$, if m and n are both even and $m < n$, and $w = 0$, otherwise. We construct a non-extendable unlocked matching in $K_m \square K_n$ of size $\lfloor \frac{3m+n-7}{2} \rfloor + w + 1$. Conversely, we prove that, for $10 \leq m \leq n$ and mn even, every unlocked matching of $K_m \square K_n$ of size at most $\lfloor \frac{3m+n-7}{2} \rfloor + w$ extends to a perfect matching. Thus, the upper bound is tight.

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König-Egerváry Index under Vertex Coalescence

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Keywords: independent set; matching; König-Egerváry; coalescence

Throughout this paper G is a finite, undirected, loopless graph without multiple edges, with vertex set $V(G)$ of cardinality $n(G) = |V(G)|$. The *independence number* $\alpha(G)$ is the size of a maximum independent set, and the *matching number* $\mu(G)$ is the size of a maximum matching of G . A vertex $v \in V(G)$ is α -critical if $\alpha(G - v) < \alpha(G)$, and is μ -critical provided $\mu(G - v) < \mu(G)$. The *König-Egerváry index* (or *König deficiency* [1]) of graph G is $\kappa(G) = n(G) - \alpha(G) - \mu(G)$ [5]. In general, $\kappa(G) \geq 0$. In particular, if $\kappa(G) = 0$, then G is a *König-Egerváry graph* (KE for short) [3, 4, 7]; if $\kappa(G) = 1$, then G is a *1-KE graph* [6]. Let G_1 and G_2 be two vertex-disjoint graphs. The graph $(G_1 \div G_2)(v, u)$ obtained from G_1 and G_2 by identifying $v \in V(G_1)$ and $u \in V(G_2)$ is the *vertex coalescence* of G_1 and G_2 via v and u [2]. This is a particular case of the clique coalescence operation, where two graphs are merged by identifying a complete subgraph present in both. In this paper, we present some necessary and/or sufficient conditions for a vertex coalescence of two graphs to be KE or 1-KE.

1 Results

Theorem 1 *Let $H = (G_1 \div G_2)(v, u)$ and $a = 1$, when at least one of v, u is α -critical and $a = 0$, otherwise; while $m = 1$, when both v, u are μ -critical and $m = 0$, otherwise. Then $\kappa(H) = \kappa(G_1) + \kappa(G_2) + a + m - 1$.*

It is worth mentioning that the coalescence of a non-KE graph and a KE graph may be a KE graph; e.g., the coalescence of K_3 and K_2 via any two vertices is a KE graph.

Corollary 2 *If G_1, G_2 are not KE, then for any pair v, u of vertices the graph $H = (G_1 \div G_2)(v, u)$ is not KE as well.*

Proof. Since G_1 and G_2 are not KE, we know that $\kappa(G_1) \geq 1, \kappa(G_2) \geq 1$. Further, in accordance with Theorem 1, we obtain

$$\begin{aligned}\alpha(H) + \mu(H) &= n(H) - \kappa(H) = n(H) - (\kappa(G_1) + \kappa(G_2) + a + m - 1) \leq \\ &= n(H) - (1 + 1 + a + m - 1) = n(H) - 1 - a - m < n(H),\end{aligned}$$

i.e., H is not a KE graph. ■

For instance, $(C_{2p+1} \div C_{2q+1})(v, u)$, $(C_{2p+1} \div K_n)(v, u)$, $(K_m \div K_n)(v, u)$, for $p, q \geq 1, m, n \geq 3$, are not König-Egerváry graphs.

Corollary 3 *Let $H = (G_1 \div G_2)(v, u)$, where G_1 and G_2 are KE graphs. Then H is either a KE graph or a 1-KE. Moreover, H is a 1-KE graph if and only if both v and u are μ -critical and at least one of v and u is α -critical.*

Corollary 4 *Let $H = (G_1 \div G_2)(v, u)$ and assume that none of v and u is α -critical, while at most one of v, u is μ -critical. Then the following are true:*

- (i) *if G_1 is 1-KE and G_2 is KE, then H is KE;*
- (ii) *if both G_1 and G_2 are 1-KE, then H is 1-KE.*

Problem 5 *Given an integer q , determine when the coalescence has the KE index equal to q .*

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Transversally Transitive Unmatching Edge Graphs [‡]

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Keywords: Biclique graph; Triangle-free graph; Perfect Matching; Comparability graph

A *biclique* is a maximal set of vertices that induces a complete bipartite subgraph. A biclique graph is the intersection graph of bicliques of some graph [3]. A graph is a *mutually included biclique graph* of some graph G if it is a spanning subgraph of the biclique graph of G such that its vertices share an edge if and only if the corresponding bicliques of G respectively admit the partitions (X_1, Y_1) and (X_2, Y_2) where $X_1 \subset X_2$ and $Y_2 \subset Y_1$ [2]. We call each element X and Y of a partition (X, Y) of a biclique a *biclique part* [1]. A graph is a TTUE (*Transversally Transitive Unmatching Edge*) graph if it admits a perfect matching M such that for every path (v_1, v_2, v_3, v_4) where $\{v_2, v_3\} \in M$ there is an edge $\{v_1, v_4\} \notin M$. A graph is a NIC (*Neighborhood Intersection Closed*) graph if the set of neighborhoods of all of its vertices is closed under intersection. A graph with a unique perfect matching is a UPM graph. For a graph G with a perfect matching M , we denote by G/M the graph obtained from G by contracting all edges in M .

In a previous work [1], we have presented a type of poset (*partially ordered set*) that describes the relation of inclusion between biclique parts of a K_3 -free (*triangle-free*) graph and called such poset a *skew-IIC* poset. Given any K_3 -free graph G and a skew-IIC poset $\mathcal{P}$ that describes the relation of inclusion between bicliques parts of G , we also have shown how to construct a K_3 -free graph from $\mathcal{P}$ such that its biclique graph is isomorphic to the biclique graph of G and that the class formed by such graphs is a proper subclass of K_3 -free graphs. In this work, we prove that this subclass is equivalent to K_3 -free $\cap$ TTUE $\cap$ NIC, which is also the same as K_3 -free $\cap$ UPM $\cap$ NIC. We show that a graph is a comparability graph if and only if it is isomorphic

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to G/M for some bipartite TTUE graph G with perfect matching M . We also show that a graph is a mutually included biclique graph of a bipartite graph if and only if it is isomorphic to G/M for some bipartite TTUE $\cap$ NIC graph G with perfect matching M .

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Mixed cages

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Keywords: Cage; Mixed Cage; Lower bound; Upper bound.

An (r, z, g) -mixed graph is a graph that is r -regular in edges, z -regular in arrows (both in-regularity and ex-regularity), and has a girth g . An (r, z, g) -mixed cage is an (r, z, g) -mixed graph of minimum order, denoted by $n(r, z, g)$.

A cycle in a mixed graph is a sequence of vertices $(x_0, x_1, \dots, x_n)$ where $x_0 = x_n$ and such that for all $i \in \{0, 1, \dots, n-1\}$ we have either $x_i x_{i+1}$ (edge) or $x_i \rightarrow x_{i+1}$ (arrow in this direction). Such a cycle is said to be of length n (necessarily $n \geq 3$). The girth of a mixed graph G is the length of the shortest cycle of G . Mixed cages were introduced by Araujo, Hernández and Montellano in [1], who established conditions for existence and provided a lower bound for the order of $(r, 1, g)$ -mixed cages, which we refer to as the AHM bound.

In this work, we extend the AHM bound for $z \geq 1$, obtaining the extended AHM bound $eAHM(r, z, g)$. We apply this bound to the study of mixed cages of girth 4.

Theorem 0.1. *We have $n(1, 2, 4) = 8$. Also, for $(r, z) \neq (1, 2)$, we get:*

$$\begin{aligned} r \leq 2z - 2 &\implies n(r, z, 4) \geq 3r + 2z + 2, \\ r \geq 2z - 2 &\implies n(r, z, 4) = 2r + 4z. \end{aligned}$$

These results use an upper bound for the girth-four case that results from a construction given by Jajcayová and Jajcay in [3].

In [2] we presented a code that constructs (normal) cages using exhaustive searches with symmetry reductions; this code was adapted in this work for the construction of mixed cages. The order of the $(1, z, 4)$ -mixed cages obtained by our program coincides with the order of the circulant mixed graphs presented as an upper bound in [1]. Moreover, the same idea generalizes to all $g \geq 3$ and still matches all of our computed mixed cages and lower bounds. We call this graph the AHM circulant and denote it by $AHM(z, g)$.

Definition 0.2. Let $z \geq 1$ and $g \geq 3$. Define

$$N(z, g) = \begin{cases} z(g-1) + 2 & z \text{ even,} \\ (z+1)(g-1) & z \text{ odd,} \end{cases}$$

and let $m = \frac{N(z, g)}{2}$. The AHM circulant is defined as

$$\text{AHM}(z, g) = \vec{C}_{2m}(1, 2, \dots, \lceil \frac{z}{2} \rceil; \pm m; m+1, m+2, \dots, m + \lfloor \frac{z}{2} \rfloor).$$

In the previous definition, the semicolons are there just for readability, separating three distinct patterns in the jumps. Also, note that $m \equiv -m \pmod{2m}$, and hence this jump would produce pairs of antiparallel arcs, which are forbidden; we mean that as edges connecting antipodal vertices. The ‘ $\pm$ ’ sign is there for emphasis.

Theorem 0.3. $\text{AHM}(z, g)$ is a $(1, z, g)$ -mixed graph, hence $n(1, z, g) \leq N(z, g)$.

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Session 8: Games, domination, alliances, and propagation

Normal Domination Partizan Game in Stars

João Marcos Brito, Edileudo Moreira Filho, Jefer Paz and Rudini Sampaio 132

On the Complexity of Fall Coloring for Subclasses of Chordal Graphs

Ana Silva, Julio Araujo, Sheila Almeida and Vinicius dos Santos 134

O jogo da posição geral em alguns produtos fortes de grafos

Douglas Bessa and Hebert Silva 136

Global defensive alliance in lexicographic product of stars with paths or cycles

Mitre C. Dourado and Alessandra B. Verissimo 138

Contributions toward the Flood-It Problem in co-comparability graphs

Luiza Santos, Sheila de Almeida and Mayara Omai 140

Optimal containment for the firefighter problem on circular and Möbius ladders

Wagner da Rocha and Diana Sasaki 142

Normal Domination Partizan Game in Stars

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Keywords: Domination game; Normal game; Combinatorial Game Theory; partizan games; split graphs; complete bipartite graphs; stars

The Domination game on graphs was introduced in 2010 by Brešar et al. [2] and, after this, many publications investigated it, including a book, “*Domination Game Played on Graphs*” [4] of 2021. It was proved PSPACE-complete in the optimization variant in 2016 by Brešar et al. [3], and proved PSPACE-complete in the normal variant in 2026 by Brito et al. [5]. For more information on combinatorial games in general and the domination game in particular, also see the recent book “*Theory of Combinatorial Games in Graphs*” [1].

In this game, Alice and Bob alternately select vertices of a graph in such a way that every vertex selected must dominate at least one vertex not dominated by the vertices selected before in the game. The game ends when the set of selected vertices form a dominating set. In the normal variant, the last to play wins. This is an impartial game in the sense that there are no special vertices that only one of the players can play. In this paper, instead of the impartial game, we investigate the partizan version of this game in the normal variant, using the Combinatorial Game Theory. In the partizan version, the vertices of the graph are already colored with A , B , or C , so Alice (resp. Bob) can only select vertices colored with A (resp. B) or C . In this paper, we determine the winner of the Normal Partizan Domination game in graphs whose connected components are complete split graphs and complete bipartite graphs, which includes star forests, for any initial coloring of its vertices.

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On the Complexity of Fall Coloring for Subclasses of Chordal Graphs

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Keywords: graph coloring; fall coloring; chordal graphs

Let $G = (V, E)$ be a simple graph. A *proper k -coloring* of G is a function $f : V \rightarrow \{1, \dots, k\}$ such that $f(u) \neq f(v)$ for every $uv \in E$. Given a proper coloring f of G , a vertex $v \in V$ is a *b-vertex* in f if it has neighbors in every color class but its own. A proper k -coloring f of G is a *fall k -coloring* if every $v \in V$ is a b-vertex in f . The *fall achromatic number* of G , denoted by $\psi_f(G)$, is the maximum integer k such that G admits a fall k -coloring. One should notice that $\psi_f(G) \leq \delta(G) + 1$. Consequently, if G admits no fall coloring, its fall achromatic number is undefined. Observe also that a fall coloring of G is equivalent to a partition of $V(G)$ into independent dominating sets, which is known in the literature as an *idomatic partition*.

The concept of fall coloring was introduced by Dunbar et al. [2], who established its foundational properties and proved that deciding whether a graph G admits a fall k -coloring is an NP-complete problem. The parameter is closely related to the well-known b-chromatic number of a graph [4], but evaluating the fall achromatic number involves the more stringent condition that all vertices of the graph must act as b-vertices.

Due to the hardness of the problem on general graphs, the computational complexity of deciding if a graph admits a fall k -coloring has been extensively studied for several restricted graph classes over the years. The problem remains NP-complete even when restricted to bipartite graphs [2, 5], 3-regular graphs [6], and chordal graphs [7]. On the positive side, polynomial-time algorithms to determine the fall achromatic number have been successfully obtained for a few specific classes, notably trees [2] and cographs [7].

For a chordal graph G , if G has a fall k -coloring, then $k = \delta(G) + 1 = \omega(G) = \chi(G)$. Testing this condition is sufficient for split [1] and strongly chordal [3] graphs, implying polynomial-time solvability for these classes as well. However, Silva [7] proved that determining whether a chordal graph G admits a fall coloring is NP-complete. The NP-completeness reduction for chordal graphs does not trivially extend to more restricted classes.

Recall that chordal graphs are precisely the intersection graphs of subtrees of a given tree. In this work, we approach the path graphs hierarchy, which are defined as the intersection graphs of (directed) paths in (oriented) trees. The hierarchy is as follows.

Interval $\subset$ **RDV** $\subset$ **In-Out RDV** $\subset$ **Directed Path** $\subset$ **Undirected Path**.

We provide new results on the complexity of the problem when constrained to the path graphs hierarchy.

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O jogo da posição geral em alguns produtos fortes de grafos

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Keywords: Conjunto de posição geral; Jogo imparcial; Jogo normal; Jogo misère; Produto forte de grafos

O *Jogo da Posição Geral* é jogado em um grafo G pelos jogadores Alice e Bob que se alternam selecionando vértices de G . A seleção de um vértice é uma *jogada legal* se o vértice não tiver sido selecionado previamente e o conjunto de vértices selecionados até então forma um conjunto de posição geral em G . Existem duas variantes para o jogo da posição geral, o jogador que selecionar o último vértice é o vencedor da *variante normal* e é o perdedor da *variante misère*. Um conjunto de vértices $S \subseteq V(G)$ é um *conjunto de posição geral* caso não existam três vértices em S que estejam em uma mesma geodésica. Chandran, Klavžar e Sampaio [1] mostraram que o jogo da posição geral, tanto na variante normal quanto na variante misère, são PSPACE-completos mesmo em grafos com diâmetro no máximo 4.

Klavžar et al. [2] provaram que para a variante normal do jogo da posição geral em produtos cartesianos, Alice vence jogo em $K_n \square K_m$ se e apenas se tanto n quanto m forem ímpar, Bob vence em $K_n \square G$ se n é par e G conexo, em produtos lexicográficos Bob vence o jogo em $G \circ K_n$ se e apenas se Bob vence em G ou n é par. Outros resultados para a variante misère do jogo da posição geral em produtos cartesianos foram obtidos em [1], no caso em que $n, m \geq 2$, Bob vence em $K_n \square K_m$ se e apenas se ou $n = 2$ e m é ímpar, ou $n = 3$ e m é par, caso $n \geq 3$ e $m \geq 2$, então Bob vence em $P_n \square P_m$, caso $n \geq 3$ e $m \geq 2$, então Bob vence em $C_n \square P_m$ se e apenas se n é ímpar, em produtos lexicográficos com G sendo um grafo conexo e $n \geq 1$, Bob vence em $G \circ K_n$ se e apenas se Bob vence na variante misère em G e n é ímpar.

O produto forte $G \boxtimes H$ dos grafos G e H possui o conjunto de vértices $V(G) \times V(H)$, os vértices $(g_1, h_1), (g_2, h_2)$ são adjacentes em $G \boxtimes H$ se algum dos três casos for satisfeito: $g_1 g_2 \in E(G)$ e $h_1 = h_2$, $h_1 h_2 \in E(H)$ e $g_1 = g_2$, ou $g_1 g_2 \in E(G)$ e $h_1 h_2 \in E(H)$.

Chandran, Klavžar e Sampaio [1] identificaram a ausência de resultados para o jogo da posição geral para ambas variantes em produto forte de gra-

fos, que seriam de interesse para pesquisas futuras. Então, neste trabalho investigamos o jogo da posição geral em produto forte de grafos completos e grafos simples arbitrários: $K_n \boxtimes G$.

Teorema 1. *Na variante normal (resp. misère) no jogo da posição geral jogada em $K_n \boxtimes G$ entre um grafo completo K_n , e um grafo simples arbitrário G , se n é ímpar e se um dos jogadores tiver uma estratégia vencedora para G , então este jogador também tem uma estratégia vencedora em $K_n \boxtimes G$.*

Demonstração. Para $g \in V(G)$ fixo, o subgrafo induzido pelo conjunto de vértices $\{(g, k) | k \in V(K)\}$ é uma camada K^g . Para duas camadas K^{g_i} e K^{g_j} grafos completos e pela definição de produto forte se $v_1, v_2 \in V(K^{g_i} \cup K^{g_j})$, então $v_1 v_2 \in E(K_n \boxtimes G)$, para $g_i, g_j \in V(G)$.

Então, para os vértices $v_3, v_4 \in V(K^g)$ e para $v_5, v_6 \notin V(K^g)$, existe uma geodésica entre algum par dos vértices v_3, v_5 e v_6 que contém esses três vértices, se e somente se existe uma geodésica entre algum par dos vértices v_4, v_5 e v_6 que contém esses três vértices. Logo, v_3 já foi ou pode ser jogado, se e somente se v_4 já foi ou pode ser jogado.

Assim, quem possui estratégia de vitória (vencedor) em G a adapta de modo que uma K^g com vértice jogado em $K_n \boxtimes G$ corresponde a g jogado em G . Assim, caso o outro jogador (perdedor) jogue em um vértice de camada já jogada, o vencedor joga em um vértice desta mesma camada. E caso o perdedor jogue em um vértice de camada que não tinha sido jogada, enquanto possível, o vencedor joga em um vértice da camada correspondente ao vértice que jogaria em G . $\square$

Teorema 2. *No jogo da posição geral jogado no produto forte $K_n \boxtimes G$ entre um grafo completo K_n , e um grafo simples arbitrário G , se n par, Bob vence a variante normal enquanto Alice vence a variante misère.*

Demonstração. A quantidade de vértices por camada é par, logo ao fim do jogo a quantidade de vértices selecionados será par. $\square$

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Global defensive alliance in lexicographic product of stars with paths or cycles[‡]

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Keywords: global defensive alliance; lexicographic product

A *path* is a sequence of distinct vertices $(v_1, \dots, v_n)$, with $n \geq 1$, such that each consecutive pair of vertices is adjacent. The vertices v_1 and v_n are called the *endpoints* of the path. A *cycle* is a circular sequence of vertices $(v_1, \dots, v_n)$, with $n \geq 3$, such that each vertex is adjacent to its predecessor and successor, and all vertices are distinct. A *star* consists of a central vertex v_0 adjacent to every other vertex $v_1, \dots, v_{n-1}$ where the vertices $v_1, \dots, v_{n-1}$ are not adjacent to each other. Define $\mathcal{P} = \{P_n : n \geq 2\}$, $\mathcal{C} = \{C_n : n \geq 3\}$ and $\mathcal{S} = \{S_n : n \geq 2\}$ as the classes of nontrivial path graphs, nontrivial cycle graphs and nontrivial star graphs, respectively.

The *lexicographic product* of graphs $G_n = (V_1, E_1)$ and $H_m = (V_2, E_2)$ is the graph $F = G_n \circ H_m$ with vertex set $V = V_1 \times V_2$ and edge set $E = \{(u_1, u_2)(v_1, v_2) : (u_1v_1 \in E_1) \text{ or } (u_1 = v_1 \text{ and } u_2v_2 \in E_2)\}$. The *open neighborhood* and the *closed neighborhood* of $v \in V$ are, respectively, $N(v) = \{u \in V : vu \in E\}$ and $N[v] = N(v) \cup \{v\}$. As defined in [1], a vertex $v \in T$ is *defended* in $T \subseteq V$ if $|N[v] \cap T| \geq |N[v] - T|$. If every vertex in T is defended, T is denominated a *defensive alliance*. A *dominating set* is a subset of vertices such that every vertex in the graph either belongs to the dominating set or is adjacent to at least one vertex in it. If a defensive alliance T is also a dominating set, then T is denominated a *global defensive alliance*. The minimum number of vertices of a *global defensive alliance* is denoted as $\gamma_a(G)$. In [3] the value $\gamma_a(G_n \circ H_m)$ where $G_n, H_m \in \mathcal{P} \cup \mathcal{C}$ was obtained. In the present work, we obtain $\gamma_a(G_n \circ H_m)$ where $G_n \in \mathcal{S}$ and $H_m \in \mathcal{P} \cup \mathcal{C}$.

Let $F = S_n \circ H_m$. A copy H_m in the product $F = S_n \circ H_m$ is called the *central copy* if it corresponds to the universal vertex of star S_n . Otherwise, the copy is referred to as a *leaf copy*.

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Note that the vertices of the central copy have more neighbors than those of the leaf copies. Lemmas 1 and 2 proved that when the global defensive alliance T is minimum, approximately half the quantity of the vertices of the central copy are in T .

Lemma 1 *Let T be a minimum global defensive alliance of $F_1 = S_n \circ P_m$ where $S_n \in \mathcal{S}$ and $P_m \in \mathcal{P}$. If q_1 is the number of vertices of T in the central copy of F_1 , then $\frac{m-2}{2} \leq q_1 \leq \frac{m}{2}$.*

Lemma 2 *Let T be a minimum global defensive alliance of $F_2 = S_n \circ C_m$ where $S_n \in \mathcal{S}$ and $C_m \in \mathcal{C}$. If q_2 is the number of vertices of T in the central copy of F_2 , then $\frac{m-1}{2} \leq q_2 \leq \frac{m+1}{2}$.*

Let T be a global defensive alliance. If a leaf copy contains at least one vertex but fewer than m vertices in T , this copy is called an *incomplete leaf copy*. The global defensive alliance T is minimum when at most one leaf copy is incomplete. Theorems 1 and 2 determine, respectively, $\gamma_a(F_1)$ and $\gamma_a(F_2)$ according to the parities of n and m .

Theorem 1 *Let $F_1 = S_n \circ P_m$ where $S_n \in \mathcal{S}$ and $P_m \in \mathcal{P}$. Thus, $\gamma_a(F_1) = \frac{nm-k}{2}$ where $k = 0$ if n and m are even; $k = 1$ if n and m are odd; $k = 2$, otherwise.*

Theorem 2 *Let $F_2 = S_n \circ C_m$ where $S_n \in \mathcal{S}$ and $C_m \in \mathcal{C}$. Thus, $\gamma_a(F_2) = \frac{nm-k}{2}$ where $k = 0$ if n and m are even; $k = 2$, otherwise.*

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Contributions toward the Flood-It Problem in co-comparability graphs

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The Flood-It Game is a combinatorial puzzle whose objective is to turn an initially multicolored surface monochromatic. The original game is played on a board with colored cells where one cell is designated as the pivot. A move in the game consists of changing the color of the pivot to a color c chosen by the player. This induces a set of cells consisting of the pivot and all cells that are connected to it through a monochromatic path of color c , called the *pivot's monochromatic component*. Note that the size of the pivot's monochromatic component may grow with each move. Thus, the game ends as soon as the board is completely flooded, that is, monochromatic. There are two versions of the game: one in which the pivot remains the same cell throughout the game, called the *fixed-pivot version*, and another in which the pivot may be chosen at each move and can correspond to any cell on the board, called the *free-pivot version*.

The game board can be modeled as a connected graph such that each board cell represents one of the vertices of the graph, and an edge exists between two vertices if and only if the corresponding cells are adjacent. *The Flood-It Problem* consists of finding the minimum number of moves to flood a given graph. The problem represents a significant computational challenge. It is NP-hard in its general form [3], meaning that no efficient polynomial-time algorithm is known for solving all instances. The free-pivot Flood-It Problem is notoriously difficult, even when restricted to grids with three or more colors [1]. Fleischer and Woeginger [2] proved that the fixed-pivot Flood-It Problem is NP-hard when restricted to trees and split graphs.

Despite of the problem's complexity, Fleischer and Woeginger [2] presented a polynomial-time algorithm for the Flood-it Problem on co-comparability graphs with a fixed pivot. *Comparability graphs* are graphs that

admit transitive orientation to its edges [4]. Graphs whose complement is a comparability graph are *co-comparability graphs*.

In a recent study, Lorenzi and Almeida [5] identified a case in which the algorithm proposed by Fleischer and Woeginger [2] enters a recursive dependency cycle and fails to terminate. During the development of this work, we identify two additional structural limitations of the algorithm. As a result, we propose a polynomial-time solution based on multi-source shortest-path techniques that addresses the fundamental issues identified in previous approaches to the Flood-It Problem, yielding a polynomial-time algorithm for the fixed-pivot Flood-It Problem on co-comparability graphs.

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Optimal containment for the firefighter problem on circular and Möbius ladders

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Keywords: Firefighter problem; cubic graph; circular ladder graph; Möbius ladder graph

The firefighter problem is a discrete-time graph protection problem introduced by Hartnell in 1995 [1]. A fire breaks out at a vertex of a graph. At each time step, a firefighter is deployed to protect one non-burning vertex, after which the fire spreads to all unprotected neighbors of the currently burning vertices. The process terminates when no further spread is possible. The objective is to maximize the number of vertices that are ultimately saved. Beyond its intrinsic graph-theoretic interest, the model captures the containment of spreading phenomena in networks, including the propagation of rumours, viruses, and epidemics, among other spreading phenomena [2].

The problem is computationally difficult even on restricted graph classes. MacGillivray and Wang showed that the associated decision problem is NP-complete for bipartite graphs [3], while Finbow *et al.* proved that the optimization problem is NP-hard for trees of maximum degree 3 [4]. This motivates the study of structured graph families.

In this work we determine the optimal number of vertices saved in the firefighter problem for the circular ladder graphs CL_{2n} (also called prism graphs) and the Möbius ladder graphs M_{2n} . The graph CL_{2n} consists of two copies of cycles on n vertices connecting their corresponding vertices, while M_{2n} is obtained from a cycle on $2n$ vertices by adding edges (chords) between opposite vertices. Both families are vertex-transitive; moreover, CL_{2n} is bipartite when n is even, whereas M_{2n} is bipartite when n is odd [5].

For an instance (G, v) of the firefighter problem, let $MVS(G, v)$ denote the maximum number of vertices of G that can be saved when the fire starts at v . To guide the formulation of our main result, we developed a branch-and-bound algorithm that explores all possible plays and prunes any partial play

that cannot improve the best solution found so far. The following theorem summarizes the results.

Theorem 1. *Let CL_{2n} be a circular ladder graph and let M_{2n} be a Möbius ladder graph, both with $|V| = 2n$. For every integer $n \geq 3$, every $v \in V(CL_{2n})$, and every $w \in V(M_{2n})$, we have*

$$MVS(CL_{2n}, v) = MVS(M_{2n}, w) = 2n - 6.$$

In particular, exactly six vertices burn under an optimal strategy in each family. Thus, in these vertex-transitive families, the optimum depends only on the order of the graph and not on the initial vertex. The proof follows a standard approach in the firefighter literature, combining explicit strategies with matching upper bounds.

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